Exact results for some non-integrable models of interacting spins and bosons

EXACT RESULTS FOR SOME NON-INTEGRABLE MODELS OF INTERACTING SPINS AND BOSONS

R. F. Bishop and C. Emary

Department of Physics, UMIST
(University of Manchester Institute of Science and Technology)
P. O. Box 88, Manchester M60 1QD, UK

1. INTRODUCTION

There exists a large number of models of common interest in the fields of quantum optics, quantum electronics, solid-state optoelectronics and allied areas that are generically of the form of a system of fermions or quantum spins interacting with bosonic degrees of freedom such as photons or phonons. Even though many simple models in this class are non-integrable in general, the possibility does nevertheless exist of being able to find exact solutions at a set of isolated points, i.e., for certain specific values of the coupling parameters. Such isolated points at which exact solutions are available are nowadays denoted as Juddian points after Judd who first obtained them in the case of the Jahn-Teller model [1].

Previous work has obtained such Juddian solutions for a limited class of Hamiltonians, typically by the use of power series and Neumann series in the bosonic field [2, 3]. Such approaches are generally not very intuitive, and the algebraic analysis involved is usually rather complicated. These features also make the methods of solution difficult to generalise to other model Hamiltonians. By contrast, we describe here the use of a much simper, more intuitive and broader approach [4, 5]. It is based upon a simple (Bogoliubov) canonical transformation of the bosonic field variables, which is itself quite intimately connected with the underlying group-theoretical structure of the models. We demonstrate its use by specifically considering two simple, non-integrable, non-adiabatic Hamiltonians of the above-mentioned kind. A great attraction of our approach is that it is easy to generalise.

The two important models in quantum optics to which we apply our approach both describe the interaction of light with matter, with the interaction being mediated by the exchange of either one or two photons. For the one-photon exchange case, the model is the well-known Rabi Hamiltonian [6]. Both models are generally non-

integrable for arbitrary coupling strengths, but become exactly soluble when the common rotating-wave approximation (RWA) is made. Thus, for example, under the RWA the Rabi Hamiltonian reduces to the familiar Jaynes-Cummings model [7].

In this contribution we describe in detail the energy spectra of both the non-integrable one- and two-photon models, and make a full comparison between each of these spectra and their corresponding counterparts when the RWA is made. We then proceed to demonstrate the existence of a series of isolated, exact solutions for each of the original models.

2. THE HAMILTONIANS

We consider a single-mode field of frequency ω, described by the annihilation and creation operators, b and $b^\dagger$, which obey the usual Bose commutation relation, $\left[b, b^\dagger\right] = 1$. The spin (or two-level system) is described by the Pauli matrices satisfying the SU(2) commutation relations, $[\sigma_k, \sigma_l] = 2i\varepsilon_{klm}\sigma_m$, where $k, l, m \in \{1 \equiv x, 2 \equiv y, 3 \equiv z\}$ and ε_{klm} is the anti-symmetric Levi-Civita symbol. We define the raising and lowering operators as $\sigma_\pm \equiv \sigma_x \pm i\sigma_y$ and the energy-splitting of the two spin states is taken to be ω_0. We use the following Hamiltonian to describe the two different interactions between the field and the spin which we employ here to exemplify our approach,

$$H_k = \frac{1}{2}\omega_0\sigma_z + \omega b^\dagger b + g\left(b^{\dagger k} + b^k\right)(\sigma_+ + \sigma_-)\,; \quad k = 1, 2, \tag{1}$$

where g is the coupling strength of the atom to the field. With $k = 1$, the interaction is of the dipole type and H_1 becomes the Rabi Hamiltonian (RH). With $k = 2$ the interaction describes two-photon processes, and H_2 is referred to as the two-photon Rabi Hamiltonian (TPRH) [8]. Both Hamiltonians H_k are, in general, non-integrable. For the systems to be on-resonance we have the condition $\omega_0 = k\omega$. There is a conserved quantity associated with H_k, given by

$$\Pi_k \equiv \exp\left[i\frac{\pi}{k}\left(b^\dagger b + \frac{k}{2}(\sigma_z + 1)\right)\right], \tag{2}$$

such that $[\Pi_k, H_k] = 0$. The existence of this operator splits the Hilbert space of H_k into $2k$ sub-sectors. Thus Π_1 has the eigenvalues $\pi_1 = \pm 1$, while Π_2 has the eigenvalues $\pi_2 = \pm 1$ and $\pi_2 = \pm i$.

Whereas H_1 is valid for arbitrary values of g, H_2 is physical, in the sense of possessing normalisable eigenfunctions, only if the condition $|\frac{4g}{\omega}| < 1$ is satisfied. This is a consequence of the intimate relation between H_2 and the squeezed states, as will be demonstrated in Sec. 3.

Rotating-Wave Approximation In quantum optics, and indeed elsewhere, models such as H_k are often treated in the rotating-wave approximation (RWA). Although useful at low coupling, there are many problems associated with the RWA, and its range of validity is often much less than is generally assumed [9, 10]. In the RWA,

H_k becomes

$$H_k^{\mathrm{RWA}} = \frac{1}{2}\omega_0\sigma_z + \omega b^\dagger b + g\left({b^\dagger}^k\sigma_- + b^k\sigma_+\right); \quad k = 1, 2. \tag{3}$$

For $k = 1$, this is just the well-known Jaynes-Cummings (JC) model, and $k = 2$ corresponds to the two-photon Jaynes-Cummings (TPJC) model. Although we shall only consider $k = 1, 2$ here, it should be noted that H_k^{RWA} does exist for all k, in contrast with H_k from which it is derived - a stark example of the effect that making the RWA can have. The Hamiltonian H_k^{RWA} is exactly soluble for arbitrary parameter values, since here the excitation number,

$$\hat{N}_k \equiv b^\dagger b + \frac{k}{2}\left(\sigma_z + 1\right), \tag{4}$$

is conserved, $\left[\hat{N}_k, H_k^{\mathrm{RWA}}\right] = 0$. This divides the full Hilbert space into k decoupled manifolds plus a denumerably infinite number of two-dimensional sub-spaces, in each of which the system can be simply diagonalised. The energy eigenvalues for the $k = 1$ Jaynes-Cummings model are given by

$$E_{\mathrm{JC}}^{(0)} = -\frac{1}{2}\omega_0,$$

$$E_{\mathrm{JC}}^{(n,\pm)} = \left(n - \frac{1}{2}\right)\omega \pm \frac{1}{2}\sqrt{\left(\omega_0 - \omega\right)^2 + 16g^2 n}; \quad n \geq 1. \tag{5}$$

On scaled resonance, $\omega = \omega_0 = 1$, the eigenvalues and eigenstates assume particularly simple forms:

$$E_{\mathrm{JC}}^{(0)} = -\frac{1}{2}; \quad \text{eigenstate} \quad |0, \downarrow\rangle,$$

$$E_{\mathrm{JC}}^{(n,\pm)} = \left(n - \frac{1}{2}\right) \pm 2g\sqrt{n}; \quad n \geq 1; \quad \text{eigenstates} \quad \frac{1}{\sqrt{2}}\left(|n, \downarrow\rangle \pm |n-1, \uparrow\rangle\right). \tag{6}$$

Similarly, the eigenenergies of the $k = 2$ two-photon Jaynes-Cummings model are given by

$$E_{\mathrm{TPJC}}^{(0)} = -\frac{1}{2}\omega_0, \qquad E_{\mathrm{TPJC}}^{(1)} = \omega - \frac{1}{2}\omega_0,$$

$$E_{\mathrm{TPJC}}^{(n,\pm)} = (n-1)\,\omega \pm \frac{1}{2}\sqrt{\left(\omega_0 - 2\omega\right)^2 + 16g^2 n\,(n-1)} \quad n \geq 2. \tag{7}$$

Again, on scaled resonance, $2\omega = \omega_0 = 1$, the eigenvalues and eigenstates become

$$E_{\mathrm{TPJC}}^{(0)} = -\frac{1}{2}; \quad \text{eigenstate} \quad |0, \downarrow\rangle$$

$$E_{\mathrm{TPJC}}^{(1)} = 0; \quad \text{eigenstate} \quad |1, \downarrow\rangle \tag{8}$$

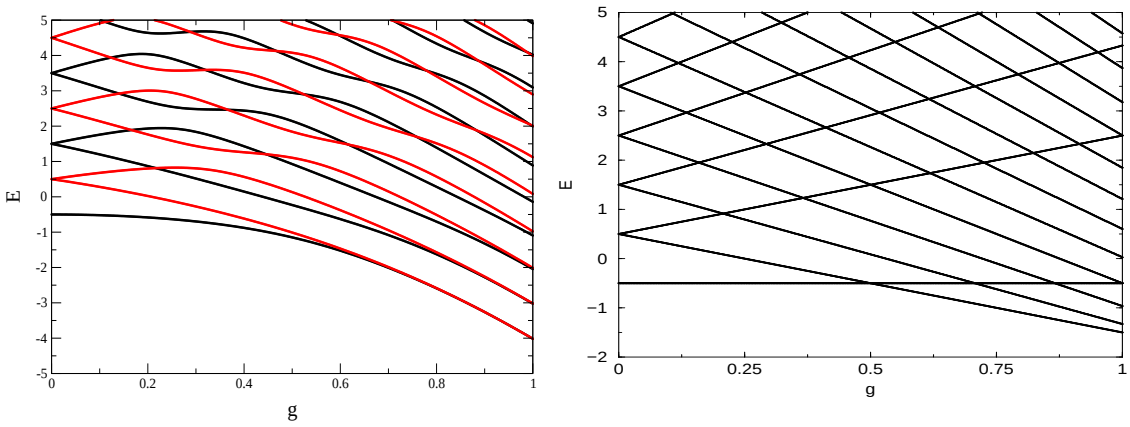

Figure 1. Spectra of the Rabi Hamiltonian H_1 and its RWA-counterpart H_1^{RWA} on scaled resonance $\omega = \omega_0 = 1$.

$$E_{\mathrm{TPJC}}^{(n,\pm)} = \frac{1}{2}(n-1) \pm 2gn\sqrt{1-\frac{1}{n}}; \quad n \geq 2; \quad \text{eigenstates} \quad \frac{1}{\sqrt{2}}\left(|n,\downarrow\rangle \pm |n-2,\uparrow\rangle\right).$$

Spectra Whereas the spectra for the RWA Hamiltonians may be determined analytically, we must use numerical techniques to study the systems without the RWA. Numerical diagonalisation is adequate for this task and in Figs. (1) and (2) we plot typical spectra obtained in this way for both the one- and two- photon Hamiltonians. These are to be compared and contrasted with the corresponding RWA spectra plotted alongside.

Considering the $k = 1$ results we see that, whereas the RWA spectrum is characterised by the large number of level-crossings characteristic of integrable systems, the spectrum of the full Hamiltonian has a reduced number of level-crossings and shows a series of marked avoided crossings, which produce a "braid-like" structure with pairs of eigenvalues intertwining with each other as the coupling is increased. From Fig. 2. we observe a similar situation with respect to the formation of braids by level repulsion, but in addition we see that both the full and RWA spectra undergo a dramatic change at a critical value of coupling. For the full Hamiltonian, the collapse of the spectrum is because the eigenfunctions become un-normalisable above $g = 0.125$ (on resonance). This feature was referred to earlier, and will be explained more fully below. On the other hand, in the RWA we see that for a coupling of $g \approx 0.25$ all the $E_{\mathrm{TPJC}}^{(n,-)}$ eigenvalues converge, with the result that at this point the ground-state energy abruptly changes from having a value of $-\frac{1}{2}$ to being unbounded from below. Whilst the eigenfunctions still remain normalisable throughout this change, this situation is clearly nonphysical and the validity of the TPJC Hamiltonian is also restricted, in this case to the range $|\frac{2g}{\omega}| < 1$. So although the ranges of validity of both H_2 and H_2^{RWA} are constrained, the reason is completely different in the two cases.

3. ISOLATED EXACT SOLUTIONS

We now demonstrate the existence of isolated exact solutions for the Hamiltonians H_k, $k = 1, 2$. These will be seen to occur at the level crossings in the spectra.

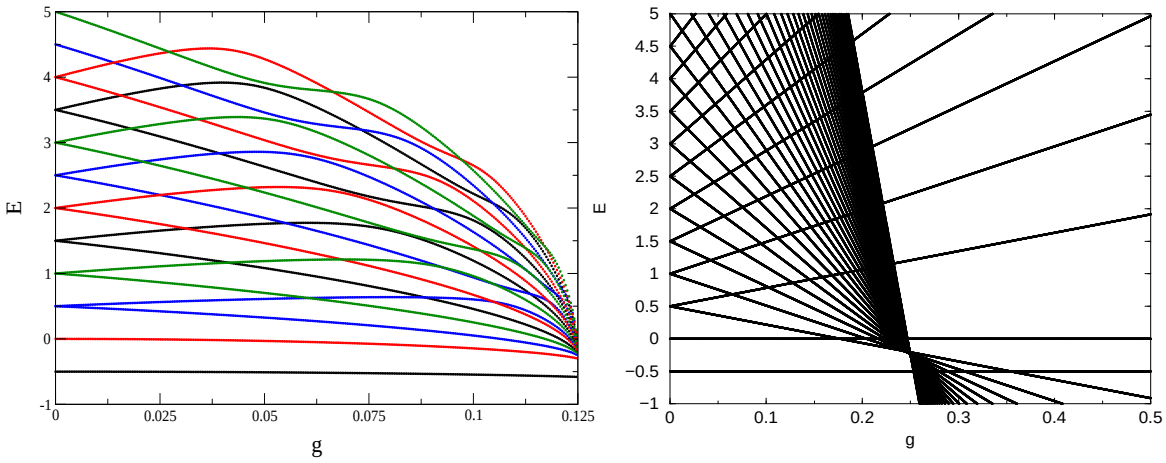

Figure 2. Spectra of the two-photon Rabi Hamiltonian H_2 and its RWA-counterpart H_2^{RWA} on scaled resonance $2\omega = \omega_0 = 1$.

We begin by rescaling the Hamiltonians as $H_k = \omega \tilde{H}_k$, where

$$\tilde{H}_k = \tilde{\omega}\sigma_z + b^\dagger b + \lambda\left(b^{\dagger k} + b^k\right)\sigma_x; \quad k = 1, 2, \tag{9}$$

and where $\tilde{\omega} \equiv \frac{\omega_0}{2\omega}$ and $\lambda \equiv \frac{2g}{\omega}$. Note that for $k = 2$, H_2 remains physical provided that $|\lambda| < 1/2$. We begin by expressing the Pauli matrices in a representation in which σ_x is diagonal. We choose $\sigma_x = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\sigma_y = \left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)$, and $\sigma_z = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. In terms of the two-component wavefunction, $|\Psi\rangle = \left(\begin{smallmatrix} |\Psi_1\rangle \\ |\Psi_2\rangle \end{smallmatrix}\right)$, the time-independent Schrödinger equation for the system, $\tilde{H}_k|\Psi\rangle = \tilde{E}|\Psi\rangle$, then reads

$$\begin{aligned} \tilde{\omega}|\Psi_2\rangle + \left(b^\dagger b + \lambda\left(b^{\dagger k} + b^k\right) - \tilde{E}\right)|\Psi_1\rangle = 0, \\ \tilde{\omega}|\Psi_1\rangle + \left(b^\dagger b - \lambda\left(b^{\dagger k} + b^k\right) - \tilde{E}\right)|\Psi_2\rangle = 0. \end{aligned} \tag{10}$$

We next perform a Bogoliubov transformation of the field mode from the operators b and $b^\dagger$ to a description in terms of new bosonic operators, a and $a^\dagger$, which obey the same commutation relation as before, namely $\left[a, a^\dagger\right] = 1$ [11]. The most general linear Bogoliubov transformation may be viewed as a rotation plus translation of the original oscillator Hilbert space to the new oscillator space,

$$\begin{aligned} a &= e^{-i\beta}\left(1 - |\sigma|^2\right)^{-1/2}\left(b - \sigma b^\dagger - z\right); \\ a^\dagger &= e^{i\beta}\left(1 - |\sigma|^2\right)^{-1/2}\left(b^\dagger - \sigma^* b - z^*\right), \end{aligned} \tag{11}$$

where σ and z are complex numbers describing the amplitudes of the rotation and translation respectively. The parameter β is a simple, and usually rather unimportant, phase factor. From the outset it is important to note the restriction $|\sigma| < 1$ in order to preserve the unitarity of the transformation.

If we set $\sigma = \beta = 0$, the resulting transformation is of the type that may be used to diagonalise the Hamiltonian of the displaced harmonic oscillator, the eigenstates of which are know to be the (Glauber) coherent states. With $z = 0$, the

Bogoliubov transformation becomes that used in diagonalising the squeezed harmonic oscillator, the eigenstates of which are the squeezed states. The constraint $|\sigma| < 1$ then corresponds to the requirement that the coefficients of both x^2 and p_x^2 in the squeezed oscillator are positive. It is simple to see how this condition is translated into the current two-photon model. If one considers Eq. (10) for $k = 2$ with $\tilde{\omega} = 0$, one obtains the Schrödinger equations for two decoupled squeezed oscillators, and the condition on the squeezing parameter σ translates to a constraint of the coupling $|\lambda| < 1/2$.

In the following we shall use both of these two specific forms, the displacement and the squeezing transformations, to find isolated, exact solutions for the one- and two- photon RH respectively.

One Photon For the RH, $k = 1$, we set $\sigma = \beta = 0$ in Eq. (11), giving us the Bogoliubov transformation of coherent states,

$$b^\dagger = a^\dagger + \lambda, \quad b = a + \lambda. \tag{12}$$

The vacuum state of the a-type bosons, $a|0,\lambda\rangle = 0$, in the original b representation is the coherent state $|\lambda\rangle$, $b|\lambda\rangle = \lambda|\lambda\rangle$, with amplitude given by the scaled coupling λ. With this transformation Eqs. (10) become

$$\begin{aligned} \tilde{\omega}|\Psi_2\rangle + \left\{a^\dagger a + 2\lambda\left(a^\dagger + a\right) + 3\lambda^2 - \tilde{E}\right\}|\Psi_1\rangle = 0, \\ \tilde{\omega}|\Psi_1\rangle + \left\{a^\dagger a - \lambda^2 - \tilde{E}\right\}|\Psi_2\rangle = 0, \end{aligned} \tag{13}$$

where the kets $|\Psi_{1,2}\rangle$ are now in the transformed representation. For these kets we choose the Ansätze

$$|\Psi_1\rangle = \sum_{n=0}^{N-1} p_n|n;\lambda\rangle = \sum_{n=0}^{N-1} p_n \frac{\left(a^\dagger\right)^n}{\sqrt{n!}}|0;\lambda\rangle = P_{N-1}\left(a^\dagger\right)|0;\lambda\rangle; \tag{14}$$

$$|\Psi_2\rangle = \sum_{n=0}^{N} q_n|n;\lambda\rangle = \sum_{n=0}^{N} q_n \frac{\left(a^\dagger\right)^n}{\sqrt{n!}}|0;\lambda\rangle = Q_N\left(a^\dagger\right)|0;\lambda\rangle; \tag{15}$$

where $|n;\lambda\rangle$ are number states of the displaced bosons, $a^\dagger a|n;\lambda\rangle = n|n;\lambda\rangle$, and we have introduced the polynomials P_{N-1} and Q_N of order $N-1$ and N respectively. Making these substitutions we have

$$\begin{aligned} \tilde{\omega}\sum_{n=0}^{N} q_n|n;\lambda\rangle + \sum_{n=0}^{N-1} p_n\left(n + 3\lambda^2 - \tilde{E}\right)|n;\lambda\rangle \\ +2\lambda\sum_{n=0}^{N-1} p_n\sqrt{n+1}|n+1;\lambda\rangle + 2\lambda\sum_{n=1}^{N-1} p_n\sqrt{n}|n-1;\lambda\rangle = 0, \\ \tilde{\omega}\sum_{n=0}^{N-1} p_n|n;\lambda\rangle + \sum_{n=0}^{N} q_n\left(n - \lambda^2 - \tilde{E}\right)|n;\lambda\rangle = 0. \end{aligned} \tag{16}$$

Considering the highest number state, $|N;\lambda\rangle$, in the second of these equations, we see that for this equation to hold we require

$$\left(N-\lambda^2-\tilde{E}\right)q_N=0. \tag{17}$$

Since $q_N\neq 0$ by Ansatz, we obtain a determination of the energy

$$\tilde{E}=N-\lambda^2. \tag{18}$$

This equation identifies the Juddian baseline energies, along which the Juddian solutions lie. Comparing the coefficients of the remaining number states gives us $2N+1$ linear equations for the $2N+1$ coefficients $(p_m, 0\leq m\leq N-1)$ and $(q_k, 0\leq k\leq N)$. To obtain non-trivial solutions, we clearly require the determinant of this equation set to be zero. This gives the compatibility condition, providing the locations of the Juddian points. The first two conditions ($N=1,2$) have the explicit forms

$$\tilde{\omega}^2+4\lambda^2=1, \qquad N=1, \tag{19}$$

$$\tilde{\omega}^4+\left(12\lambda^2-5\right)\tilde{\omega}^2+32\lambda^4-32\lambda^2+4=0, \qquad N=2. \tag{20}$$

Thus, for a given N, we have a polynomial of Nth order in λ^2 and $\tilde{\omega}^2$. Note that there is always an additional solution with $\tilde{\omega}=0$ to the compatibility condition, since this value decouples the equations (16). Each of these has N roots for λ^2 in terms of $\tilde{\omega}^2$, which all turn out to be real, thus giving the location of N Juddian solutions.

These solutions may also be found independently by using the displaced bosons

$$b^\dagger=a^\dagger-\lambda, \quad b=a-\lambda. \tag{21}$$

and interchanging the roles of $|\Psi_1\rangle$ and $|\Psi_2\rangle$. This demonstrates the degeneracy of the spectra at these Juddian points, and shows that they occur at the level crossings in the spectra.

Two Photon For the TPRH case we shall utilise squeezed bosons to find isolated exact solutions. We set $z=\beta=0$ in Eq. (11), obtaining

$$b=\left(1-\sigma^2\right)^{-1/2}\left(c+\sigma c^\dagger\right); \quad b^\dagger=\left(1-\sigma^2\right)^{-1/2}\left(\sigma c+c^\dagger\right), \tag{22}$$

where c and $c^\dagger$ are the squeezed bosonic operators. We now choose σ to be real and given by

$$\sigma=\frac{\Omega-1}{2\lambda}; \quad \Omega\equiv\sqrt{1-4\lambda^2}. \tag{23}$$

Making this Bogoliubov transformation in Eq. (10) with H_2, we obtain

$$\tilde{\omega}|\Psi_2\rangle+\left\{\Omega c^\dagger c-\left[\tilde{E}+\frac{1}{2}-\frac{\Omega}{2}\right]\right\}|\Psi_1\rangle=0,$$

$$\tilde{\omega}|\Psi_1\rangle+\frac{1}{\Omega}\left\{-\sqrt{1-\Omega^2}\left(c^{\dagger\,2}+c^2\right)+\left(2-\Omega^2\right)c^\dagger c\right.$$

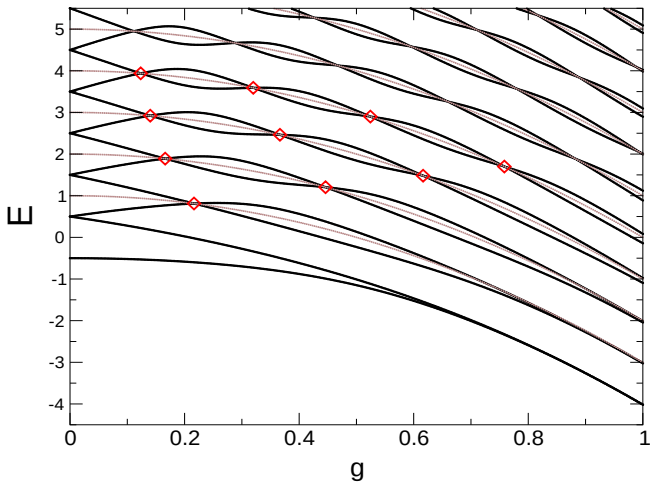

Figure 3. The first ten Juddian points of the Rabi Hamiltonian (diamonds). Also plotted are the energy levels obtained by numerical diagonalisation (dark lines), and the Juddian base-lines (light lines). The Hamiltonian is resonant; $\omega = \omega_0 = 1$.

$$+\frac{1}{2}(1-\Omega)(2+\Omega) - \Omega\tilde{E}\Big\}|\Psi_2\rangle = 0. \tag{24}$$

In a fashion similar to the $k = 1$ case, we now choose simple Ansätze for $|\Psi_1\rangle$ and $|\Psi_2\rangle$ in terms of the squeezed number states;

$$|\Psi_1\rangle = \sum_{n=0}^{N} p_n |n;\sigma\rangle, \qquad |\Psi_2\rangle = \sum_{m=0}^{N-2} q_m |m;\sigma\rangle. \tag{25}$$

We then proceed in direct analogy to the one-photon case, This determines the energy baselines, along which the solutions lie, to be

$$\tilde{E} = -\frac{1}{2} + \left(N + \frac{1}{2}\right)\Omega. \tag{26}$$

We also obtain the compatibility conditions which provide the locations of the solutions in λ–$\tilde{\omega}$ space, which we shall not reproduce here. As in the $k = 1$ case, further solutions degenerate in energy with those above may be found by using a squeezed representation with parameter $-\sigma$ rather than σ.

4. RESULTS

In Figs. 3 and 4 we plot the location of the lowest Juddian points for each of the two Hamiltonians, plotted against their respective energy schema and energy baselines.

The Juddian points occur at the level crossings in the spectra. Thus we see that they occur when two solutions of different values π_k become degenerate in energy, and it is this degeneracy that is the key to the existence of the Juddian solutions. The displaced and squeezed number states used in constructing the Ansätze above are not

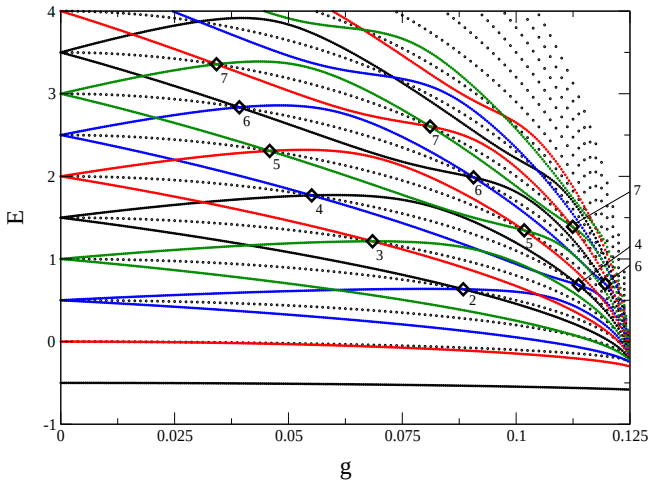

Figure 4. The first twelve Juddian points (diamonds) of the two-photon Rabi Hamiltonian determined by the method outlined in the text, plotted against the energy spectrum determined numerically (solid lines). Also plotted are the energy baselines (dotted lines). Each point is labeled with its order N. The Hamiltonian is resonant; $2\omega = \omega_0 = 1$.

states of definite π_k, and it is precisely because we can construct wavefunctions of mixed π_k that allows us to find such simple Ansätze at the Juddian points.

In the one-photon case, the solutions found by the above method cover each and every level-crossing in the spectrum. For the TPRH, however, only a certain subset of crossings are described by the above method. Considering the quantum numbers π_2 of the two intersecting lines at each crossing, we see that the above type of solution can only describe the crossings of states having $\pi_2 = +1$ with ones having $\pi_2 = -1$, and of crossings of states having $\pi_2 = +i$ with ones having $\pi_2 = -i$. The remaining four types of possible crossings are not described. This series of crossings can be understood by considering the operator $\Pi_2^2 = \exp\left(i\pi b^\dagger b\right)$, which obviously commutes with the Hamiltonian. From considering the eigenvalues of this operator we see that the Juddian solutions we have described occur between levels which have the same value of Π_2^2. Thus although the Ansätze of the Juddian solutions above are not eigenstates of Π_2, they are eigenstates of the square of this operator.

The reason why the above Ansätze can describe these solutions and not the others is as follows. The solutions that we have been able to find occur at crossings between energy eigenfunctions that are both composed of either all even or all odd number states. At the Juddian points these two eigenstates become degenerate in energy and thus, to find the energy at the level-crossing, we may form a linear superposition of the two eigenstates, which will, in general, not be an eigenstate of Π_2^2. Because the degenerate energy eigenstates are both "odd" or both "even", the formation of the superposition allows the individual terms in one wavefunction to add to the terms in the other. If we form the superposition correctly, the resultant wavefunction may have a form much simpler than the constituent wavefunctions, which is exactly the case in the Ansatz chosen above.

The solutions that we have been unable to find with the above method occur at the level-crossings between eigenstates one of which is composed of only odd number

states, the other composed only of even number states. Consequently, no superposition of these states will lead to a reduction in the complexity of either wavefunction and we have been unable to find simple Ansätze at these level-crossings.

5. CONCLUSIONS

We have demonstrated the utility of a rather general method employing Bogoliubov transformations of the bosonic mode to find some exact solutions of some typical non-integrable Hamiltonians which couple the bosonic field to quantum spin-half (or two-level) systems via terms either linear or bilinear in the field mode. The method is easy to generalise to other related models. Examples include models with more that one type of boson or field mode and/or with couplings to spins of quantum number $s > 1/2$ (or, equivalently, to n-level (atomic) systems with $n > 2$). Such broad classes of models nowadays provide a unifying framework in which to describe such important collective effects as superradiance in atomic asses and dissipation in arrays of quantum dots. It is our hope that the approach outlined in the present paper will be helpful for the analysis of such problems.

Finally, we note that while our approach does not appear to be immediately extensible to describe the missing crossing points in the TPRH energy spectrum, for example, it may still be possible that exact solutions can also be found at these points. Although there is no *a priori* reason to expect that such solutions must exist, we observe from the numerically generated results that the remaining level-crossings, for the resonant case $2\omega_0 = \omega = 1$, for example, appear very accurately to lie on simple baselines given by

$$\tilde{E} = -\frac{1}{2} + N\Omega, \qquad N = 2, 3, \ldots, \tag{27}$$

where, as before, $\Omega = \sqrt{1 - 4\lambda^2}$. These baselines are so similar to the baselines for the Juddian solutions found above in Eq. (26) as to strongly suggest that similar Juddian solutions might also be obtainable at these remaining level-crossings. Further work in this direction therefore seems well merited.

ACKNOWLEDGMENTS

C.E. acknowledges the financial support of a research studentship from the Engineering and Physical Sciences Research Council (E.P.S.R.C.) of Great Britain.

REFERENCES

[1] B. R. Judd, J. Chem. Phys. **67**, 1174 (1977); J. Phys. C: Solid State Phys. **12**, 1685 (1979).

[2] H. G. Reik, H. Nusser, and L. A. Amarante Ribeiro, J. Phys. A: Math. Gen. **15**, 3491 (1982); H. G. Reik and M. Doucha, Phys. Rev. Lett **57**, 787 (1986);

H. G. Reik, P. Lais, M. E. Stützle, and M. Doucha, J. Phys. A: Math. Gen **20**, 6327 (1987).

[3] M. Kus, J. Math. Phys. **26**, 2792 (1985); M. Kùs and M. Lewenstein, J. Phys. A: Math. Gen. **19**, 305 (1986).

[4] C. Emary and R. F. Bishop, J. Math. Phys. **43**, 3916 (2002).

[5] C. Emary and R. F. Bishop J. Phys. A **35**, 8231 (2002).

[6] I. I. Rabi, Phys. Rev. **51**, 652 (1937).

[7] E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963).

[8] K. M. Ng, C. F. Lo, and K. L. Liu, Eur. Phys. J. D **6**, 119 (1999); C. F. Lo, K. L. Liu, and K. M. Ng, Europhys. Lett. **42**, 1 (1998).

[9] C. Emary, PhD Thesis, UMIST, Manchester, 2001.

[10] R. F. Bishop and C. Emary, J. Phys. A **34**, 5635 (2001).

[11] R. F. Bishop and A. Vourdas, J. Phys. A: Math. Gen. **19**, 2525 (1986).