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Belief rule-based system for portfolio optimisation with nonlinear cash-flows and constraints

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ABSTRACT

A belief rule-based (BRB) system is a generic nonlinear modelling and inference scheme. It is based on the concept of belief structures and evidential reasoning (ER), and has been shown to be capable of capturing complicated nonlinear causal relationships between antecedent attributes and consequents. The aim of this paper is to develop a BRB system that complements the RiskMetrics WealthBench system for portfolio optimisation with nonlinear cash-flows and constraints. Two optimisation methods are presented to locate efficient portfolios under different constraints specified by the investors. Numerical studies demonstrate the effectiveness and efficiency of the proposed methodology.

1. Introduction

Portfolio optimisation is concerned with maximising the wealth of investors and managing the risk of an integrated portfolio over a given holding period. In modern portfolio theory, the seminal Markowitz's mean–variance model provides a useful theoretical framework for solving portfolio optimisation problem (Markowitz, 1952; Michaud, 1989). A trade-off is made between return and risk in investment and asset allocation decisions. Among the set of investment portfolios with the same level of return, an investor should choose an "efficient" portfolio that has the smallest variance or risk, as all the other portfolios are "inefficient". However, apart from the Gaussian assumption, the original Markowitz's optimisation scheme ignores environmental factors (e.g., taxes and transaction costs) (Campbell and Viceira, 2002). In multi-period portfolio optimisation problems, scenario simulation analysis is used to forecast expected returns (Markowitz and Perold, 1981; Koskosidis and Duarte, 1997). As such, a series of theoretical work has been conducted to improve portfolio mean–variance analysis (Black and Litterman, 1992; Tütüncü and Koenig, 2004; Ehrgott et al., 2004; Jacobs et al., 2005; Kim et al., 2005; Idzorek, 2006; Çelikyurta and Özekici, 2007; DeMiguel et al., 2009; Xidonas et al., 2011). At the same time, various simulation techniques using historical data or Monte Carlo algorithm have also been used to forecast returns of portfolios with nonlinear instruments such as options (RiskMetrics, 1999, 2004; Palmquist et al., 2001; Glasserman, 2004; Brandt et al., 2005; Guastaroba et al., 2009).

Most portfolio decisions begin with asset class allocation, which is an important prerequisite to long-term forecasting of individual asset returns and risks. An asset class is a set of assets (e.g., shares, bonds) that has some common return characteristics (Greer, 1997). It is widely believed that asset class returns can be forecasted more accurately than individual asset returns which tend to revert to asset class returns in the long-term (RiskMetrics, 2004). In allocating investment to a number of asset classes, the objectives of portfolio optimisation depend on investors' financial targets and cash-flow constraints. In long-term asset allocation, the amount invested usually varies, as investors both accumulate and invest savings, and as they make withdrawals to pay for expenses (RiskMetrics, 2003). Under these nonlinear cash-flows and constraints, the relationship between a set of asset class weights (i.e., a portfolio) and the distribution of its possible returns is highly nonlinear (Campbell and Viceira, 2002; RiskMetrics, 2004; Gondzio and Grothey, 2007). In order to identify the efficient allocations of asset class weights for maximising returns and minimising risks, there is a need to develop new methods that can be used to model and optimise in such a highly nonlinear setting. This paper develops a novel portfolio modelling and optimisation method using a belief rule-based (BRB) system. The BRB system extends traditional IF-THEN rule based model using the concept of belief structures and evidential reasoning (ER). It is capable of capturing complicated causal relationships with different types of information under uncertainties (Yang and Singh, 1994; Yang et al., 2006, 2007) and has been widely applied in nonlinear system modelling and decision support systems (Xu et al., 2007; Zhou et al., 2009; Chen et al., 2011).

In this paper, the BRB system works by combining high dimensional return distributions of assets and portfolios to deduce return distribution of a new portfolio of a given set of investment weights.

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Table 1
Belief rule-based system and portfolio optimisation.

BRB system	Portfolio optimisation
Input and parameters	
Antecedent attributes	Investment weights $w_1, \ldots, w_i, \ldots, w_I$ for I assets.
Referential value $A_i = \{A_{i,j}; j = 1, \ldots, J_i\}$	Referential weights that w_i may take.
Antecedent in the kth rule	Investment weight of the kth referential portfolio $(w_{k,1}, w_{k,2}, \ldots, w_{k,I})$, $w_{k,i} \in A_i$.
Consequent in the kth rule $\{(D_1, p_{k,1}), (D_2, p_{k,2}), \ldots, (D_N, p_{k,N})\}$	Return distribution of the kth referential portfolio with probability $p_{k,n}$ associated with referential return D_n.
Rule weight θ_k, $k = 1, \ldots, K$	Relative importance of the kth referential portfolio. For uninformative rule weight, $\theta_k = 1$ for all k[a].
Attribute weights $\delta_{i,k}$	Relative importance of asset i in the kth referential portfolio. For uninformative attribute weight, $\delta_{i,k} = 1$ for all i[b].
Output	
Belief distribution $\{(D_1, p_1^*), \ldots, (D_n, p_n^*), \ldots, (D_N, p_N^*)\}$	BRB inference on return distribution of target portfolio P^* with specific investment weights.

[a] In this paper the rule weight is set equal to 1, i.e. uninformative rule weight. Although it is not implemented in this paper, unequal rule weights can be used to reflect the importance (or unimportance) of the kth referential portfolio based on expert knowledge.
[b] In portfolio optimisation, the attribute weight (i.e., the relative importance of asset classes) is set equal to 1. Unequal attribute weights reflect the importance (or unimportance) of the ith asset in the kth referential portfolio. Attribute weights are always non-negative.

The goal of this paper is not to propose any portfolio strategy. The BRB system is an intelligent inference scheme for projecting portfolio return distribution and thereby allows us to trace the efficient frontier and locate the optimal portfolio for a given set of investment objectives. The method works for any investment objectives, cash flow constraints and utility preference. The rest of the paper is organised as follows. The portfolio optimisation problem is modelled using a BRB system, and the detailed procedures are described in the following section. In Section 3, two portfolio optimisation methods are presented to search for efficient portfolios under different constraints. In Section 4, numerical studies are conducted to demonstrate the effectiveness and efficiency of the developed BRB system for portfolio optimisation. The paper is concluded in Section 5.

2. Portfolio modelling using a BRB system

Belief rules are the key constituents of a BRB system, and they can be regarded as extended IF-THEN rules. In a belief rule, each antecedent attribute takes a referential value, and each possible consequent is associated with a belief degree (Yang et al., 2006). To build a BRB system for portfolio optimisation, we map the relationships between BRB systems and portfolio optimisation in Table 1 below.

From Table 1, with "∧" denotes "and", the kth ($k = 1, \ldots, K$) referential portfolio, i.e., the belief rule k in the BRB system, can be defined as

$$\text{Rule } k: \begin{cases} \text{IF}(w_1 \text{ is } w_{k,1}) \wedge (w_2 \text{ is } w_{k,2}) \wedge \cdots \wedge (w_I \text{ is } w_{k,I}), \\ \text{THEN}\{(D_1, p_{k,1}), (D_2, p_{k,2}), \ldots, (D_N, p_{k,N})\}, \quad \left(\sum_{n=1}^{N} p_{k,n} \leqslant 1\right), \\ \text{with rule weight } \theta_k, \\ \text{and attribute weight } \delta_{1,k}, \delta_{2,k}, \ldots, \delta_{I,k}, \quad k \in \{1, \ldots, K\}. \end{cases} \tag{1}$$

To use the BRB system to solve the portfolio optimisation problem involves the following steps: (1) constructing K number of referential portfolios; (2) calculating activation weights for a smaller set of K_P referential portfolios; (3) generating new portfolio P^* through BRB inference.

2.1. Constructing referential portfolios

2.1.1. Generating referential portfolios

In portfolio optimisation, the investment weights sum to unity. In addition, investors may also specify upper or lower bounds on the investment weights. For example, an investor may specify that investment in large cap stocks must be at least 20%, or the investment weight for equity be between 30% and 50%. Let lb_i and ub_i represent, respectively, the lower and upper bounds on the weight of the ith asset. Then the linear and unity constraints can be represented as[1]

$$lb_i \leqslant w_i \leqslant ub_i \tag{2}$$

$$\sum_{i=1}^{I} w_i = 1 \tag{3}$$

From Eq. (3), we can reduce the dimension of the solution space by one, and the weight w_I of the Ith asset can be represented as

$$w_I = 1 - \sum_{i=1}^{I-1} w_i \tag{4}$$

which means that the number of independent weights becomes $I - 1$.

Since the constraints on investment weights are often individually specified, there might be conflicting and redundant constraints.[2] Thus, the lower and upper bounds $\overline{lb}_i$ and $\overline{ub}_i$ for each asset can be updated as

$$\overline{lb}_i = \max\left(lb_i, 1 - \sum_{i'=1, i' \neq i}^{I} ub_{i'}\right) \tag{5}$$

$$\overline{ub}_i = \min\left(ub_i, 1 - \sum_{i'=1, i' \neq i}^{I} lb_{i'}\right) \tag{6}$$

With the updated and internally consistent weight constraints, the set of referential values $A_i = \{A_{i,j}; j = 1, \ldots, J_i\}$ can be generated based on $\overline{lb}_i$ and $\overline{ub}_i$, and the combinations of J_i for $i = 1, \ldots, I - 1$. As an illustrative example suppose there are three assets with the following investment weight constraints:

[1] First of all, w_i is typically set within a certain range to avoid corner solution or solutions that are meaningless, e.g. it may not be feasible for $w_{i=1}$ to equal to some positive million and $w_{i=2}$ to equal to some negative million. In practice, there is usually the implementation rule that the proportion invested in a particular asset class is capped. Here, the numerical example is based on safer wealth management and hence there is no short selling position which is typically associated with a more aggressive and speculative strategy. But the method here also works for negative investment weights.

[2] Any weight constraints (other than upper and lower bounds and summing to unity) will not be imposed on the initial referential portfolios. Otherwise, the subsequent inferences will be based on constrained distribution and partial information. The investment weight constraints can be easily accommodated in portfolio optimisation however.

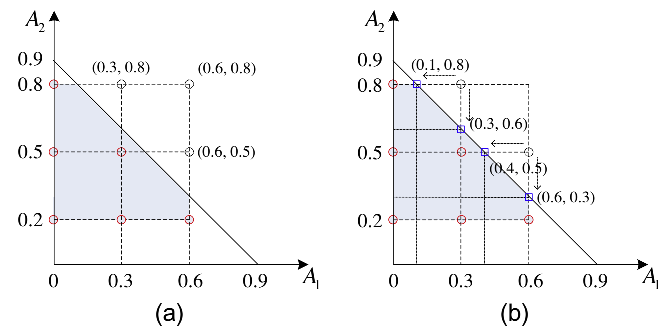

Fig. 1. Three assets example with projecting new referential points.

$$0 \leqslant w_1 \leqslant 0.6, \quad 0.2 \leqslant w_2 \leqslant 0.8, \quad 0.1 \leqslant w_3 \leqslant 0.8$$

Since w_3 can be expressed in terms of w_1 and w_2, we can use the combination of the referential values of w_1 and w_2 to construct the initial set of referential portfolios. Assume the referential values of w_1 are {0,0.3,0.6} and those of w_2 are {0.2,0.5,0.8}. Without the unity constraint, 3×3 referential portfolios will be generated as shown in Fig. 1a. Note that in Fig. 1 the feasible space is shaded and the feasible referential portfolios are shown by circles.

However, since the referential values are generated only for two independent variables, we need to check the weight for the third variable. It is obvious in Fig. 1a that some of the referential points, i.e., (0.3,0.8), (0.6,0.5) and (0.6,0.8), are not within the feasible solution space which is constrained by the bounds of the third variable.

$$\sum_{i=1}^{I-1} w_i \leqslant 1 - \overline{lb}_I \tag{7}$$

$$\sum_{i=1}^{I-1} w_i \geqslant 1 - \overline{ub}_I \tag{8}$$

If we simply delete the infeasible referential points, the BRB inference accuracy near those deleted referential points will decrease. Hence, some new referential points are generated to replace the infeasible ones as shown in Fig. 1b. To replace the infeasible referential point (0.3,0.8), two new referential points (0.1,0.8) and (0.3,0.6) are generated by projecting the infeasible referential point onto the boundary of the feasible space. Similarly, two new referential points (0.4,0.5) and (0.6,0.3) are generated to replace the infeasible referential point (0.6,0.5). No new referential points are needed for the infeasible referential point (0.6,0.8), since its projection overlaps with the referential points (0.1,0.8) and (0.6,0.3).

The above algorithm works fine for the three-asset example. For a multi-asset portfolio optimisation problem, a heuristic method is developed to generate new referential points to replace infeasible ones in an effective way. If a referential point violates the constraint (7), we raise w_I to $\overline{lb}_I$ by reducing the weights of the other $I-1$ assets one by one to produce a maximum of $I-1$ possible new referential points for replacement. In the above example, the referential point (0.3,0.8) is infeasible, as the sum of two referential values is 0.3 + 0.8 = 1.1, which is larger than $1 - \overline{lb}_I = 1 - 0.1 = 0.9$. Therefore, we use the heuristic method above to generate new referential points for replacement. We can reduce either the weight of w_1 to $1 - \overline{lb}_I - w_2 = 1 - 0.\ 1 - 0.8 = 0.1$ or the weight of w_2 to $1 - \overline{lb}_I - w_1 = 1 - 0.1\ -0.3 = 0.6$. As a result, two new referential points (0.1,0.8) and (0.3,0.6) are generated to replace the infeasible referential point (0.3,0.8). However, if changing the weight w_i from $A_{i,j}$ to $A_{i,j-1}$ is still not enough to raise w_I to $\overline{lb}_I$, this particular replacement is simply discarded because it will lead to redundant replacement. In Fig. 1b, for example, the replacement can be ignored for the infeasible referential point (0.6,0.8).

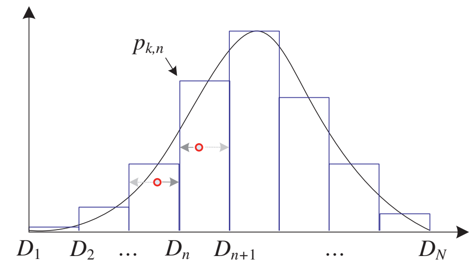

Fig. 2. Probability distribution on buckets of referential returns.

2.1.2. Calculating probability distributions for referential portfolios

For each of the K referential portfolios, we use the RiskMetrics WealthBench (RM-WB) platform (RiskMetrics, 2004) to simulate a set of $L = 500$ 30-year returns $\dot{R}_k = \{\dot{R}_{k,l}, l = 1, \ldots, L$ and $k = 1, \ldots, K\}$, where L is the number of sample paths.[3] We use $\dot{R}$ with a dot to indicate portfolio returns generated by simulation. For each path, the portfolio return is a cumulative outcome of monthly returns based on the assumed dynamics of the constituent assets, projected tax rules and investor's cash flows requirements. With the set of simulated returns, the mean and variance of the portfolio can be calculated as

$$E(\dot{R}_k) = \frac{1}{L}\sum_{l=1}^{L}\dot{R}_{k,l} \tag{9}$$

$$var(\dot{R}_k) = \frac{1}{L}\sum_{l=1}^{L}(\dot{R}_{k,l})^2 - \left(\frac{1}{L}\sum_{l=1}^{L}\dot{R}_{k,l}\right)^2 \tag{10}$$

Next, as shown in Fig. 2, we group the simulated portfolio returns $\dot{R}_k$ into $N-1$ buckets using N referential returns D_1, $D_2, \ldots, D_N$ covering the entire range of all K referential portfolio returns. It is worth noting that the buckets could be non-uniformly distributed, and that the probability $p_{k,n}$ is on the referential return D_n, instead of on the bucket $[D_n, D_{n+1}]$.

For the kth referential portfolio, we can simply put $\dot{R}_{k,l}$ into the buckets and obtain the probability of each bucket. However, the expected return cannot be recovered accurately with such a scheme. Therefore, we use the proportional allocation method below to produce the probability $p_{k,n}$:

$$p_{k,n} = \frac{1}{L}\left(\sum_{l_{n-1}\in L_{n-1}}(\dot{R}_{k,l_{n-1}} - D_{n-1})/(D_n - D_{n-1}) + \sum_{l_n\in L_n}(D_{n+1} - \dot{R}_{k,l_n})/(D_{n+1} - D_n)\right), \quad 1 \leqslant n \leqslant N \tag{11}$$

[3] The returns on different asset classes (see Footnote 5) are simulated using simple shifted lognormal distribution using volatility–covariance estimates for different time horizons (short term, 1 year, 3 years, and 5 years and above). The mean drifts are formulated with a long-term view of current investments taking into account inflation forecasts over the same period. The cash flows are simulated sequentially at monthly interval using these input and chosen dynamics, and the output are bucketed into "accounts" (classification includes taxable, tax-deferred, non-taxable, locked taxable for assets with sales restriction, locked (municipal) bond used for income). These accounts have specific rules concerning cash flow, taxation and inter-account transfer. The terminal cash flows are aggregated before the calculation of terminal wealth. It is the annualized returns and risk estimates of the portfolios that have been shown to have qualified the constraints that are fed into the BRB optimisation algorithms here. Portfolio moment characteristics (e.g. mean and variance) are then calculated for all qualified portfolios. The portfolio return is simply the annualized growth rate of the wealth invested over the investment horizon, i.e. 30 years in this case.

where L_n is the number of simulated returns that fall into the bucket $[D_n, D_{n+1}]$, i.e., $D_n \leqslant \dot{R}_{k,l_n} < D_{n+1}, \forall l_n \in L_n$ and $L_0 = L_N = 0$. The probability $\sum_{n=1}^{N} p_{k,n} = 1$, and it is possible that some $p_{k,n} = 0$. From here, the mean and variance of the kth referential portfolio can be recovered as:

$$E(R_k) = \sum_{n=1}^{N} D_n p_{k,n} \tag{12}$$

$$var(R_k) = \sum_{n=1}^{N} (D_n)^2 p_{k,n} - \left(\sum_{n=1}^{N} D_n p_{k,n}\right)^2 \tag{13}$$

Here we use R_k without a dot to represent portfolio returns derived from the probability distribution of the tabulated buckets above in contrast to those with a dot calculated straight from the simulation. From Eq. (11), it can be shown that the mean returns calculated from Eqs. (9) and (12) are identical, but not the variances calculated in Eqs. (10) and (13). The difference between the statistical variance and the approximated variance is related to the number of the buckets. Appendix A shows that the two variances converge as N tends to infinity. As such, we need to make a trade-off between computational complexity and inference accuracy. The referential returns $D_1, D_2, \ldots, D_M$ and the associated probabilities $p_{k,n}$ for each referential portfolio are the key ingredients in producing return distributions of new portfolios.

2.2. Calculating activation weights

Given a new set of investment weights for a new portfolio P^*, now we can use the BRB system constructed above to derive its return distribution. First, we have to assess the matching degree between the given investment weights and those of the kth referential portfolios and calculate the activation weight λ_k for a smaller set of most relevant referential portfolios. As discussed before, we must first check that the referential investment weights do not violate Eqs. (7) and (8). Otherwise they will be replaced by projected weights as explained in the previous section. Note that this will then lead to a different portfolio of course. As the set of referential weights might be revised, the original matching method in (Yang et al., 2006) is no longer applicable. Here, we use the normalised Euclidian distance to calculate the activation weight of the kth ($k \in K_P$) referential portfolio as shown below:

$$\lambda_k = \frac{\sqrt{\sum_{i=1}^{I-1}(w_i - w_{k,i})^2}}{\sum_{l \in K_P}\left[\sqrt{\sum_{i=1}^{I-1}(w_i - w_{l,i})^2}\right]} \tag{14}$$

where K_P represents a set of referential portfolios on a specific polyhedron. In the numerical example in the previous section, if a new portfolio has the asset weights w_1 and w_2 as (0.4,0.3), the five referential portfolios (0.3,0.2), (0.3,0.5), (0.4,0.5), (0.6,0.2) and (0.6,0.3) on a polygon will be activated.

2.3. Generating new portfolios with BRB inference

Once the activations weights λ_k are obtained, the inferences on the return distribution of new portfolio P^* can be summarised using a belief rule expression matrix as show in Table 2 below:

From here the ER approach is used to combine activated rules to infer the return distribution of portfolio P^*:

$$\{(D_1, p_1^*), \ldots, (D_n, p_n^*), \ldots, (D_N, p_N^*)\} \tag{15}$$

where p_n^* represents the combined probability assigned to D_n.

The ER approach is a generic evidence-based multi-criteria decision analysis approach for dealing with problems having both quantitative and qualitative criteria (Yang and Singh, 1994). The kernel of the ER approach is a recursive reasoning algorithm which is developed on the basis of Dempster–Shafer (D–S) theory (Dempster, 1968; Shafer, 1976) and decision theory (Yoon and Hwang, 1995; Giovanni and Lurdes, 2009). The implementation procedure of the ER approach is summarised as follows (Yang and Xu, 2002). First, for each referential portfolio k with activation weight λ_k, we calculate the weighted probability mass for each bucket $m_{k,n} = \lambda_k p_{k,n}$. Next, we calculate

Table 2
Belief rule expression matrix.

Referential Portfolio	Activation weight	Belief degree (or probability)					
		D_1	D_2	…	D_n	…	D_N
1	λ_1	$p_{1,1}$	$p_{1,2}$	…	$p_{1,n}$		$p_{1,N}$
2	λ_2	$p_{2,1}$	$p_{2,2}$	…	$p_{2,n}$		$p_{2,N}$
⋮	⋮						
3	λ_k	$p_{k,1}$	$p_{k,2}$	…	$p_{k,n}$		$p_{k,N}$
⋮	⋮						
K_P	λ_{K_p}	$p_{K_p,1}$	$p_{K_p,2}$	…	$p_{K_p,n}$		$p_{K_p,N}$

$$\bar{m}_k = 1 - \sum_{n=1}^{N} m_{k,n} = 1 - \lambda_k \sum_{n=1}^{N} p_{k,n} = 1 - \lambda_k, \quad \text{and} \quad \bar{m} = \mu\left[\prod_{k=1}^{K_P} \bar{m}_k\right]$$

where μ is a normalising factor to ensure probabilities sum to 1, $\bar{m}_k$ represents the amount of weighted probability mass in the final output not contributed by the referential portfolio k, and $\bar{m}$ reflects the aggregation of the weighted influence exerted by the set of K_P referential portfolios. If $\bar{m} = 0$, one of the referential portfolios strongly dominates the outcome. If $\lambda_k = 1/K_P$ for all k, $\bar{m} = \mu \left(1 - K_P^{-1}\right)^{K_P}$, then all the referential portfolios are at equal distance to the target portfolio P^* and exert equal influence on the outcome.

Then, the final output distribution can be inferred using the analytical ER algorithm (Wang et al., 2006),

$$m_n = \mu\left[\prod_{k=1}^{K_P}(m_{k,n} + \bar{m}_k) - \prod_{k=1}^{K_P}\bar{m}_k\right], \quad \text{and}$$

$$p_n^* = \frac{m_n}{1 - \bar{m}} = \frac{\prod_{k=1}^{K_P}(m_{k,n} + \bar{m}_k) - \prod_{k=1}^{K_P}\bar{m}_k}{1/\mu - \prod_{k=1}^{K_P}\bar{m}_k}$$

For $\sum_{n=1}^{N} p_n^* = 1$,

$$\mu = \left[\sum_{n=1}^{N}\prod_{k=1}^{K_P}(m_{k,n} + \bar{m}_k) - (N-1)\prod_{k=1}^{K_P}\bar{m}_k\right]^{-1}$$

After substituting all the intermediate variables, p_n^* can now be written in terms of activation rule weights and referential portfolio probabilities as follows:

$$p_n^* = \frac{\prod_{k=1}^{K_P}(\lambda_k p_{k,n} + 1 - \lambda_k) - \prod_{k=1}^{K_P}(1 - \lambda_k)}{\sum_{j=1}^{N}\prod_{k=1}^{K_P}(\lambda_k p_{k,j} + 1 - \lambda_k) - N\prod_{k=1}^{K_P}(1 - \lambda_k)} \tag{16}$$

If $\lambda_k > 0$ and $p_{k,n} > 0$, then p_n^* must be greater than 0.

Finally, the mean and variance of the new portfolio P^* can be calculated as

$$E(P^*) = \sum_{n=1}^{N} D_n p_n^* \tag{17}$$

$$var(P^*) = \sum_{n=1}^{N} (D_n)^2 p_n^* - \left(\sum_{n=1}^{N} D_n p_n^*\right)^2 \tag{18}$$

3. Portfolio optimisation methods

The BRB system can be used to identify efficient portfolios in two ways. Section 3.1 identifies the optimal portfolio by first constructing the efficient frontier using the means and variances of portfolios produced in Section 2.3 and then searching along the efficient frontier using the objective function supplied by the investors. The second method, as described in Section 3.2, locates the optimal portfolio directly using the BRB procedures together with a nonlinear optimiser.

3.1. Constructing the efficient frontier

In Markowitz's mean–variance analysis, the outcomes are usually represented as an efficient frontier, on which portfolios have maximum return for a given level of risk, or minimum risk for a given level of return. A rational investor would select a portfolio on the efficient frontier based on his risk-return preferences, or select the "market" portfolio if a risk free rate exists with unlimited amount of borrowing and lending. To trace an efficient frontier, the parametric Markowitz's mean–variance framework may be used in portfolio optimisation. However, the key problem here is that portfolio's return and variance are not a straightforward function of the constituents' return and variance because of the nonlinear cash flows and differential tax treatments. Thus, in this study we generate the efficient frontier using the initial referential portfolios from Section 2.1 and the portfolios generated through the BRB system in Section 2.3. A portfolio P is efficient if there is no other portfolio P' with both higher mean and lower variance of returns than those of the portfolio P. Under the investor's specified portfolio weight constraints, we eliminate all inefficient portfolios, and use only the set of efficient portfolios for constructing the efficient frontier. Along the efficient frontier, the optimal portfolio can be located once the investor's objective function is specified.

It is worth noting that the precision of the solution found along an efficient frontier depends on the granularity of interpolation points in the solution space. We can also use large granularity to roughly locate the optimal portfolio and then apply small granularity to further interpolate the target zone in order to produce more accurate solutions.

3.2. Specifying the objective function

Using Eqs. (14)–(18), we can develop the following portfolio optimisation procedures for different objective functions. Two example objective functions are listed below:

- *Maximising* expected return – Since all the investment weights and cash flow constraints are already incorporated into portfolio cash flow simulation, the optimal portfolio is the one on the efficient frontier that has the highest return that satisfies the risk tolerance constraint which is usually stated in terms of the variance of expected return.
- *Maximising* probability of meeting cash flow goals – Assume that the goals are reasonable, which means that the average return after meeting all the cash flows requirements is positive. Then, we choose the portfolio that has the highest probability of having non-negative return.

3.2.1. Mean–variance efficient set of portfolios

For an investor who wishes to find a portfolio P with the highest expected return under a given level of risk σ, the objective function is

$$\text{Max } E(P) \text{ subject to } \text{var}(P) \leqslant \sigma \tag{19}$$

From Eqs. 17 and 18, 19 can be translated into

$$\begin{aligned} &\max_{w_i} \quad \sum_{n=1}^{N} D_n p_n^* \\ &\text{s.t.} \quad \overline{lb}_i \leqslant w_i \leqslant \overline{ub}_i, \quad \sum_{i=1}^{I-1} w_i \leqslant 1 - \overline{lb}_I, \quad \sum_{i=1}^{I-1} w_i \geqslant 1 - \overline{ub}_I, \\ &\text{and} \sum_{n=1}^{N} (D_n)^2 p_n^* - \left(\sum_{n=1}^{N} D_n p_n^*\right)^2 \leqslant \sigma \end{aligned}$$

Alternatively, we can also formulate the model to minimise risk under a given level of expected return R^*:

$$\text{Min } \text{var}(P) \text{ subject to } E(P) \geqslant R^* \tag{20}$$

which can be translated into

$$\begin{aligned} &\min_{w_i} \quad \sum_{n=1}^{N} (D_n)^2 p_n^* - \left(\sum_{n=1}^{N} D_n p_n^*\right)^2 \\ &\text{s.t.} \quad \overline{lb}_i \leqslant w_i \leqslant \overline{ub}_i, \quad \sum_{i=1}^{I-1} w_i \leqslant 1 - \overline{lb}_I, \quad \sum_{i=1}^{I-1} w_i \geqslant 1 - \overline{ub}_I, \\ &\text{and} \sum_{n=1}^{N} D_n p_n^* \geqslant R^* \end{aligned}$$

3.2.2. Models for optimising probability-based risk measures

The probability level-based risk measures, such as value-at-risk (VaR), risk of loss, and shortfall risk, are important for risk management and risk regulation (Gaivoronski and Pflug, 2005). The optimisation models with such constraints can be formulated as,

$$\text{Max } E(P) \text{ subject to } prob(P < q) = p \tag{21}$$

Again, using Eqs. (17) and (18), the problem represented by Eq. (21) can be translated into,

$$\begin{aligned} &\max_{w_i} \quad \sum_{n=1}^{N} D_n p_n^* \\ &\text{s.t.} \quad \overline{lb}_i \leqslant w_i \leqslant \overline{ub}_i, \quad \sum_{i=1}^{I-1} w_i \leqslant 1 - \overline{lb}_I, \quad \sum_{i=1}^{I-1} w_i \geqslant 1 - \overline{ub}_I \\ &\text{and} \sum_{n=1}^{N} (p_n^* \times I(D_n < q)) \leqslant p \end{aligned}$$

where q is the loss limit, and $I(D_n < q)$ is equal to 1 if $D_n < q$ and 0 otherwise, p is the probability of VaR which is usually set at 5% or 1%.

The nonlinear optimisation can be solved using gradient-based search methods or nonlinear optimisation software packages, such as the *fmincon* function in the Optimization Toolbox of Matlab (Coleman et al., 1999).

4. Numerical studies

In this section, two numerical studies are conducted to illustrate the procedure of using the BRB system to solve portfolio optimisation problems.[4]

[4] The version of Matlab used in all computations in this paper is 7.1.0.246 (R14) Service Pack 3, and the operating system is Windows XP on Intel (R) Core (TM)2 Quad 2.50 GHz processor. We did not make any changes to the *fmincon* function code in the Optimization Toolbox of MATLAB. The initial conditions (or options) in Matlab are set as follows. `options = optimset('MaxFunEvals',5000 * length(x0), 'MaxIter', 5000,'TolX',1e−12); [x,fval] = fmincon('BRB_Obj',x0,A, b,Aeq,beq,lb,ub,'BRB_Cons',options).`

4.1. Three-asset-class example

Suppose that we select the three-asset-class example from the RM-WB platform (RiskMetrics, 2004), namely, *US large cap growth*, *US large cap value*, and *US small cap*.[5] The lower bound (*lb*) and upper bound (*ub*) are given in Table 3.

As discussed in Section 2, we first check whether there is any conflict among the investment weight constraints, and then generate a set of referential portfolios. We denote the investment weights for the three asset classes as w_1, w_2 and $1 - w_1 - w_2$ respectively.

In this example, there is no conflict among the upper and lower bound of weight constraints. Suppose five referential values are uniformly selected for the two asset weights w_1 andw_2, we then have $\dot{K} = 5 \times 5 = 25$ referential portfolios, and none of them violates the inequality constraints given by Eqs. (7) and (8). Here we use $\dot{K}$ with a dot to denote the original set of weight combinations, and K without a dot to indicate asset weight combinations generated for producing BRB referential portfolios. The number of buckets used to group the consequential portfolio returns is $N = 1000$.

To generate additional portfolios for constructing the efficient frontier, we select 21 points uniformly from each feasible interval of w_1 and w_2. This leads to $K = 21 \times 21 = 441$ new portfolios and their return distributions can be inferred using the return distributions of the referential portfolios. To check the accuracy of the BRB inference results, we use RM-WB to simulate the distributions for the 441 portfolios. The results are presented in Fig. 3.

As shown in Fig. 3, the BRB system can closely replicate the nonlinear relationship between the asset weight combination and the mean and risk of portfolio returns. The maximum absolute error between RM-WB and BRB outputs is less than 0.2%. Using these additional portfolio inference points, we can get the approximate efficient frontier as shown in Fig. 4.

Along the efficient frontier, the optimal portfolio can easily be found with a given level of expected return or risk. In this example, the computational time is given in Table 4.

It is clear that the BRB system proposed in this paper is very efficient for generating the efficient frontier. The RM-WB takes 17 min to simulate all 441 portfolio returns compared with 75 s when using the BRB procedures proposed above.

4.2. Nine-asset-class example

This section solves a 9-asset-class asset allocation problem which includes *World equity (ex US)*, *US large cap growth*, *US large cap value*, *US midcap*, *US smallcap*, *Cash*, *US bonds*, *US muni bonds*, *World bonds (ex US)*.[6] The investment weight constraints are given in Table 5.

In this case, the upper bound of *Cash* is revised to 0.55 according to Eqs. (5) and (6). In consideration of the inference accuracy, the numbers of referential values for the first $I - 1 = 8$ asset classes are 2, 3, 2, 2, 2, 5, 2 and 2 respectively. These referential values are positioned evenly in the feasible range of each asset weight.

Table 3
Lower and upper bounds of investment weights.

	US large cap growth	US large cap value	US small cap
lb	0	0	0.18
ub	0.14	0.68	1

This leads to 960 initial weight combinations, of which 874 points are infeasible. Hence, 418 projected rule points are generated leading to $\dot{K} = 960 - 874 + 418 = 504$ referential portfolios in total. The number of buckets used to group the consequential portfolio returns is $N = 100$.

Further, we uniformly select 3, 3, 3, 2, 2, 7, 2 and 3 points from the feasible intervals of asset class weights. $K = 1775$ new portfolios are generated for the purpose of locating the efficient frontier. Fig. 5 compares the mean and standard deviation associated with the 1775 portfolios produced from RM-WB simulation and the proposed BRB procedures.

In Fig. 5, the maximum absolute error between RM-WB and BRB outputs is less than 7%. The corresponding efficient frontier is shown in Fig. 6.

Along the efficient frontier, we can search for the optimal portfolios under given return or risk levels. Table 6 lists the computational time for RM-WB simulation and BRB portfolios for this 9-asset-class example.

By avoiding a full scale simulation, the BRB procedures developed in this paper can reduce more than two thirds of the computation time in this 9-asset-class example.

4.3. Optimal portfolio weights

For a given level of risk, we can locate an approximate portfolio that generates the highest amount of return from the set of 1775 portfolios produced in Section 4.2. Alternatively, we can also use Eq. (19) in Section 3.2 together with a nonlinear optimiser to find the optimum weights. The initial condition can be randomly generated or taken from the optimum portfolio from Section 4.2. Fig. 7 shows the generated optimal portfolios under three different risk levels.

Under the risk level 1.0×10^5, the optimal portfolio has investment weights {0.2, 0.15, 0, 0.17, 0.0647, 0, 0.25, 0, 0.1653}.[7]

Since the relationship between the portfolio return and asset weight combination is highly nonlinear along the efficient frontier, the local linearisation and perturbation methods have been used to approximate the optimal portfolio in the literatures (Speranza, 1993; Judd, 1996; RiskMetrics, 2004).[8] Here, we pick up 10 portfo-

[5] Many fund managers, including the RM-WB system, would first make asset allocation prior to actual asset selection when forming investment portfolio. For the US investors, the distinct group of investment class that are stable and not too correlated includes US Large Cap Growth, US Large Cap Value, US Mid Cap, US Small Cap, International Equity (excluding US), US Bonds, International Bonds and Cash. Typically these asset classes will be represented and analysed using one of the investable indices from S& P-Barra or MSCI. We have avoided using e.g. IBM stock to represent US equity as IBM may not be a good representation of US equity class as a whole. However, whether asset class or specific asset should be used in the example is not a major issue here. The focus of this paper is on the methodology and not on the actual identity of asset or portfolio. The asset class can be replaced by any specific asset.

[6] See Footnote 5 for some explanations for these asset classes.

[7] Following the suggestion by one of the referees, we have also investigated a few heuristic based optimisation methods such as genetic algorithm, simulated annealing and neighbourhood perturbation methods. We have implemented some test cases and show, given efficient starting value (based on the closest reference portfolio), Matlab mathematical algorithm, sequential quadratic programming, outperformed the solutions obtained from genetic algorithm. The simulated annealing method is a single solution-based method, and is not very effective for nonlinear continuous optimisation. The neighbourhood perturbation method proposed by Angelelli and Ortobelli (2009) is not appropriate because of the boundary and other nonlinear constraints.

[8] The working of RM-WB is exactly the opposite of our BRB approach and is based on local optimisation through linear approximation. If there are N assets, simulations are first performed to generate portfolio return distributions for a given portfolio weights P as well as $P + \varepsilon_1, P + \varepsilon_2, \ldots, P + \varepsilon_N$ where ε_i is a small increment applied to amount invested in asset i. Then $W_P(\varepsilon_i)$, the impact of ε_i on terminal wealth, is calculated path by path to produce a local covariance matrix and a mean vector. This information is used to calculate the mean and variance of any new portfolios *near P* under the constraint that the perturbative weight is not too big. The best portfolio is then selected among the set of new portfolios and P. So RM-WB works by finding the local solution around P, and then expands the solution coverage by having many different Ps. BRB makes no attempt to find local optimum but draw inference on return distributions for new portfolios that cover the entire solution space by synthesising the return distributions information from a set of Ps.

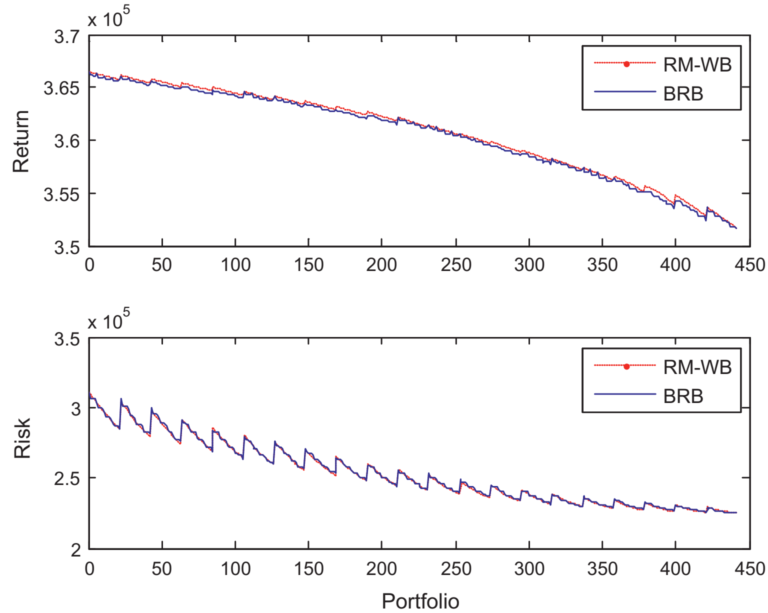

Fig. 3. Comparison between RM-WB and BRB for 3-asset-class portfolios.

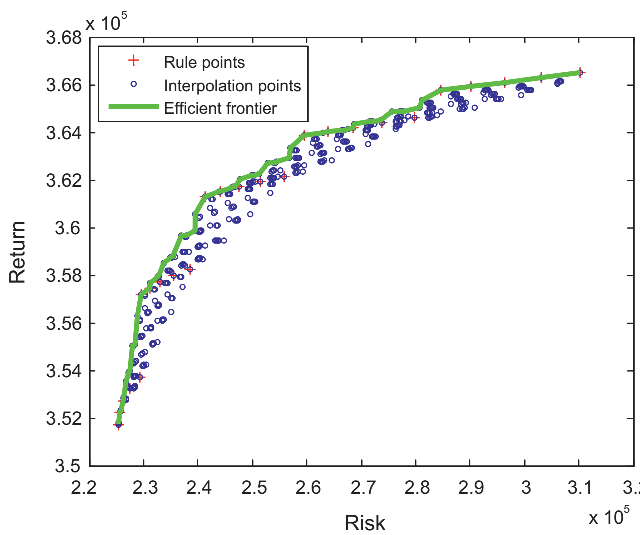

Fig. 4. BRB Efficient frontier for 3-asset-class example.

Table 4
Computational time for 3-asset-class example.

RM-WB simulation $K = 441$	Portfolio optimisation using BRB		
	Rule base generation using RM-WB $\dot{K} = 25$	BRB inference $K = 441$	BRB in total
17 minutes	60 seconds	15 seconds	75 seconds

Table 5
Lower and upper bounds of investment weights in nine asset classes.

	World equity (ex US)	US large cap growth	US large cap value	US mid cap	US small cap	Cash	US bonds	US muni bonds	World bonds (ex US)
lb	0	0.15	0	0	0.05	0	0.1	0	0.15
ub	0.2	0.45	0.21	0.17	0.25	0.9	0.25	0.2	0.45

lios from the interpolated efficient frontier around the risk level 1.0×10^5. The weights of asset classes and the return and risk of RM-WB and BRB outputs are shown in Table 7.

The stacked area graph in Fig. 8 shows that the 10 portfolios have very different investment weights although they are similar to each other in terms of portfolio expected return and risk. This result also indicates that the relationship between the investment weights and the portfolio returns is highly nonlinear along the efficient frontier.

Given that the asset weights surrounding the optimal solution are not continuous, it is unlikely that an optimal strategy based on the 'local linearisation' scheme will be effective in locating the optimal solution. The BRB nonlinear inference is more appropriate in this case.

5. Concluding remarks

The study applies a BRB system to solve a high-dimensional nonlinear portfolio optimisation problem under nonlinear cash flows and constraints. Two methods are proposed; one locates the optimal portfolio by first constructing the efficient frontier using the portfolios generated by the BRB inference and then searching along the efficient frontier using the objective function supplied by the investors. The second method finds the optimal portfolio directly using the BRB procedures in conjunction with a nonlinear optimiser. The BRB algorithm used in this paper is different from other numerical methods:

- the BRB algorithm does not require an explicit function for portfolio return distribution (e.g. mean and variance) in terms of investment weights and the constituents' return distributions;[9]

[9] For an example, the portfolio variance is $\sigma_P^2 = \sum_i \sum_j w_i w_j \rho_{ij} \sigma_i \sigma_j$ in the Gaussian case which is often used in the optimisation routine. With nonlinear cash flows, portfolio variance cannot be expressed in such a functional form and has to be determined from the portfolio returns directly instead. Indeed, portfolio returns are not a linear function of constituent assets' returns either because of nonlinear tax treatment and investor's specific cash flow requirements and constraints. RM-WB works by linearising portfolio mean and variance locally (see footnote 7).

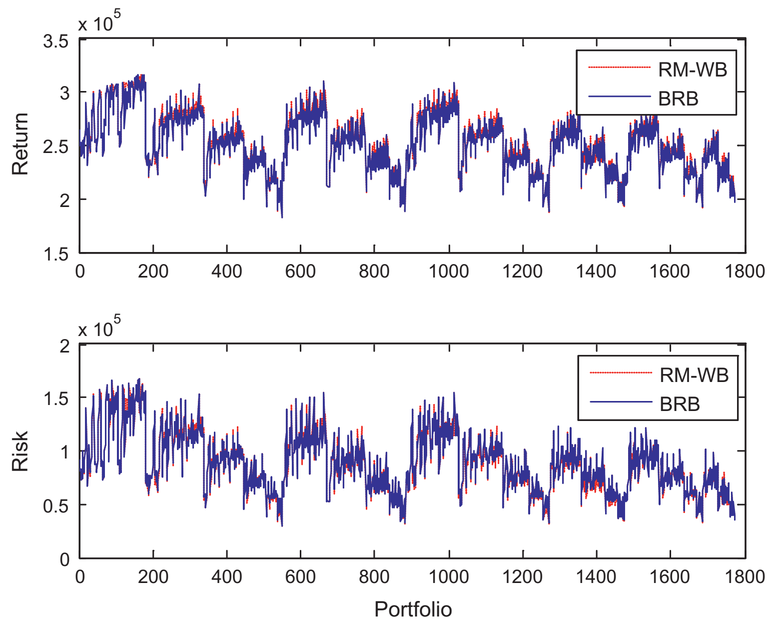

Fig. 5. Comparison between RM-WB and BRB with 9-asset-class portfolios.

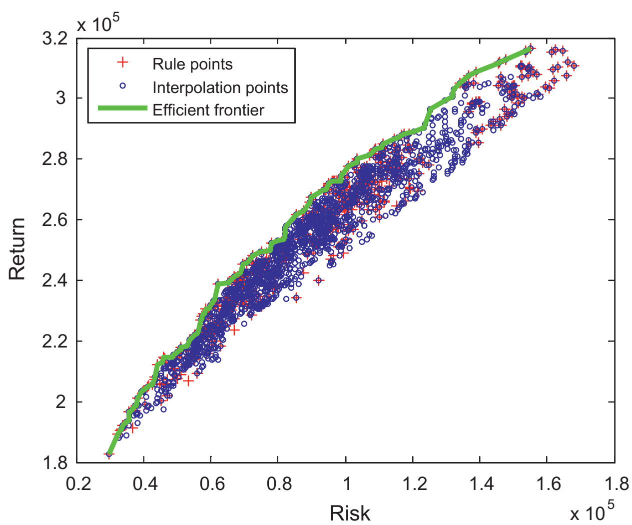

Fig. 6. BRB efficient frontier for 9-asset-class example.

Table 6
Computational time for 9-asset-class example.

RM-WB simulation $K = 1775$	Portfolio optimisation using BRB		
	Rule base generation using RM-WB $\acute{K} = 504$	BRB inference $K = 1775$	BRB in total
59 minutes	1010 seconds	80 seconds	18 minutes

- the BRB procedures can also incorporate uncertain and incomplete information about input return distribution. We have not exploited this feature since we have complete information about input return distribution in our numerical examples.

In addition, the BRB portfolio optimisation can also capture complicated nonlinear relationships between portfolio investment weights and portfolio return distribution. Numerical studies show that the BRB solution is more appropriate and efficient compared with the 'local linearisation' method commonly used in practice.

This paper considers a general long-term asset class allocation problem. In portfolio optimisation; only Eqs. (14) and (16) are needed in projecting return distribution for any new targeted

Table 7
Ten portfolios nearest the optimal solution.

Asset class									RM-WB		BRB	
World equity (ex US)	US large cap growth	US large cap value	US midcap	US small cap	Cash	US bonds	US muni bonds	World bonds (ex US)	Return	Risk	Return	Risk
0.2	0.15	0	0.17	0.05	0	0.1	0	0.33	273,912	100,524	273,912	100,605
0.2	0.15	0.21	0	0.05	0	0.24	0	0.15	274,320	98,634	274,320	98,715
0.2	0.15	0.03	0.17	0.05	0	0.25	0	0.15	277,485	101,024	277,485	101,099
0.1	0.15	0.13	0.17	0.05	0	0.25	0	0.15	279,392	104,636	271,595	99,824
0.095	0.15	0.105	0	0.25	0	0.25	0	0.15	28,2176	105,308	271,415	98,341
0.2	0.15	0	0	0.25	0.092	0.158	0	0.15	272,527	99,942	272,280	100,359
0.2	0.15	0.21	0	0.05	0	0.1	0.1	0.19	273,350	99,157	272,462	98,081
0.2	0.15	0	0	0.25	0.05	0.1	0.1	0.15	274,620	100,901	274,871	101,586
0.2	0.15	0	0	0.25	0.092	0.1	0.058	0.15	271,747	99,603	27,1918	100,176
0.2	0.15	0.21	0	0.05	0	0.14	0.1	0.15	272,992	97,867	272,845	98,578

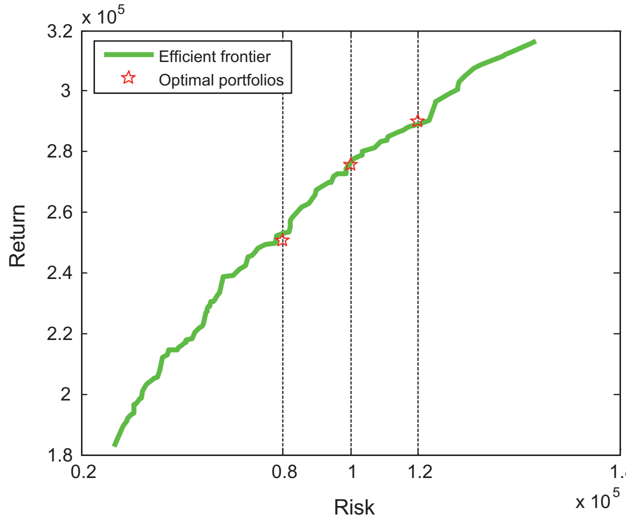

Fig. 7. Optimal portfolios under different risk levels.

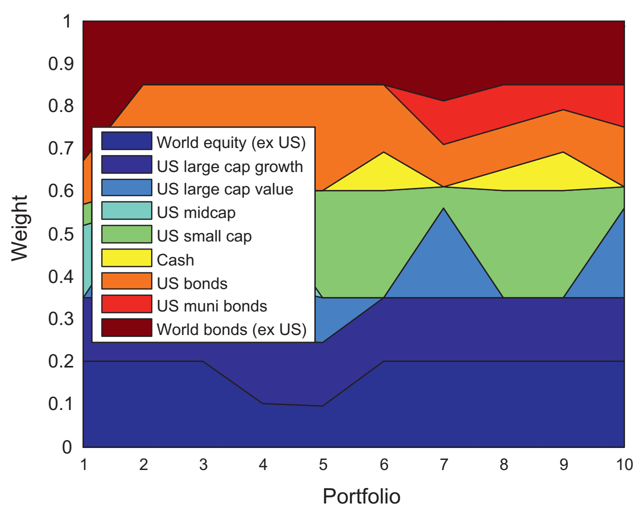

Fig. 8. Investment weights of 10 portfolios closest to the optimal solution.

portfolios. The number of constituent assets has little impact on Eq. (16). It is obvious in Eq. (14) that the number of asset classes mainly affects the calculation of activation weights, which only involves some basic arithmetic operations. In the numerical study in Section 4, most of the computation time is spent on simulating the cash flows for the referential input portfolios which has to be done in RM-WB system. The BRB optimisation itself is very efficient even for interpolating 1775 points in the 9-asset-class example. Thus we can conclude that the methodology is efficient for large-size problems as the number of asset classes increases. Future research could consider portfolio optimisation with multiple objectives of minimising Value at Risk or expected shortfall, and maximising sharp ratio which will further exploits the strength of BRB in handling nonlinear cash flows and constraints.

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Appendix A. Proof of estimation error of risk

According to Eq. (11), it is straightforward to prove that

$$E(R_k) = \sum_{n=1}^{N} D_n p_{k,n} = E(\dot{R}_k) = \frac{1}{L}\sum_{l=1}^{L} \dot{R}_{k,l}$$

The error between Eqs. (13) and (10) can be calculated as follows

$$\mathrm{var}(R_k) - \mathrm{var}(\dot{R}_k) = \sum_{n=1}^{N} (D_n)^2 p_{k,n} - \frac{1}{L}\sum_{l=1}^{L} (\dot{R}_{k,l})^2 \tag{A1}$$

In the equation above, we consider the simulated returns $\dot{R}_{k,l}, l = 1, \ldots, L$ one by one. Without loss of generality, suppose an simulated return $D_n \leqslant \dot{R}_{k,l} \leqslant D_{n+1}$. The error ξ in Eq. (A1) incurred by $\dot{R}_{k,l}$ can be calculated as

$$\begin{aligned} \xi &= (D_n)^2 p_{k,n} + (D_{n+1})^2 p_{k,n+1} - (\dot{R}_{k,l})^2 \\ &= (D_n)^2 \tfrac{D_{n+1} - \dot{R}_{k,l}}{D_{n+1} - D_n} + (D_{n+1})^2 \tfrac{\dot{R}_{k,l} - D_n}{D_{n+1} - D_n} - (\dot{R}_{k,l})^2 \\ &= \dot{R}_{k,l}(D_{n+1} + D_n) - D_{n+1}D_n - (\dot{R}_{k,l})^2 \end{aligned}$$

Let ζ denote the width of the bucket $[D_n, D_{n+1}]$.ζ decreases with the increase of the number of uniformly distributed buckets, and $\dot{R}_{k,l} - D_n \leqslant \zeta$. We replace D_{n+1} by $D_{n+1} = D_n + \zeta$. Thus, we have

$$\begin{aligned} \xi &= 2\dot{R}_{k,l}D_n + \dot{R}_{k,l}\zeta - (D_n)^2 - D_n\zeta - (\dot{R}_{k,l})^2 \\ &= 2\dot{R}_{k,l}D_n - (\dot{R}_{k,l})^2 - (D_n)^2 + \dot{R}_{k,l}\zeta - D_n\zeta \\ &= \zeta(\dot{R}_{k,l} - D_n) - (\dot{R}_{k,l} - D_n)^2 \end{aligned}$$

It is obvious that $\lim_{\zeta\to 0}\xi = 0$. ξ is monotonically increasing with respect to ζ.

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