Predicting the time at which a Lévy process attains its ultimate supremum

Erik Baurdoux* & Kees van Schaik†

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Abstract

We consider the problem of finding a stopping time that minimises the L^1-distance to θ, the time at which a Lévy process attains its ultimate supremum. This problem was studied in [12] for a Brownian motion with drift and a finite time horizon. We consider a general Lévy process and an infinite time horizon (only compound Poisson processes are excluded, furthermore due to the infinite horizon the problem is only interesting when the Lévy process drifts to $-\infty$). Existing results allow us to rewrite the problem as a classic optimal stopping problem, i.e. with an adapted payoff process. We show the following. If θ has infinite mean there exists no stopping time with a finite L^1-distance to θ, whereas if θ has finite mean it is either optimal to stop immediately or to stop when the process reflected in its supremum exceeds a positive level, depending on whether the median of the law of the ultimate supremum equals zero or is positive. Furthermore, pasting properties are derived. Finally, the result is made more explicit in terms of scale functions in the case when the Lévy process has no positive jumps.

Keywords: Lévy processes, optimal prediction, optimal stopping.

Mathematics Subject Classification (2000): 60G40, 62M20

1 Introduction

This paper addresses the question of how to predict the time a Lévy process attains its ultimate supremum with an infinite time horizon. (Due to the jumps a Lévy process can experience, the word "attains" is used here in a slightly broader sense than when the driving process is continuous, cf. Section 3). That is, we aim to find a stopping time that

*Department of Statistics, London School of Economics. Houghton street, London, WC2A 2AE, United Kingdom. E-mail: e.j.baurdoux@lse.ac.uk

†School of Mathematics, University of Manchester. Oxford Road, Manchester, M13 9PL, United Kingdom. E-mail: kees.vanschaik@manchester.ac.uk

is within the set of all stopping times closest (in L^1 sense) to the time the Lévy process attains its ultimate supremum. This is an example of an *optimal prediction problem*. It is related to classic and well studied optimal stopping problems, however the key difference is that the payoff process is not adapted to the filtration generated by the driving stochastic process. Indeed, in our case the time the Lévy process attains its ultimate supremum is not known (with absolute certainty) at any (finite) time t. However as time progresses more information about the time of the ultimate supremum becomes available. Examples of optimal prediction problems where this is not the case include the "hidden target" type problems studied in [24].

In recent years optimal prediction problems have received considerable attention, see e.g. [2, 4, 10–13, 15–17, 24, 27]. One reason is that such problems have very relevant applications in fields like engineering, finance and medicine. Prominent examples concern the optimal time to sell an asset (in finance) or the optimal time to administer a drug (in medicine).

The papers referred to above are mainly concerned with optimal prediction problems driven by Brownian motion (with drift), particularly with a finite time horizon. Some exceptions are [2] which deals with random walks, [15] with mean-reverting diffusions and [4] deals with spectrally positive stable processes. The same problem as we consider was studied in [12], however for a Brownian motion with drift and a finite time horizon. In that paper more general Lévy processes are also briefly mentioned, and it is interesting to note that the structure of the solution for the finite horizon case suggested there is consistent with our results.

In this paper the driving process is a general Lévy process X drifting to $-\infty$, i.e. $\lim_{t\to\infty} X_t = -\infty$ a.s. (otherwise the problem we consider is trivial as we will briefly point out in the sequel). We are interested in solving

$$\inf_{\tau} \mathbb{E}[|\theta - \tau|], \tag{1.1}$$

where θ is the time X attains its ultimate supremum (cf. Section 3 for details) and the infimum is taken over all stopping times τ with respect to the filtration generated by X. Following [12, 27], due to the stationary independent increments of X (1.1) can be expressed as an optimal stopping problem driven by the reflected process Y given by $Y_t = \overline{X}_t - X_t$ for all $t \geq 0$, where $\overline{X}_t := \sup_{s\leq t} X_s$ denotes the running supremum of X. This allows us to show the following. If θ has infinite mean then (1.1) is degenerate in the sense that it equals ∞. Now suppose θ has finite mean. If the law of the ultimate supremum $\overline{X}_\infty := \lim_{t\to\infty} \overline{X}_t$ has an atom in 0 of size at least $1/2$ then $\tau = 0$ is optimal in (1.1), otherwise the infimum in (1.1) is attained by the first time Y enters an interval $[y^*, \infty)$ for some y^* strictly larger than the median of the law of $\overline{X}_\infty$. We derive pasting properties and in the special case that X is spectrally negative we obtain (semi-)explicit expressions for (1.1) and y^* in terms of scale functions.

The rest of this paper is organised as follows. In Section 2 we discuss some preliminaries and some technicalities to be used later on. In Section 3 we prove our main result. Finally, in Section 4 we make our result more explicit in the case X is spectrally negative.

2 Preliminaries

Let $X = (X_t)_{t\geq 0}$ be a Lévy process starting from 0 defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, where $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by X which is naturally enlarged (cf. Definition 1.3.38 in [7]). Recall that a Lévy process is characterised by stationary, independent increments and paths which are right continuous and have left limits, and its law is characterised by the characteristic exponent Ψ defined through $\mathbb{E}[e^{\mathrm{i}zX_t}] = e^{-t\Psi(z)}$ for all $t \geq 0$ and $z \in \mathbb{R}$. According to the Lévy-Khintchine formula there exist $\sigma \geq 0$, $a \in \mathbb{R}$ and a measure Π (the Lévy measure) concentrated on $\mathbb{R}\setminus\{0\}$ satisfying $\int_{\mathbb{R}}(1\wedge x^2)\,\Pi(dx) < \infty$ (the tuple (σ, a, Π) is usually refered to as the *Lévy triplet*) such that

$$\Psi(z) = \frac{\sigma^2}{2}z^2 + \mathrm{i}az + \int_{\mathbb{R}} \left(1 - e^{\mathrm{i}zx} + \mathbf{1}_{\{|x|<1\}}\mathrm{i}zx\right)\,\Pi(dx)$$

for all $z \in \mathbb{R}$. For further details see e.g. the textbooks [6, 20, 26].

We denote the running supremum at time t by $\overline{X}_t = \sup_{s\leq t} X_s$ for all $t \geq 0$, so that $\overline{X}_\infty := \lim_{t\to\infty} \overline{X}_t$ is the ultimate supremum of X. As is well known, see e.g. Theorem 12 on p. 267 in [6], we have

$$\overline{X}_\infty < \infty \text{ a.s.} \quad \Leftrightarrow \quad \lim_{t\to\infty} X_t = -\infty \text{ a.s.} \quad \Leftrightarrow \quad \int_1^\infty \frac{1}{s}\mathbb{P}(X_s \geq 0)\,\mathrm{d}s < \infty. \tag{2.1}$$

In Section 3 we look at the problem of predicting the time of the ultimate supremum of X, which is defined as

$$\theta := \inf\{t \geq 0 \,|\, \overline{X}_t = \overline{X}_\infty\}$$

(cf. the discussion in Section 3). By the Sparre Andersen identity (cf. Lemma 15 on p. 170 in [6]) we have

$$\mathbb{E}[\theta] < \infty \quad \Leftrightarrow \quad \int_0^\infty \mathbb{P}(X_s \geq 0)\,\mathrm{d}s < \infty. \tag{2.2}$$

In Section 3 the presence of an atom in 0 in the law of $\overline{X}_\infty$ plays a prominent role. As is well known, this is related to a property called *(ir)regularity upwards.* Denoting

$$\tau^+(x) := \inf\{t > 0 \,|\, X_t > x\},$$

X is said to be regular upwards if $\tau^+(0) = 0$ a.s. – i.e. if X enters $(0, \infty)$ immediately after starting from 0; otherwise (then $\tau^+(0) > 0$ a.s.) X is said to be irregular upwards. Similarly, X is said to be regular (resp. irregular) downwards if $-X$ is regular (resp.

irregular) upwards. Theorem 6.5 in [20] classifies regularity upwards in terms of the Lévy triplet. It is a well known rule of thumb that the solution to an optimal stopping problem driven by X exhibits so-called smooth or continuous pasting dependent on whether this property holds or not, see e.g. [1] and the references therein. See also Theorem 7. The following lemma concerns the connection between an atom in 0 in the law of $\overline{X}_\infty$ and (ir)regularity upwards.

Lemma 1. *Suppose X is not a compound Poisson process and drifts to $-\infty$. The law of $\overline{X}_\infty$ has an atom in 0 if and only if X is irregular upwards.*

Proof. For any $q > 0$ let $\mathrm{e}(q)$ be a random variable independent of X following an exponential distribution with mean $1/q$. From the Wiener-Hopf factorisation (see in particular part (ii) and (iii) of Theorem 6.16 in [20] e.g.) we know that for any $\beta > 0$

$$\mathbb{E}\left[e^{-\beta \overline{X}_{\mathrm{e}(q)}}\right] = \exp\left(\int_0^\infty \int_{(0,\infty)} \frac{1}{t} e^{-qt}\left(e^{-\beta x} - 1\right) \mathbb{P}(X_t \in dx)\, \mathrm{d}t\right).$$

Then

$$\begin{aligned}\mathbb{P}(\overline{X}_\infty = 0) &= \lim_{\beta\to\infty} \mathbb{E}\left[e^{-\beta \overline{X}_\infty}\right] = \lim_{\beta\to\infty}\lim_{q\downarrow 0} \mathbb{E}\left[e^{-\beta \overline{X}_{\mathrm{e}(q)}}\right] \\ &= \exp\left(-\int_0^\infty \int_{(0,\infty)} \frac{1}{t}\mathbb{P}(X_t \in dx)\, \mathrm{d}t\right) = \exp\left(-\int_0^\infty \frac{1}{t}\mathbb{P}(X_t > 0)\, \mathrm{d}t\right),\end{aligned}$$

hence $\mathbb{P}(\overline{X}_\infty = 0) > 0$ iff

$$\int_0^1 \frac{1}{t}\mathbb{P}(X_t > 0)\, \mathrm{d}t < \infty \quad \text{and} \quad \int_1^\infty \frac{1}{t}\mathbb{P}(X_t > 0)\, \mathrm{d}t < \infty. \tag{2.3}$$

Indeed, the second integral is finite since we assumed (2.1) while the first integral is finite iff X is irregular upwards (cf. [20] Theorem 6.5). □

Next, we show that when X is not compound Poisson, the atom in 0 identified in the above Lemma 1 is the only possible atom in the law of $\overline{X}_\infty$. Recall that X is said to *creep upwards* if for some (and then all) $x > 0$ it holds $\mathbb{P}(X_{\tau^+(x)} = x) > 0$. For instance all Lévy processes with a Gaussian component and those of bounded variation with a positive drift creep upwards, see e.g. Theorem 7.11 in [20].

Lemma 2. *Suppose X is not a compound Poisson process (still drifting to $-\infty$). The distribution function F of $\overline{X}_\infty$ is continuous on $\mathbb{R}_{\geq 0}$. Furthermore, if X creeps upwards then F is Lipschitz continuous on $\mathbb{R}_{\geq 0}$.*

Proof. From Wiener-Hopf theory we know that $\overline{X}_\infty$ is equal in law to H_{e}, where H is the ascending ladder height process and e is an exponentially distributed random variable independent of X and with parameter $\kappa(0,0)$, where κ denotes the Laplace exponent of the ladder process (see e.g. Chapter 6 in [20] for further details). The Laplace exponent ψ of H given by

$$\psi(z) = -\frac{1}{t}\log \mathbb{E}\left[e^{-zH_t}\right] \quad \text{for all } t > 0$$

can be expressed as

$$\psi(z) = d_H z + \int_{\mathbb{R}_{>0}} \left(1 - e^{-zx}\right) \Pi_H(\mathrm{d}x),$$

where $d_H \geq 0$ is the drift and the Lévy measure Π_H satisfies $\int_{\mathbb{R}_{>0}}(1 \wedge x)\Pi_H(\mathrm{d}x) < \infty$.

From [6] Proposition 17 on p. 172 and Theorem 19 on p. 175 it follows that X creeps upwards iff $d_H > 0$, and if this is the case F has a bounded, continuous and positive density on $\mathbb{R}_{>0}$.

Henceforth suppose $d_H = 0$, it remains to show that F is continuous on $\mathbb{R}_{\geq 0}$ when X is not compound Poisson. If $\Pi_H(\mathbb{R}_{>0}) = \infty$ it is not difficult to see that F is continuous, cf. Theorem 5.4 (i) in [20]. Let now $\Pi_H(\mathbb{R}_{>0}) < \infty$. Then H is a compound Poisson process (with jump distribution Π_H times a constant). In this case $\mathbb{P}(\overline{X}_\infty = 0) = \mathbb{P}(H_{\mathrm{e}} = 0) > 0$ and hence in particular X is irregular upwards by the above Lemma 1. Denote by $\widehat{H}$ the ladder height process of $-X$ which is a subordinator (without killing as X drifts to $-\infty$). Note that $\widehat{H}$ can not be compound Poisson, because if it was then by the same argument as above X would be irregular downwards in addition to irregular upwards and hence X would be compound Poisson which we excluded. So either $d_{\widehat{H}} > 0$ or $\Pi_{\widehat{H}}(\mathbb{R}_{>0}) = \infty$ (or both) must hold, which in turn implies (analogue to above, cf. Theorem 5.4 (i) in [20]) that the renewal measure $\widehat{U}$ of $\widehat{H}$ given by

$$\widehat{U}(\mathrm{d}x) = \int_0^\infty \mathbb{P}(\widehat{H}_t \in \mathrm{d}x)\,\mathrm{d}t$$

has no atoms. The 'equation amicale inversée' from [28] reads

$$\Pi_H((y,\infty)) = \int_{\mathbb{R}_{\geq 0}} \Pi((x+y,\infty))\widehat{U}(\mathrm{d}x)$$

and shows that Π_H has no atoms since $\widehat{U}$ has no atoms. Hence, H is compound Poisson with a continuous jump distribution. It is now a straightforward exercise to find an expression for the law of H_{e} (by conditioning on the number of jumps H experiences before e) and to deduce that the continuous jump distribution garantees that the distribution function of H_{e} is continuous on $\mathbb{R}_{\geq 0}$. □

Note that the above result is not sharp: there are obvious examples of Lévy processes not creeping upwards for which F is nevertheless Lipschitz on $\mathbb{R}_{\geq 0}$, for instance when X is a compound Poisson process with positive, exponentially distributed jumps plus a negative drift. In this case, when $\mathbb{E}[X_1] < 0$ so that $\overline{X}_\infty < \infty$ a.s. it holds $\mathbb{P}(\overline{X}_\infty = 0) > 0$ (by Lemma 1 above) while $\overline{X}_\infty$ has a positive, bounded and continuous density on $\mathbb{R}_{>0}$

(cf. Theorem 2 in [23]). For an interesting study of the law of the supremum of a Lévy process we refer to [9].

Remark 3. *Some examples of Lévy processes with two-sided jumps for which the density of $\overline{X}_\infty$ is known (semi-)explicitly are Lévy processes with arbitrary negative jumps and phase-type positive jumps (cf. [23]) and the class of so-called meromorphic Lévy processes which have jumps consisting of a possibly infinite mixture of exponentials (cf. [19]). If X has no positive jumps it is well known that $\overline{X}_\infty$ follows an exponential distribution (cf. Section 4), while if X has no negative jumps, scale functions may be used to describe the law of $\overline{X}_\infty$ (cf. [20]).*

We conclude with a technical result that will be of use later. Recall that $\tau^+(x) = \inf\{t > 0 \,|\, X_t > x\}$ and similarly $\tau^-(x) = \inf\{t > 0 \,|\, X_t < x\}$.

Lemma 4. *Suppose that X is regular downwards, then for any $c > 0$*

$$\limsup_{\varepsilon \downarrow 0} \frac{\mathbb{P}(\tau^+(c-\varepsilon) < \tau^-(-\varepsilon))}{\varepsilon} > 0.$$

Proof. From p. 10 in [8] we know that

$$\lim_{\varepsilon \downarrow 0} \frac{\mathbb{P}(\tau^+(c-\varepsilon) < \tau^-(-\varepsilon))}{h(\varepsilon)} = n(H > c),$$

where n denotes the excursion measure, H the height of a generic excursion and h is the renewal function of the downward ladder height process. This function h is subadditive and satisfies $h(0) = 0$ when X is regular downwards. As h is not a constant function, it also holds that

$$\limsup_{\varepsilon \downarrow 0} \frac{h(\varepsilon)}{\varepsilon} > 0.$$

Indeed, if this limsup were 0, then $h'(0+) = 0$ which would imply that h would be the zero function since h is non-decreasing and for any $y \geq 0$

$$\frac{h(y+\varepsilon) - h(y)}{\varepsilon} \leq \frac{h(\varepsilon)}{\varepsilon}.$$

This concludes the proof of the Lemma. □

3 Predicting the time of the ultimate supremum

Define the time where the ultimate supremum of the Lévy process X is (first) attained by θ, that is

$$\theta := \inf\{t \geq 0 \,|\, \overline{X}_t = \overline{X}_\infty\}$$

where we understand $\inf \emptyset = \infty$. Note that "attained" is used in a loose sense here. Indeed, if X has negative jumps it might happen that the ultimate supremum is never attained. However, the above definition ensures that we have $X_\theta = \overline{X}_\infty$ on the event $\{X_\theta \geq X_{\theta-}\}$ while $X_{\theta-} = \overline{X}_\infty$ on the event $\{X_\theta < X_{\theta-}\}$. Furthermore, when X is not a compound Poisson process, the set $\{t \geq 0 \,|\, \overline{X}_t = \overline{X}_\infty\}$ is a singleton, see e.g. p. 158 of [20].

Our aim in this section is to find a stopping time as close as possible to θ, that is, we consider the optimal stopping problem:

$$\inf_\tau \mathbb{E}[|\theta - \tau|], \tag{3.1}$$

where the infimum is taken over all $\mathbf{F}$-stopping times.

As is well known (see Theorem 7.1 in [20]), either $\lim_{t\to\infty} X_t = -\infty$, $\lim_{t\to\infty} X_t = \infty$ or $\limsup_{t\to\infty} X_t = -\liminf_{t\to\infty} X_t = \infty$ a.s. In the latter two cases we have $\theta = \infty$ a.s. and hence (3.1) is degenerate. Henceforth in this section we assume that

$$X \text{ drifts to } -\infty \text{ and } \theta \text{ has finite mean}$$

(we will deal with the case that θ has infinite mean in Proposition 8). Recall that these properties were discussed in Section 2.

As $\mathbb{E}[\theta] < \infty$ and $\mathbb{E}[|\theta - \tau|] \geq |\mathbb{E}[\theta] - \mathbb{E}[\tau]|$ we can without loss of generality consider the infimum in (3.1) over all stopping times with finite mean.

Recall that we denote by F the distribution function of $\overline{X}_\infty$. Furthermore we introduce for any $y \geq 0$ the process Y^y, which is X reflected in its running supremum, started from y:

$$Y^y_t = (y \vee \overline{X}_t) - X_t \quad \text{for all } t \geq 0.$$

Note that Y^y is a strong Markov process.

Following [12] Lemma 2 and [27] Lemma 1 we rewrite the expectation in (3.1) as an expectation involving an $\mathbf{F}$-adapted process which will allow us to apply standard optimal stopping techniques. We include the proof for completeness.

Proposition 5. *For any stopping time τ with finite mean we have that*

$$\mathbb{E}[|\theta - \tau|] = 2\mathbb{E}\left[\int_0^\tau F(Y^0_t)\mathrm{d}t\right] + \mathbb{E}[\theta] - \mathbb{E}[\tau] = \mathbb{E}\left[\int_0^\tau \left(2F(Y^0_t) - 1\right)\,\mathrm{d}t\right] + \mathbb{E}[\theta].$$

Proof. We have

$$\begin{aligned} |\theta - \tau| &= \theta - \tau + 2(\tau - \theta)\mathbf{1}_{\{\theta \leq \tau\}} \\ &= \theta - \tau + 2\int_0^\tau \mathbf{1}_{\{\theta \leq t\}}\,\mathrm{d}t. \end{aligned}$$

Applying Fubini's Theorem twice we deduce that

$$\begin{aligned}\mathbb{E}\left[\int_0^\tau \mathbf{1}_{\{\theta\le t\}}\,\mathrm{d}t\right] &= \int_0^\infty \mathbb{E}\left[\mathbf{1}_{\{t<\tau\}}\mathbf{1}_{\{\theta\le t\}}\right]\,\mathrm{d}t \\ &= \int_0^\infty \mathbb{E}\left[\mathbf{1}_{\{t<\tau\}}\mathbb{E}\left[\mathbf{1}_{\{\theta\le t\}}\,|\,\mathcal{F}_t\right]\right]\,\mathrm{d}t \\ &= \mathbb{E}\left[\int_0^\tau \mathbb{P}\left(\theta\le t\,|\,\mathcal{F}_t\right)\,\mathrm{d}t\right].\end{aligned}$$

Furthermore, for any $t \geq 0$,

$$\begin{aligned}\mathbb{P}\left(\theta\le t\,|\,\mathcal{F}_t\right) &= \mathbb{P}\left(\sup_{s\ge t} X_s \le \overline{X}_t \,\middle|\, \mathcal{F}_t\right) \\ &= \mathbb{P}\left(S + X_t \le \overline{X}_t \ |\ \mathcal{F}_t\right) \\ &= F(Y_t^0),\end{aligned}$$

where S denotes an independent copy of $\overline{X}_\infty$. We conclude that when τ has finite mean

$$\mathbb{E}|\theta-\tau| = 2\mathbb{E}\left[\int_0^\tau F(Y_t^0)\mathrm{d}t\right] + \mathbb{E}[\theta] - \mathbb{E}[\tau] = \mathbb{E}\left[\int_0^\tau \left(2F(Y_t^0)-1\right)\,\mathrm{d}t\right] + \mathbb{E}[\theta].$$

□

Hence, by defining a function V on $\mathbb{R}_{\geq 0}$ as

$$V(y) = \inf_\tau \mathbb{E}\left[\int_0^\tau \left(2F(Y_t^y)-1\right)\,\mathrm{d}t\right] \tag{3.2}$$

we have that an optimal stopping time for $V(0)$ is also optimal in (3.1). Therefore let us analyse the function V.

Inspecting the integrand in (3.2) makes it clear that a quantity of interest is the (lower) median of the law of $\overline{X}_\infty$, that is:

$$\mathrm{m} := \inf\{z \ge 0\,|\,F(z)\ge 1/2\} \in \mathbb{R}_{\ge 0}.$$

(It is interesting to compare this with the "median rule" in [24] where the "hidden target" is assumed to be independent of the underlying process X). If $\mathrm{m}=0$, that is if $\mathbb{P}(\overline{X}_\infty = 0) \ge 1/2$, it is easy to see it is optimal to stop immediately.

Proposition 6. *The time $\tau = 0$ is optimal in (3.2) for all $y \ge 0$ if and only if* $\mathrm{m}=0$. *In this case $V(y)=0$ for all $y\ge 0$.*

Proof. Suppose $\mathrm{m} = 0$. This implies $F(z) \ge 1/2$ for all $z > 0$ and in particular also $F(0) \ge 1/2$ (by right continuity). Hence $2F(z)-1 \ge 0$ for all $z \ge 0$ and the result follows. Next suppose $\mathrm{m} > 0$. Denoting $\sigma(x) = \inf\{t\ge 0\,|\,Y_t^0 \ge x\}$ we have

$$V(0) \leq \mathbb{E}\left[\int_0^{\sigma(\mathrm{m})} \big(2F(Y_t^0) - 1\big)\ \mathrm{d}t\right] < 0,$$

since $\sigma(\mathrm{m}) > 0$ a.s. by right continuous paths of Y^0 and $F < 1/2$ on $[0, \mathrm{m})$. $\square$

We now turn our attention to the more interesting case $\mathrm{m} > 0$. Note that it is still possible that F has a discontinuity in 0 with size less than 1/2. Recall our standing assumptions that X drifts to $-\infty$ and θ has finite mean. Recall also that Lemma 2 states that F is Lipschitz continuous on $\mathbb{R}_{\geq 0}$ at least when X creeps upwards. In the result below we denote by V'_- and V'_+ the right and left and right derivative of V, respectively.

Theorem 7. *Suppose that X is not a compound Poisson process and is such that $\mathrm{m} > 0$. Then there exists an $y^* \in [\mathrm{m}, \infty)$ such that an optimal stopping time in (3.2) is given by*

$$\tau^* = \inf\{t \geq 0 \,|\, V(Y_t^y) = 0\} = \inf\{t \geq 0 \,|\, Y_t^y \geq y^*\}.$$

Furthermore V is a non-decreasing, continuous function satisfying the following:

(i) if X is regular downwards and F is Lipschitz continuous on $\mathbb{R}_{\geq 0}$, then $y^* > \mathrm{m}$ and $V'_-(y^*) = V'_+(y^*) = 0$ (smooth pasting);

(ii) if X is irregular downwards then $y^* > \mathrm{m}$ is the unique solution on $\mathbb{R}_{>0}$ to the equation (in y)

$$\mathbb{E}\left[\int_0^{\sigma_+(y)} (2F(Y_u^y) - 1)\ \mathrm{d}u\right] = 0$$

where $\sigma_+(y) = \inf\{t > 0 \,|\, Y_t^y > y\}$. Furthemore, when F' exists and is positive on $\mathbb{R}_{>0}$ smooth pasting does not hold, i.e. $V'_-(y^*) > V'_+(y^*) = 0$.

Proof. We break the proof up in a five steps.

Step 1 (V is non-decreasing and $V(x) \in (-\infty, 0]$).

Denote the payoff process when started from $y \geq 0$ by L^y, i.e.

$$L_t^y = \int_0^t (2F(Y_u^y) - 1)\ \mathrm{d}u \quad \text{for all } t \geq 0.$$

Clearly V is non-decreasing (due to monotonicity of Y^y in y and $z \mapsto 2F(z) - 1$) and takes values in $\mathbb{R}_{\leq 0}$. Indeed we can see that $V(0) > -\infty$ as follows. Note that

$$V(0) \geq -\mathbb{E}\left[\int_0^\infty \mathbf{1}_{\{Y_u^0 \leq \mathrm{m}\}}\ \mathrm{d}u\right] = -\mathbb{E}[\rho(\mathrm{m})],$$

where

$$\rho(x) = \sup\{t \geq 0 \,|\, Y_t^0 \leq x\}$$

denotes the last time the reflected process is in $[0,x]$. Let us show that $\rho(x)$ has finite mean for any $x>0$. Note that $\rho(x)=\theta+\zeta(x)$, where $\zeta(x)$ is the time the final excursion of Y^0 leaves $[0,x]$. As we assume that θ has a finite mean, the results in [22] allow us to deduce that the post-maximum process has the same law as $-X$ conditioned to stay positive. Therefore $\zeta(x)$ is the last passage time over the level x of $-X$ conditioned to stay positive. From p. 357 in [14] and Lemma 4 in [5] we know that $\zeta(x)$ is equal in distribution to $\overline{g}_{\hat{\tau}^+(x)}$, where $\hat{\tau}^+(x)$ denotes the first passage time of $-X$ over level x and $\overline{g}_t$ denotes the time of the last maximum of $-X$ prior to time $t>0$. Therefore

$$\mathbb{E}[\zeta(x)]=\mathbb{E}[\overline{g}_{\hat{\tau}^+(x)}]\leq\mathbb{E}[\hat{\tau}^+(x)]<\infty,$$

where the last inequality holds since $-X$ drifts to $+\infty$, see for example [6] Proposition 17 on p. 172.

Step 2 (V is continuous).

To prove that V is continuous we introduce the notation

$$\sigma(x):=\inf\{t\geq 0\,|\,Y_t^y\geq x\} \tag{3.3}$$

and we first show there exists an $\bar{y}$ (large enough) such that for all $y\geq 0$

$$V(y)=\inf_{\tau\wedge\sigma(\bar{y})}\mathbb{E}\left[L_\tau^y\right]. \tag{3.4}$$

Indeed for any $y\geq 0$ and $\tau\geq\sigma(\bar{y})$

$$\mathbb{E}[L_\tau^y]=\mathbb{E}\left[\int_0^{\sigma(\bar{y})}(2F(Y_t^y)-1)\,\mathrm{d}u\right]+\mathbb{E}\left[\int_{\sigma(\bar{y})}^{\tau}(2F(Y_u^y)-1)\,\mathrm{d}u\right].$$

Analogously to the situation in Step 1,

$$\mathbb{E}\left[\int_{\sigma(\bar{y})}^{\tau}(2F(Y_u^y)-1)\,\mathrm{d}u\right]\geq-\mathbb{E}[\rho(\mathrm{m})]$$

with $\mathbb{E}[\rho(\mathrm{m})]<\infty$ and

$$\begin{aligned}\mathbb{E}\left[\int_0^{\sigma(\bar{y})}(2F(Y_u^y)-1)\,\mathrm{d}u\right] &\geq -\mathbb{E}[\rho(k)]+\mathbb{E}\left[\int_{\rho(k)}^{\rho(\bar{y})}(2F(Y_u^y)-1)\,\mathrm{d}u\right]\\ &\geq -\mathbb{E}[\rho(k)]+\frac{1}{2}\mathbb{E}\left[\int_{\rho(k)}^{\rho(\bar{y})}\,\mathrm{d}u\right],\end{aligned} \tag{3.5}$$

where $k:=\inf\{z\geq 0\,|\,F(z)\geq 3/4\}$. As $\mathbb{E}[\rho(k)]<\infty$ and $\mathbb{E}[\rho(\bar{y})]$ can be made arbitrarily large by increasing $\bar{y}$ we see that $\bar{y}$ exists such that for all $y\geq 0$ and $\tau\geq\sigma(\bar{y})$ it holds that $\mathbb{E}[L_\tau^y]>0$, implying (3.4).

Now, since L^y is a continuous process and

$$\mathbb{E}\left[\sup_{t\geq 0}\left|L^y_{t\wedge\sigma(\bar{y})}\right|\right] \leq \mathbb{E}[\sigma(\bar{y})] < \infty \tag{3.6}$$

it is clear that the optimum in (3.4) is attained. As F is continuous on $\mathbb{R}_{\geq 0}$ (cf. Lemma 2) it is uniformly continuous on $[0,\bar{y}]$. Take any $\varepsilon > 0$. Let $\delta > 0$ be such that for all $y_1, y_2 \in [0,\bar{y}]$ with $|y_1 - y_2| < \delta$ it holds $|F(y_1) - F(y_2)| < \varepsilon$. For any $y \geq 0$ we have, where τ_y is the optimal stopping time when starting from y and we use $Y^{y+\delta}_t - Y^y_t \leq \delta$ for all $t \geq 0$:

$$V(y+\delta) - V(y) \leq \mathbb{E}[L^{y+\delta}_{\tau_y}] - \mathbb{E}[L^y_{\tau_y}] \leq 2\mathbb{E}\left[\int_0^{\sigma(\bar{y})} \left(F(Y^{y+\delta}_u) - F(Y^y_u)\right) \mathrm{d}u\right] \leq 2\varepsilon\mathbb{E}[\sigma(\bar{y})],$$

establishing the continuity of V as $\mathbb{E}[\sigma(\bar{y})] < \infty$.

Step 3 (Stopping region of the form $[y^*,\infty)$).

Following the usual arguments from general theory for optimal stopping, taking into account (3.6) and that L is continuous, an optimal stopping time for (3.2) is given by $\tau^* = \inf\{t \geq 0 \,|\, \widehat{L}^y_t = L^y_t\}$, where the Snell envelope $\widehat{L}^y$ satisfies, on account of the Markov property of (L^y, Y^y) :

$$\widehat{L}^y_t = \operatorname*{ess\,inf}_{\tau\geq t} \mathbb{E}\left[L^y_\tau \,|\, \mathcal{F}_t\right] = L^y_t + \operatorname*{ess\,inf}_{\tau\geq t} \mathbb{E}\left[L^y_\tau - L^y_t \,|\, \mathcal{F}_t\right] = L^y_t + V(Y^y_t)$$

and hence indeed $\tau^* = \inf\{t \geq 0 \,|\, V(Y^y_t) = 0\}$. See e.g. Theorem 2.4 on p. 37 in [25] for details. The properties of V we have established in Step 1 and Step 2 ensure that we may also write $\tau^* = \sigma(y^*)$ (as defined in (3.3)), where

$$y^* := \inf\{y \geq 0 \,|\, V(y) = 0\} \in [0,\bar{y}].$$

It is immediate that $y^* \geq \mathrm{m}$, since if $y^* < \mathrm{m}$ we could pick any $y \in (y^*, \mathrm{m})$ and derive $V(y) \leq \mathbb{E}[L^y_{\sigma(\mathrm{m})}] < 0$ which contradicts with $V(y) = 0$.

Step 4 (Continuous vs. smooth fit).

First, suppose X is regular downwards. As V is non-decreasing, for $V'_-(y^*) = 0$ it is enough to show that

$$\limsup_{\varepsilon\downarrow 0} \frac{V(y^*) - V(y^* - \varepsilon)}{\varepsilon} \leq 0. \tag{3.7}$$

Denoting for any x, y

$$\sigma(x,y) = \inf\{t \geq 0 \,|\, Y^x_t \geq y\}$$

we know from Step 3 that for any $\varepsilon > 0$

$$V(y^* - \varepsilon) = \mathbb{E}[L^{y^*-\varepsilon}_{\sigma(y^*-\varepsilon,y^*)}] \quad \text{and} \quad V(y^*) \leq \mathbb{E}[L^{y^*}_{\sigma(y^*-\varepsilon,y^*)}]$$

and hence, where C is a Lipschitz constant of F

$$V(y^*) - V(y^* - \varepsilon) \leq 2\mathbb{E}\left[\int_0^{\sigma(y^*-\varepsilon,y^*)} \left(F(Y_u^{y^*}) - F(Y_u^{y^*-\varepsilon})\right) \mathrm{d}u\right]$$

$$\leq 2\mathbb{E}\left[\int_0^{\sigma(y^*-\varepsilon,y^*)} C\left(Y_u^{y^*} - Y_u^{y^*-\varepsilon}\right) \mathrm{d}u\right] \leq 2C\varepsilon\mathbb{E}[\sigma(y^*-\varepsilon, y^*)].$$

The result (3.7) now follows by remarking that

$$\sigma(y^* - \varepsilon, y^*) \leq \inf\{t \geq 0 \,|\, y^* - \varepsilon - X_t \geq y^*\} \leq \inf\{t \geq 0 \,|\, X_t \leq -\varepsilon\} \downarrow 0 \text{ a.s. as } \varepsilon \downarrow 0$$

on account of X being regular downwards.

Next, suppose X is irregular downwards. With the notation $\sigma_+(y) = \inf\{t \geq 0 \,|\, Y_t^y > y\}$ we have $\sigma_+(y) > 0$ a.s. for all $y > 0$ (as X is irregular downwards and $\overline{X}_t \downarrow 0$ as $t \downarrow 0$) and for $y < y^*$ we may write (by a similar argument as in Step 3):

$$V(y) = \mathbb{E}\left[L^y_{\sigma_+(y)} + V(Y^y_{\sigma_+(y)})\right].$$

Letting $y \uparrow y^*$ and using that $V(y) = 0$ for all $y \geq y^*$ we see (recall V is bounded and continuous, and (3.6) holds)

$$V(y^*-) = \mathbb{E}\left[\int_0^{\sigma_+(y^*)} \left(2F(Y_u^{y^*}) - 1\right) \mathrm{d}u\right]. \tag{3.8}$$

By continuity of V we have $V(y^*-) = 0$ and hence the above rhs vanishes. Furthermore, since the integrand is monotone in y and for $y_1 < y_2$ we have $\sigma_+(y_2) \geq \sigma_+(y_1)$, the inequality being strict with positive probability, it is clear that y^* is the unique element in $\mathbb{R}_{>0}$ for which the rhs of (3.8) vanishes.

However, smooth pasting (i.e. $V'_-(y^*) = V'_+(y^*) = 0$), does not hold when F has a positive derivative on $\mathbb{R}_{>0}$. Namely, we have for any $\varepsilon > 0$

$$V(y^* - \varepsilon) \leq \mathbb{E}\left[\int_0^{\sigma_+(y^*)} \left(2F(Y_u^{y^*-\varepsilon}) - 1\right) \mathrm{d}u\right]$$

and using (3.8) we get

$$V(y^*) - V(y^* - \varepsilon) \geq \mathbb{E}\left[\int_0^{\sigma_+(y^*)} 2\left(F(Y_u^{y^*}) - F(Y_u^{y^*-\varepsilon})\right) \mathrm{d}u\right].$$

Dividing by ε and applying Fatou's Lemma yields

$$\liminf_{\varepsilon\downarrow 0} \frac{V(y^*) - V(y^*-\varepsilon)}{\varepsilon} \geq 2\mathbb{E}\left[\int_0^{\sigma_+(y^*)} \liminf_{\varepsilon\downarrow 0} \mathbf{1}_{\{\overline{X}_u < y^* - \varepsilon\}} \frac{F(Y_u^{y^*}) - F(Y_u^{y^*} - \varepsilon)}{\varepsilon} \mathrm{d}u\right]. \tag{3.9}$$

As $Y_u^{y^*} - Y_u^{y^*-\varepsilon} = \varepsilon$ on the event $\{\overline{X}_u < y^* - \varepsilon\}$ we see that the rhs in (3.9) is indeed strictly positive since F' is.

Step 5 ($y^* > \mathrm{m}$).

Recalling from Step 3 that $y^* \geq \mathrm{m}$, it remains to show that $y^* \neq \mathrm{m}$ in cases (i) and (ii) of the theorem. First case (ii). If X is irregular downwards, then we have (recall that $\sigma_+(\mathrm{m}) = \inf\{t \geq 0 \,|\, Y_t^{\mathrm{m}} > \mathrm{m}\}$)

$$V(\mathrm{m}) \leq \mathbb{E}[L^{\mathrm{m}}_{\sigma_+(\mathrm{m})}] = \mathbb{E}\left[\int_0^{\sigma_+(\mathrm{m})} (2F(Y_u^{\mathrm{m}}) - 1)\, \mathrm{d}u\right].$$

The rhs is strictly negative on account of the following facts: it holds that $F(z) < 1/2$ for $z < \mathrm{m}$, $Y_t^{\mathrm{m}} < \mathrm{m}$ for $0 < t < \sigma_+(\mathrm{m})$ and $\sigma_+(\mathrm{m}) > 0$ a.s. (as X is irregular downwards). Hence the continuity of V implies $y^* > \mathrm{m}$.

Next case (ii), so suppose that X is regular downwards and F is Lipschitz continuous on $\mathbb{R}_{\geq 0}$. Assume we had $y^* = \mathrm{m}$. We will show that this violates smooth pasting. Recall $\tau^+(a) = \inf\{t > 0 \,|\, X_t > a\}$ and $\tau^-(a) = \inf\{t > 0 \,|\, X_t < a\}$. For all $\varepsilon > 0$ small enough, using a similar argument to that in Step 3 shows

$$V(\mathrm{m} - \varepsilon) = \mathbb{E}\left[\int_0^{\tau^-(-\varepsilon)\wedge\tau^+(\mathrm{m}-\varepsilon)} \big(2F(Y_u^{\mathrm{m}-\varepsilon}) - 1\big)\, \mathrm{d}u + V\left(Y^{\mathrm{m}-\varepsilon}_{\tau^-(-\varepsilon)\wedge\tau^+(\mathrm{m}-\varepsilon)}\right)\right].$$

As $Y_t^{\mathrm{m}-\varepsilon} \leq \mathrm{m}$ for $t < \tau^-(-\varepsilon) \wedge \tau^+(\mathrm{m} - \varepsilon)$ the first integral in the above expectation is non-positive and hence

$$\begin{aligned} V(\mathrm{m} - \varepsilon) \leq \mathbb{E}\left[V\left(Y^{\mathrm{m}-\varepsilon}_{\tau^-(-\varepsilon)\wedge\tau^+(\mathrm{m}-\varepsilon)}\right)\right] &\leq \mathbb{E}\left[\mathbf{1}_{\{\tau^+(\mathrm{m}-\varepsilon)<\tau^-(-\varepsilon)\}} V\left(Y^{\mathrm{m}-\varepsilon}_{\tau^-(-\varepsilon)\wedge\tau^+(\mathrm{m}-\varepsilon)}\right)\right] \\ &= V(0)\mathbb{P}(\tau^+(\mathrm{m} - \varepsilon) < \tau^-(-\varepsilon)). \end{aligned}$$

The result now is a consequence of $V(0) < 0$ and Lemma 4. □

We conclude this section by showing that if θ has infinite mean it is impossible to find a stopping time which has finite L^1-distance to θ. This is intuitively not very surprising given the above Theorem 7. Namely, suppose we approximate X in a suitable sense by a sequence of Lévy processes, indexed by n say, for each of which the corresponding time of the ultimate supremum θ_n has finite mean. For each element in the sequence, a stopping time minimising the L^1-distance to θ_n is the first time the reflected process exceeds a level y_n^*. Suppose the y_n^*'s have a limit y_∞^*. If y_∞^* is finite then the limit of the optimal stopping times, say $\hat{\tau}$, is the first time the reflected process associated with X exceeds the level y_∞^*. However this would mean that $\hat{\tau}$ has finite mean and hence $\mathbb{E}[|\theta - \hat{\tau}|] = \infty$. On the other hand, if y_∞^* is infinite then $\hat{\tau} = \infty$ a.s. and hence still $\mathbb{E}[|\theta - \hat{\tau}|] = \infty$.

Proposition 8. *Suppose as before that X is not a compound Poisson process and drifts to $-\infty$. Suppose now that $\mathbb{E}[\theta] = \infty$. Then (3.1) is degenerate, i.e. for all stopping times τ it holds $\mathbb{E}[|\theta - \tau|] = \infty$.*

Proof. For any $q > 0$, let $\mathrm{e}(q)$ denote an exponentially distributed random variable with mean $1/q$, independent of X. (For convenience we denote the joint law of X and $\mathrm{e}(q)$ also by $\mathbb{P}$). We denote

$$\theta^{(q)} := \theta \wedge \mathrm{e}(q)$$

and similarly for any stopping time τ we denote

$$\tau^{(q)} := \tau \wedge \mathrm{e}(q).$$

Let us assume that a stopping time $\hat{\tau}$ exists with $\mathbb{E}[|\theta - \hat{\tau}|] < \infty$ and derive a contradiction. First, note that since $|\theta^{(q)} - \hat{\tau}^{(q)}| \to |\theta - \hat{\tau}|$ a.s. as $q \downarrow 0$ and $|\theta^{(q)} - \hat{\tau}^{(q)}| \leq |\theta - \hat{\tau}|$ for all $q > 0$ (this is readily checked from the definition of $\theta^{(q)}$ and $\hat{\tau}^{(q)}$) dominated convergence yields

$$\limsup_{q\downarrow 0} \inf_{\tau} \mathbb{E}[|\theta^{(q)} - \tau^{(q)}|] \leq \lim_{q\downarrow 0} \mathbb{E}\left[|\theta^{(q)} - \hat{\tau}^{(q)}|\right] = \mathbb{E}[|\theta - \hat{\tau}|] < \infty. \tag{3.10}$$

Now, using that for any $t \geq 0$ we have

$$\mathbb{P}(\theta^{(q)} \leq t \,|\, \mathcal{F}_t) = \mathbb{P}(\mathrm{e}(q) \leq t) + \mathbb{P}(\mathrm{e}(q) > t)\mathbb{P}(\theta \leq t \,|\, \mathcal{F}_t) = 1 - e^{-qt} + e^{-qt} F(Y_t^0),$$

the same reasoning as in Proposition 5 yields for any stopping time τ

$$\begin{aligned} \mathbb{E}[|\theta^{(q)} - \tau^{(q)}|] &= \mathbb{E}[\theta^{(q)}] + \mathbb{E}\left[\int_0^{\tau^{(q)}} \left(1 + 2e^{-qu}\left(F(Y_u^0) - 1\right)\right) \mathrm{d}u\right] \\ &= \mathbb{E}[\theta^{(q)}] + \mathbb{E}\left[\int_0^{\tau} e^{-qu}\left(1 + 2e^{-qu}\left(F(Y_u^0) - 1\right)\right) \mathrm{d}u\right]. \end{aligned} \tag{3.11}$$

To examine the rhs of (3.11), define the function V_q on $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ as

$$V_q(t, y) := \inf_{\tau} \mathbb{E}\left[\int_0^{\tau} e^{-qu}\left(1 + 2e^{-q(t+u)}\left(F(Y_u^y) - 1\right)\right) \mathrm{d}u\right]. \tag{3.12}$$

Note that the mappings $t \mapsto V_q(t, y)$ for any fixed $y \geq 0$ and $y \mapsto V_q(t, y)$ for any fixed $t \geq 0$ are non-decreasing. Furthermore for any $t \geq 0$

$$V_q(t, 0) \geq -\int_0^{\infty} e^{-qu}\,\mathrm{d}u > -\infty \tag{3.13}$$

and hence V_q is a bounded function taking values in $\mathbb{R}_{\leq 0}$. It is a straightforward exercise to slightly adjust the arguments from Step 2 (using (3.13)) in the proof of the above Theorem 7 to see that V_q is a continuous function. Following the same arguments as in Step 3 of the proof of the above Theorem 7, the Snell envelope $\widehat{L}$ of the process L defined as

$$L_t = \int_0^t e^{-qu} \left(1 + 2e^{-qu} \left(F(Y_u^0) - 1\right)\right) \mathrm{d}u \quad \text{for all } t \geq 0$$

satisfies

$$\begin{aligned}\widehat{L}_t &= \operatorname*{ess\,inf}_{\tau \geq t} \mathbb{E}[L_\tau \,|\, \mathcal{F}_t] = L_t + \operatorname*{ess\,inf}_{\tau \geq t} \mathbb{E}[L_\tau - L_t \,|\, \mathcal{F}_t] \\ &= L_t + e^{-qt} \operatorname*{ess\,inf}_{\tau \geq t} \mathbb{E}\left[\int_t^\tau e^{-q(u-t)} \left(1 + 2e^{-qu} \left(F(Y_u^0) - 1\right)\right) \mathrm{d}u \,\middle|\, \mathcal{F}_t\right] \\ &= L_t + e^{-qt} V_q(t, Y_t^0)\end{aligned}$$

for any $t \geq 0$. Therefore general theory of optimal stopping dictates that a stopping time minimising the rhs of (3.11) and, equivalently, which is optimal for $V_q(0,0)$, is given by

$$\tau_q^* = \inf\{t \geq 0 \,|\, \widehat{L}_t = L_t\} = \inf\{t \geq 0 \,|\, V_q(t, Y_t^0) = 0\} = \inf\{t \geq 0 \,|\, Y_t^0 \geq b_q(t)\}, \quad (3.14)$$

where $b_q(u) := \inf\{y \geq 0 \,|\, V_q(u,y) = 0\} \in \mathbb{R}_{>0}$ for all $u \geq 0$. Note that the final step uses the monotonicity of V_q in y. (Cf. Theorem 2.4 on p. 37 in [25] for details.) Note that the monotonicity of $t \mapsto V_q(t,y)$ implies that b_q is non-increasing.

Now we are ready to return to our main argument. Taking the infimum over τ in (3.11) yields

$$\inf_\tau \mathbb{E}[|\theta^{(q)} - \tau^{(q)}|] = \mathbb{E}[\theta^{(q)}] + V_q(0,0),$$

where the infimum in the lhs (and in $V_q(0,0)$) is attained by τ_q^* as defined in (3.14). Letting $q \downarrow 0$ in this equation it follows on account of (3.10) and $\theta^{(q)} \uparrow \theta$ so $\mathbb{E}[\theta^{(q)}] \to \mathbb{E}[\theta] = \infty$ that $V_q(0,0) \to -\infty$ as $q \downarrow 0$.

Next, we show that this implies

$$\text{for any } u_0 > 0 \text{ we have } b_q(u_0) \to \infty \text{ as } q \downarrow 0. \quad (3.15)$$

Indeed, suppose this were not the case, i.e. that $(q_k)_{k \geq 0}$ exists with $q_k \downarrow 0$ as $k \to \infty$ such that $b_{q_k}(u_0) \leq a$ for some $a > 0$ and all $k \geq 0$. The monotonicity of b then implies $b_{q_k}(u) \leq a$ for all $u \geq u_0$ and $k \geq 0$. Hence for all $k \geq 0$

$$\tau_{q_k}^* \leq \inf\{u \geq u_0 \,|\, Y_u^0 \geq a\},$$

where the stopping time in the rhs has finite mean due to the fact that X drifts to $-\infty$ (see the relevant comment in Step 1 of the proof of Theorem 7). However, due to (3.12) it holds that $V_{q_k}(0,0) \geq -\mathbb{E}[\tau_{q_k}^*]$, violating $V_{q_k}(0,0) \to -\infty$ as $q_k \downarrow 0$. Hence (3.15) holds.

For any $u_0 > 0$ we see from (3.15) together with the monotonicity of b_q that

$$\inf_{u\in[0,u_0]} b_q(u) \geq b_q(u_0) \to \infty \quad \text{as } q \downarrow 0$$

and consequently

$$\mathbb{P}(\tau_q^* \leq u_0) \leq \mathbb{P}\left(\sup_{s\leq u_0} Y_s^0 \geq b_q(u_0)\right) \to 0 \quad \text{as } q \downarrow 0.$$

This implies that

$$\inf_{\tau} \mathbb{E}[|\theta^{(q)} - \tau^{(q)}|] = \mathbb{E}[|\theta^{(q)} - \tau_q^{*(q)}|] \to \infty \quad \text{as } q \downarrow 0. \tag{3.16}$$

However, this violates (3.10) and hence we have arrived at the required contradiction. Indeed, one way to see that (3.16) holds is the following. Fix some $x > 0$. For an arbitrary $\varepsilon > 0$ pick θ_0 so that $\mathbb{P}(\theta > \theta_0) \leq \varepsilon$. Then

$$\begin{aligned}
\mathbb{P}(|\theta^{(q)} - \tau_q^{*(q)}| \leq x) &\leq \mathbb{P}(\theta > \mathrm{e}(q)) + \mathbb{P}(|\theta - \tau_q^{*(q)}| \leq x) \\
&\leq \mathbb{P}(\theta > \mathrm{e}(q)) + \mathbb{P}(\theta > \theta_0) + \mathbb{P}(\tau_q^{*(q)} \leq x + \theta_0) \\
&\leq \mathbb{P}(\theta > \mathrm{e}(q)) + \mathbb{P}(\theta > \theta_0) + \mathbb{P}(\mathrm{e}(q) \leq x + \theta_0) + \mathbb{P}(\tau_q^* \leq x + \theta_0),
\end{aligned}$$

and as $q \downarrow 0$ all the terms in the final rhs vanish except for the second, which is bounded above by the arbitrarily chosen ε. □

Remark 9. *If θ has infinite mean a possibility is to replace the L^1-distance by a more interesting metric, an alternative would for instance be to consider*

$$\inf_{\tau} \mathbb{E}[|\tau - \theta| - \theta]$$

as done in [16].

4 Example: spectrally negative

One special case for which the results from Theorem 7 can be expressed more explicitly is when X is spectrally negative, i.e. when the Lévy measure Π is concentrated on $\mathbb{R}_{<0}$ but X is not the negative of a subordinator. In this section X is assumed to be spectrally negative. Further details of the definitions and properties used in this section can be found in [20] Chapter 8.

Let ψ be the Laplace exponent of X, i.e.

$$\psi(z) = \frac{1}{t} \log \mathbb{E}\left[e^{zX_t}\right].$$

Then ψ exists at least on $\mathbb{R}_{\geq 0}$, it is strictly convex and infinitely differentiable with $\psi(0) = 0$ and $\psi(\infty) = \infty$. Denoting by Φ the right inverse of ψ, i.e. for all $q \geq 0$

$$\Phi(q) = \sup\{z \geq 0 \,|\, \psi(z) = q\} \in \mathbb{R}_{\geq 0},$$

the ultimate supremum $\overline{X}_\infty$ follows an exponential distribution with parameter $\Phi(0)$ with the usual convention that $\overline{X}_\infty = \infty$ a.s. when $\Phi(0) = 0$. It follows that

$$\overline{X}_\infty < \infty \quad \Leftrightarrow \quad \Phi(0) > 0 \quad \Leftrightarrow \quad \psi'(0+) < 0. \tag{4.17}$$

If the properties in (4.17) hold then the assumptions in Theorem 7 are satisfied. Indeed, $\psi'(0+) < 0$ implies that $\mathbb{E}[X_1] < 0$ and hence X drifts to $-\infty$ (cf. Theorem 7.2 in [20]). Furthermore Corollary 8.9 in [20] yields

$$\int_0^\infty \mathbb{P}(X_t \geq 0)\,\mathrm{d}t = \lim_{q \downarrow 0} \int_0^\infty e^{-qt}\mathbb{P}(X_t \geq 0)\,\mathrm{d}t = \lim_{q \downarrow 0} \int_0^\infty \Phi'(q) e^{-\Phi(q)x}\,\mathrm{d}x$$

which is finite as $\Phi(0) > 0$, implying that θ has finite mean (cf. Section 2).

Next, we briefly introduce scale functions. The scale function W associated with X is defined as follows: it satisfies $W(x) = 0$ for $x < 0$ while on $\mathbb{R}_{\geq 0}$ it is continuous, strictly increasing and characterised by its Laplace transform:

$$\int_0^\infty e^{-\beta x} W(x)\,\mathrm{d}x = \frac{1}{\psi(\beta)} \quad \text{for } \beta > \Phi(0).$$

Furthermore on $\mathbb{R}_{>0}$ the left and right derivatives of W exist. Note that in this case X is regular (resp. irregular) downwards when X is of unbounded (resp. bounded) variation. For ease of notation we shall assume that Π has no atoms when X is of bounded variation, which guarantees that $W \in C^1(0, \infty)$. Also, when X is of unbounded variation it holds that $W(0) = 0$ with $W'(0+) > 0$ (see [21]), otherwise $W(0) = 1/\mathtt{d}$ where $\mathtt{d} > 0$ is the drift of X.

For several families of spectrally negative Lévy processes W allows a (semi-)explicit representation, see [18] and the references therein. Scale functions are a natural tool for describing several types of fluctuation identities, relevant for this paper is that the potential measure of the reflected process Y^y starting from $y \geq 0$ killed at leaving the interval $[0, a]$, i.e.

$$U_a(y, \mathrm{d}x) = \int_0^\infty \mathbb{P}(Y_t^y \in \mathrm{d}x, t < \sigma(a))\,\mathrm{d}t,$$

where $\sigma(a) = \inf\{t \geq 0 \,|\, Y_t^y \geq a\}$ can also be expressed in terms of scale functions (cf. [20] Theorem 8.11):

Lemma 10. *When X is spectrally negative the measure $U_a(y, \mathrm{d}x)$ has a density on $(0, a)$ a version of which is given by*

$$u_a(y, x) = W(a - y)\frac{W'(x)}{W'(a)} - W(x - y)$$

and only when X is of bounded variation it has an atom at zero which is then given by

$$U_a(y,\{0\}) = \frac{W(a-y)W(0)}{W'(a)}.$$

The results of Theorem 7 are expressed in terms of scale function as follows.

Corollary 11. *When X is spectrally negative and satisfies any of the properties in (4.17), then y^* is the unique solution on $\mathbb{R}_{>0}$ to the equation in y:*

$$\int_{[0,y]} (1-2e^{-\Phi(0)x})W(\mathrm{d}x) = W(0) \tag{4.18}$$

and

$$V(y) = \int_0^{y^*} \left(2e^{-\Phi(0)x}-1\right) W(x-y)\,\mathrm{d}x \quad \textit{for all } y \geq 0. \tag{4.19}$$

Proof. Denoting $\sigma(x) = \inf\{t \geq 0 \,|\, Y_t^y \geq x\}$ we have from Theorem 7 that $V(y) = V(y,y^*) := \mathbb{E}[L^y_{\sigma(y^*)}]$, where

$$\begin{aligned} V(y,y^*) &= \mathbb{E}\left[\int_0^\infty (2F(Y_t^y)-1)\mathbf{1}_{\{t<\sigma(y^*)\}}\,\mathrm{d}t\right] \\ &= \int_0^\infty \int_{[0,y^*]} (2F(x)-1)\mathbb{P}(Y_t^y \in \mathrm{d}x, t<\sigma(y^*))\,\mathrm{d}t \\ &= \int_{[0,y^*]} (2F(x)-1) \int_0^\infty \mathbb{P}(Y_t^y \in \mathrm{d}x, t<\sigma(y^*))\,\mathrm{d}t \\ &= \int_0^{y^*} (2F(x)-1)u_{y^*}(y,x)\mathrm{d}x - U_{y^*}(y,\{0\}). \end{aligned}$$

Plugging in the result from Lemma 10 yields

$$\begin{aligned} V(y,y^*) &= \int_0^{y^*} \left(1-2e^{-\Phi(0)x}\right)\left(W(y^*-y)\frac{W'(x)}{W'(y^*)} - W(x-y)\right)\mathrm{d}x - \frac{W(y^*-y)W(0)}{W'(y^*)} \\ &= \int_0^{y^*} \left(2e^{-\Phi(0)x}-1\right) W(x-y)\,\mathrm{d}x \\ &\quad + \frac{W(y-y^*)}{W'(y^*)}\left(\int_0^{y^*} \left(1-2e^{-\Phi(0)x}\right) W'(x)\,\mathrm{d}x - W(0)\right). \end{aligned}$$

If X is of bounded variation, i.e. irregular upwards, continuity of V requires in particular $V(y^*-,y^*) = V(y^*,y^*) = 0$, which readily implies that y^* solves (4.18) and that (4.19) holds, since $W(x)=0$ for $x<0$ and $W(0)>0$. If X is of unbounded variation, i.e. regular upwards, the smooth pasting condition at y^* (Theorem 7 (i)) again readily implies that y^* solves (4.18) and (4.19) holds, since in this case $W(x)=0$ for $x<0$, $W(0)=0$ and $W'(0+)>0$.

Finally it remains to show that (4.18) has at most one solution. This is straightforward since the function g defined by

$$g(y) = \int_0^y (1 - 2e^{-\Phi(0)x}) W'(x)\, \mathrm{d}x$$

satisfies $g(0) = 0$, $g'(y) < 0$ for $y \in (0, \mathrm{m})$ (here $\mathrm{m} = \log(2)/\Phi(0)$) and $g'(y) > 0$ for $y > \mathrm{m}$. □

Example 12. *Consider the jump-diffusion $X_t = \sigma B_t + \mu t - \sum_{i=1}^{N_t} Y_i$, where B is a Brownian motion, N is a Poisson process with intensity $\lambda > 0$, $(Y_i)_{i\geq 1}$ is a sequence of iid exponentially distributed random variables with parameter $\theta > 0$, $\sigma > 0$ and $\mu \in \mathbb{R}$. Then the Laplace exponent ψ is given by*

$$\psi(z) = \frac{\sigma^2}{2} z^2 + \mu z - \frac{\lambda z}{\theta + z}.$$

Choosing the parameters such that $\psi'(0) < 0$ we see that ψ has roots $\beta_1 < -\theta$, $\beta_2 = 0$ and $\beta_3 > 0$, with

$$\beta_{1,3} = -\left(\frac{\theta}{2} + \frac{\mu}{\sigma^2}\right) \pm \sqrt{\left(\frac{\theta}{2} + \frac{\mu}{\sigma^2}\right)^2 - 2\left(\frac{\mu\theta - \lambda}{\sigma^2}\right)}.$$

Furthermore

$$W(x) = C_1 e^{\beta_1 x} + C_2 + C_3 e^{\beta_3 x} \quad \text{for } x \geq 0$$

where

$$C_1 = \frac{2(\theta + \beta_1)}{\sigma^2 \beta_1(\beta_1 - \beta_3)}, \quad C_2 = \frac{2\theta}{\sigma^2 \beta_1 \beta_3} \quad \text{and} \quad C_3 = \frac{2(\theta + \beta_3)}{\sigma^2 \beta_3(\beta_3 - \beta_1)}$$

as follows directly from the definition (see also [3]). Plugging this together with $\Phi(0) = \beta_3$ into (4.18) and (4.19) leads to Figure 1.

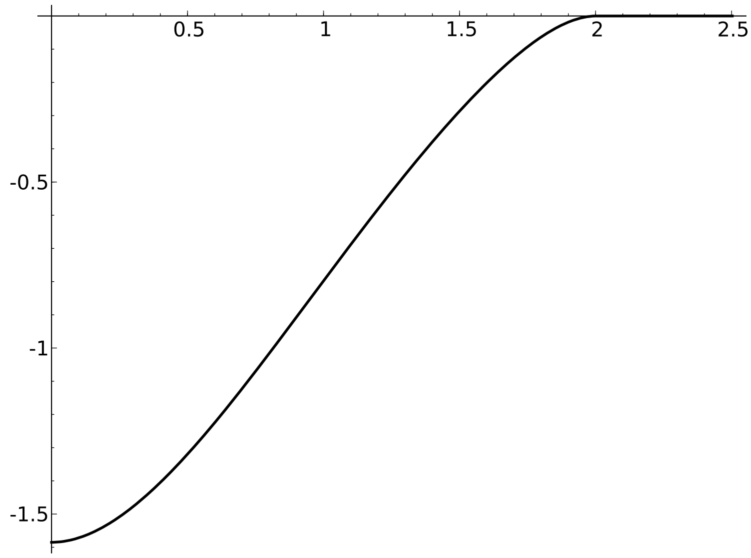

Figure 1: A plot of the value function V in the setting of Example 12, with $\sigma = \mu = 1/2$ and $\lambda = \theta = 1$. Note that $y^* \approx 2.0$.

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