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Riesz-type inequalities and maximum flux exchange flow

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Abstract

Let D stand for the open unit disc in $\mathbb{R}^d$ $(d \geq 1)$ and $(D, \mathscr{B}, m)$ for the usual Lebesgue measure space on D. Let $\mathscr{H}$ stand for the real Hilbert space $L^2(D, m)$ with standard inner product $(\cdot, \cdot)$. The letter G signifies the Green operator for the (non-negative) Dirichlet Laplacian $-\Delta$ in $\mathscr{H}$ and ψ the torsion function $G\chi_D$. We pose the following problem. Determine the optimisers for the shape optimisation problem

$$\alpha_t := \sup\Big\{(G\chi_A, \chi_A) : \, A \subseteq D \text{ is open and } (\psi, \chi_A) \leq t\Big\}$$

where the parameter t lies in the range $0 < t < (\psi, 1)$. We answer this question in the one-dimensional case $d = 1$. We apply this to a problem connected to maximum flux exchange flow in a vertical duct. We also show existence of optimisers for a relaxed version of the above variational problem and derive some symmetry properties of the solutions.

Key words: shape optimisation

Mathematics Subject Classification 2010: 35J20

1 Introduction

Let Ω stand for a bounded open set in $\mathbb{R}^d$ $(d \geq 1)$ and $(\Omega, \mathscr{B}, m)$ for the usual Lebesgue measure space on Ω. Let $\mathscr{H}$ stand for the real Hilbert space $L^2(\Omega, m)$ with standard inner product $(\cdot, \cdot)$. The letter G signifies the Green operator for the (non-negative) Dirichlet Laplacian $-\Delta$ in $\mathscr{H}$ and ψ the torsion function $G\chi_\Omega$. We pose the following problem. Determine the optimisers for the shape optimisation problem

$$\alpha_t := \sup\Big\{(G\chi_A, \chi_A) : \, A \subseteq \Omega \text{ is open and } (\psi, \chi_A) \leq t\Big\} \tag{1.1}$$

where the parameter t lies in the range $0 < t < (\psi, 1)$. We show that optimisers exist for a relaxed version of this problem and derive certain symmetry properties of the solutions when Ω is replaced by the open unit ball D. We obtain the explicit form of the optimisers in the one-dimensional case $d = 1$ for the open interval $D = (-1, 1)$.

Define

$$V_t := \Big\{ \, f \in \mathscr{H} : \, 0 \leq f \leq 1 \, m\text{-a.e. on } \Omega \text{ and } (f, \psi) \leq t \Big\}$$

for t in the range $0 < t < (\psi, 1)$ and consider the relaxed variational problem

$$\beta_t := \sup\Big\{ \, J(f) : \, f \in V_t \Big\}, \tag{1.2}$$

where $J(f) = (f, Gf)$. The first main result runs as follows.

Theorem 1.1. *For each t in the range $0 < t < (\psi, 1)$, there exists $f \in V_t$ such that $\beta_t = J(f)$.*

In case Ω is replaced by the open unit ball D centred at the origin, we can say more about the symmetry properties of optimisers. In fact,

Theorem 1.2. *Let $f \in V_t$ such that $\beta_t = J(f)$. Then f possesses circular cap symmetry.*

We now turn to the one-dimensional case $d = 1$ so that $D = (-1, 1)$ and the torsion function ψ is given explicitly by $\psi(x) = (1/2)\,(1 - x^2)$ for $x \in D$. Noting that $(\psi, 1) = 2/3$, define $\varphi : D \to (0, 2/3)$ by $\varphi(x) := (\chi_{(x,\,1)}, \psi)$, and specify $\xi_t \in (-1, 1)$ uniquely via the relation

$$\varphi(\xi_t) = t \tag{1.3}$$

for each $t \in (0, 2/3)$. Set $A_t := (\xi_t, 1)$. Then

Theorem 1.3. *For any open subset A in D satisfying $(\psi, \chi_A) \leq t$ it holds that*

$$(G\,\chi_A, \chi_A) \leq (G\,\chi_{A_t}, \chi_{A_t}),$$

and equality occurs precisely when either $A = A_t$ or $A = -A_t$.

The inequality is somewhat reminiscent of the Riesz rearrangement inequality: this justifies the epithet in the title. This problem has a probabilistic interpretation in so far as the function $G\,\chi_A$ is the expected occupation time in A spent by absorbing Brownian motion in D (associated to the Laplacian Δ). The $d \geq 2$ case has not yet been resolved. It is tempting to speculate that a hyperbolic cap optimises (1.1) in this case. Numerical evidence does not seem to bear this out, however [5].

One reason why this problem is intriguing is because of its connection to maximum flux exchange flow in a vertical duct, a model of lava flow in a volcanic vent (see [4]). In the two-dimensional case $d = 2$, we imagine a configuration of two immiscible fluids in $D \times \mathbb{R}$ with different physical characteristics in a state of steady flow. The densities of the fluids are labelled ρ, ρ' and we take $\rho > \rho'$. Each fluid has unit viscosity. With respect to cylindrical coordinates $(x, z) \in D \times \mathbb{R}$, gravity acts in the direction $(0, -1)$ according to the model. The pressure p depends only upon z and has constant gradient $\partial p / \partial z = -G$. Suppose that the fluid with density ρ occupies a region in $D \times \mathbb{R}$ with cross-section $A \subseteq D$. Restricting the problem to D, the velocity u of the components of the fluid may be described (informally) using the Navier-Stokes equation via

$$\begin{aligned} 0 &= \Delta\,u + G - \rho\,g \quad \text{on} \quad A; \\ 0 &= \Delta\,u + G - \rho'\,g \quad \text{on} \quad D \setminus A. \end{aligned}$$

Non-slip (Dirichlet) boundary conditions are imposed on the boundary of D. It is also assumed that u and its gradient are continuous on the interface between the two regions A and $D \setminus A$ (continuity of velocity and stress).

The parameter G lies in the interval $(\rho'\,g, \rho\,g)$. This allows the possibility of a bi-directional flow. Upon rescaling (and relabelling the velocities) we obtain the system

$$\begin{aligned} 0 &= \Delta\,u - \lambda - 1 \quad \text{on} \quad A; \\ 0 &= \Delta\,u - \lambda + 1 \quad \text{on} \quad D \setminus A; \end{aligned} \tag{1.4}$$

where

$$\lambda := \frac{(\rho' + \rho)g - 2\,G}{(\rho - \rho')\,g} \in (-1, 1)$$

is a proxy for the pressure gradient. Two problems arise. One is to maximise the flux $Q := (\chi_{D\setminus A},\, u)$ amongst all regions A which satisfy the flux balance condition $(u,\, 1) = 0$ with constant λ; the other in which we optimize also over λ. In detail, we seek optimisers for the problems

$$\gamma \;:=\; \sup\left\{ (\chi_{D\setminus A},\, u) : (u,\, 1) = 0,\; A \subseteq D \text{ open},\; \lambda \in (-1,\, 1) \right\}, \tag{1.5}$$

$$\gamma_\lambda \;:=\; \sup\left\{ (\chi_{D\setminus A},\, u) : (u,\, 1) = 0,\; A \subseteq D \text{ open} \right\}, \tag{1.6}$$

where in the latter λ is fixed in the interval $(-1,\, 1)$. It turns out that problem (1.1) is closely related to the two problems above. Note too that these problems have obvious analogues for the case $d = 1$.

We come to our last main result. Note that the $d = 2$ analogue is discussed as a marginal case in [4].

Theorem 1.4. *In case $d = 1$,*

(i) for each $\lambda \in (-1,\, 1)$, the problem (1.6) is optimised precisely when either $A = A_{\frac{1-\lambda}{3}}$ or $A = -A_{\frac{1-\lambda}{3}}$;

(ii) the problem (1.5) is optimised precisely when either $A = (0,\, 1)$ or $A = (-1,\, 0)$ and has optimal value $1/12$.

We give a brief sketch of the organisation of the paper. In Section 2, we obtain existence of optimisers for the relaxed problem (1.2) and derive some symmetry properties when Ω is replaced by the ball D. Sections 3 to 8 deal with the proof of Theorem 1.3. Section 9 contains an application to maximum flux exchange flow (Theorem 1.4).

2 Existence of optimisers and symmetry in a general relaxed setting

For the sake of clarity, we first of all remark that the (non-negative) Dirichlet Laplacian $(D(-\Delta),\, -\Delta)$ is associated with the Dirichlet form $(\mathscr{F},\, \mathscr{E})$ in $\mathscr{H}$ with form domain $\mathscr{F} := W_0^{1,2}(\Omega)$ and

$$\mathscr{E}(u,\, v) = (Du,\, Dv) \qquad (u, v \in \mathscr{F}).$$

We begin with the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $(f_n)_{n\in\mathbb{N}}$ be a maximising sequence for β_t. Now, V_t is weakly sequentially compact in $\mathscr{H}$. This follows by appeal to [6] Theorem 10.2.9 due to the fact that V_t is bounded, closed and convex in the reflexive Banach space $\mathscr{H}$. So we may assume that (f_n) converges weakly in $\mathscr{H}$ to some $f \in V_t$ as $n \to \infty$ after choosing a subsequence if necessary.

Put $u_n := Gf_n$. Then for each n,

$$\|u_n\|_{W_0^{1,1}(\Omega)} \le \sqrt{2\, m(\Omega)}\, \|u_n\|_{W_0^{1,2}(\Omega)}.$$

Additionally,

$$\|u_n\|^2_{W_0^{1,2}(\Omega)} = \mathscr{E}(u_n,\, u_n) + (u_n,\, u_n) = (f_n,\, Gf_n) + (Gf_n,\, Gf_n) \le (1, \psi) + (\psi, \psi).$$

In short, the sequence (u_n) is bounded in $W_0^{1,1}(\Omega)$. In case $d \ge 2$ by the Rellich-Kondrachov compactness theorem ([3] 5.7 for example), we may assume that (u_n) converges in $L^1(\Omega,\, m)$ to some element u after extracting a subsequence if necessary. In case $d = 1$, we use Morrey's inequality (see [3] 5.6.2, for example) and the Arzela-Ascoli compactness criterion to extract a uniformly convergent subsequence. The details are described in the proof of Theorem 3.1.

For each $n \in \mathbb{N}$,

$$(u_n,\, \varphi) = (G\, f_n,\, \varphi) = (f_n,\, G\,\varphi) \text{ for all } \varphi \in \mathscr{H},$$

which yields

$$(u,\,\varphi) = (f,\,G\,\varphi) = (G\,f,\,\varphi) \text{ for all } \varphi \in \mathscr{H}$$

upon taking limits. Therefore, $u = Gf$ m-a.e. on Ω. Moreover,

$$J(f) - J(f_n) = (\,u,\,f\,) - (\,u_n,\,f_n\,) = (\,u,\,f - f_n\,) + (\,f_n,\,u - u_n\,)$$

and the right-hand side converges to zero as $n \to \infty$ in virtue of the weak respectively strong $L^1(\Omega,\,m)$ (or uniform in the case $d = 1$) convergence of the sequences (f_n) respectively (u_n). As $\beta_t = \lim_{n\to\infty} J(f_n)$ it follows that $\beta_t = J(f)$. □

In the remainder of this section, we replace Ω by the open unit ball D in $\mathbb{R}^d$ centred at the origin. We first discuss the operation of polarisation for integrable functions on D (see [2] and references therein). For $\nu \in S^{d-1}$ the closed half-space $H = H_\nu$ is defined by

$$H_\nu := \big\{x \in \mathbb{R}^d \,:\, x \cdot \nu \geq 0\big\}$$

with an associated reflection

$$\tau_H \,:\, \mathbb{R}^d \to \mathbb{R}^d;\, x \mapsto x - 2\,(x \cdot \nu)\,\nu.$$

Refer to the collection of all these closed half-spaces by $\mathcal{H}$. The polarisation f_H of $f \in L^1_+(D,\,m)$ with respect to $H \in \mathcal{H}$ is defined as follows. Choose an m-version of f, which we again denote by f. Set

$$f_H(x) := \begin{cases} f(x) \wedge f(\tau_H x) & \text{for} \quad x \in D \cap H, \\ f(x) \vee f(\tau_H x) & \text{for} \quad x \in D \setminus H. \end{cases}$$

Its m-equivalence class is the polarisation of f. The definition is well-posed.

The Green kernel $G(x,\,y)$ is given by

$$G(x,\,y) = \Phi(y - x) - \Phi(|x|(y - x^*)) \text{ for } (x,\,y) \in D \times D \setminus \mathtt{d},$$

where Φ is the fundamental solution of Laplace's equation in $\mathbb{R}^d$, $\mathtt{d}$ stands for the diagonal in $D \times D$ and the decoration $*$ refers to inversion in the unit sphere. We note the inequality

$$G(x,\,y) > G(x,\,\tau_H y) \text{ for any } x,\,y \in D \cap \operatorname{int} H, \tag{2.1}$$

which follows from the strong maximum principle.

Theorem 2.1. *Let $f \in L^1_+(D,\,m)$ and $H \in \mathcal{H}$. Then $J(f) \leq J(f_H)$ with equality if and only if either $f = f_H$ or $f \circ \tau_H = f_H$ m-a.e. on D.*

Proof. We work with an m-version of f, again denoted f. Define

$$A^+ := \{x \in D \cap H \,:\, f(x) < f(\tau_H x)\}$$

and similarly B^+ but with the strict inequality replaced by the sign $>$. Put $A^- := \tau_H A^+$ and $A := A^+ \cup A^-$. Set $S := D \setminus A$. In this notation,

$$f_H = \chi_A\, f \circ \tau_H + \chi_S\, f.$$

As a consequence,

$$J(f_H) = J(\chi_A\, f \circ \tau_H) + 2\,(\chi_A\, f \circ \tau_H,\, G\chi_S f) + J(\chi_S\, f) = J(\chi_A\, f) + 2\,(\chi_A\, f \circ \tau_H,\, G\chi_S f) + J(\chi_S\, f)$$

and a similar identity holds for $J(f)$ but without composition with reflection. We may then write that

$$J(f_H) - J(f) = 2 \int_{A^+} \int_{B^+} (f(\tau_H x) - f(x))(g(x,\, y) - g(\tau_H x,\, y))(f(y) - f(\tau_H y))\, m(dy)\, m(dx).$$

It is clear from this representation with the help of (2.1) that $J(f) \leq J(f_H)$.

In the case of equality, it holds that either $m(A^+) = 0$ or $m(B^+) = 0$. In the former case, $f = f_H$ while in the latter, $f \circ \tau_H = f_H$ m-a.e. on D.

The spherical cap symmetrisation (see [7], [8], [9] for example) of $A \in \mathscr{B}$ with respect to the direction $\omega \in S^{d-1}$ is the set $A^* \in \mathscr{B}$ specified uniquely by the conditions

$$\begin{array}{lcll} A^* \cap \{0\} & = & A \cap \{0\}\,, & \\ A^* \cap \partial B(0,\, r) & = & B(r\omega,\, \rho) \cap \partial B(0,\, r) & \text{for some } \rho \geq 0, \\ \sigma_r(A^* \cap \partial B(0,\, r)) & = & \sigma_r(B(r\omega,\, \rho) \cap \partial B(0,\, r)), & \end{array}$$

for each $r \in (0,\, 1)$. Here, σ_r stands for the surface area measure on $\partial B(0,\, r)$. The spherical cap symmetrisation of $f \in L^1_+(D,\, m)$ (denoted f^* for brevity) is defined as follows. Choose an m-version of f, which we again denote by f. Let f^* be the unique function such that

$$\{f^* > t\} = \{f > t\}^* \text{ for each } t \in \mathbb{R}.$$

Its m-equivalence class is the polarisation of f. The definition is again well-posed. We also write f^* as $C_\omega f$.

Before proving Theorem 1.2, we prepare a number of lemmas. We first discuss a useful two-point inequality. We introduce the notation

$$\begin{array}{lcl} Q & := & \left\{(x_1,\, x_2) \in \mathbb{R}^2 :\, x_1 \geq 0 \text{ and } x_2 \geq 0\right\}, \\ R & := & \{(x_1,\, x_2) \in Q :\, 0 \leq x_2 < x_1\}\,, \\ S & := & \{(x_1,\, x_2) \in Q :\, 0 \leq x_1 < x_2\}\,. \end{array}$$

Equip Q with the ℓ^1-norm $\|x\|_1 := |x_1| + |x_2|$ where $x = (x_1,\, x_2) \in Q$. Define a mapping $\varphi :\, Q \to Q$ via

$$(x_1,\, x_2) \mapsto (x_1 \vee x_2,\, x_1 \wedge x_2).$$

A geometric argument establishes the following lemma.

Lemma 2.1. *For any $x,\, y \in Q$,*

$$\|\varphi x - \varphi y\|_1 \leq \|x - y\|_1$$

with strict inequality if and only if $x \in R$ and $y \in \overline{S}$ or $x \in \overline{R}$ and $y \in S$ or the same with the rôles of x and y interchanged.

For $\omega \in S^{d-1}$ introduce the collection of closed half-spaces

$$\mathcal{H}_\omega := \left\{x \in \mathbb{R}^d :\, x \cdot \nu \geq 0\right\}.$$

Lemma 2.2. *Let $f \in L^1_+(D,\, m)$ and $\omega \in S^{d-1}$. For any $H \in \mathcal{H}_\omega$,*

$$\|f_H - C_\omega f\|_{L^1(D,\, m)} \leq \|f - C_\omega f\|_{L^1(D,\, m)} \tag{2.2}$$

with strict inequality if

$$m(\{f \circ \tau_H > f\}) > 0.$$

Proof. Select an m-version of f, again denoted f. Note that $f_H^* = f^*$. By the two-point inequality Lemma 2.1,

$$|f_H(x) - f^*(x)| + |f_H(\tau_H x) - f^*(\tau_H x)| \le |f(x) - f^*(x)| + |f(\tau_H x) - f^*(\tau_H x)| \tag{2.3}$$

for $x \in D \cap H$. It only remains to integrate over $D \cap H$ to obtain the inequality.

For each $x \in D \cap H$ the pair $(f^*(x),\, f^*(\tau_H x))$ belongs to $\overline{R}$. By Lemma 2.1 the condition $(f^*(x),\, f^*(\tau_H x)) \in S$ guarantees strict inequality in (2.3). This observation leads to the criterion in the Lemma. □

The next lemma is a spherical cap symmetrisation counterpart to [2] Lemma 6.3, and extends [7] Lemma 3.9.

Lemma 2.3. *Let $f \in L^1_+(D, m)$ and $\omega \in S^{d-1}$ and assume that $f \neq C_\omega f$. Then there exists $H \in \mathcal{H}_\omega$ such that*

$$\|f_H - C_\omega f\|_{L^1(D,\, m)} < \|f - C_\omega f\|_{L^1(D,\, m)}.$$

Proof. For shortness, write f^* for $C_\omega f$. As $f \neq f^*$ there exists $t > 0$ such that

$$m(\{f > t\} \,\Delta\, \{f^* > t\}) > 0.$$

It follows that the sets $A := \{f \le t < f^*\}$ and $B := \{f^* \le t < f\}$ are disjoint and have identical positive m-measure.

We claim that there exists $H \in \mathcal{H}_\omega$ such that $m(A \cap \tau_H B) > 0$. Taking this as read, on $A \cap \tau_H B$ we have that $f^* > t \ge f^* \circ \tau_H$ so that $A \cap \tau_H B \subseteq H$. Also, $f \le t < f \circ \tau_H$ there. In short, $A \cap \tau_H B \subseteq \{f \circ \tau_H > f\} \cap H$. So $m(\{f \circ \tau_H > f\}) > 0$ and there is strict inequality in (2.2) by Lemma 2.2.

To prove the claim, we assume for a contradiction that $m(A \cap \tau_H B) = 0$ for all $H \in \mathcal{H}_\omega$. Let F be a countable dense subset in $S^{d-1} \cap H_\omega$. Then

$$m(A \cap \bigcup_{\nu \in F} \tau_{H_\nu} B) = 0.$$

Therefore, for all $r \in (0,\, 1)$, it holds that

$$\sigma_r(A_r \cap \tau_{H_\nu} B_r) = 0 \text{ for every } \nu \in F,$$

except on a λ-null set N. Here, λ stands for Lebesgue measure on the Borel sets in $\mathbb{R}$, and $A_r := A \cap \partial B(0,\, r)$ for the section of A (likewise for B_r). Let $\nu \in S^{d-1} \cap H_\omega$ with corresponding reflection $\tau = \tau_{H_\nu}$. Select a sequence (ν_j) in F which converges to ν in S^{d-1}. Write τ_j for the reflection associated to closed half-space H_{ν_j}. For $r \in (0,\, 1) \setminus N$,

$$|\sigma_r(A_r \cap \tau B_r) - \sigma_r(A_r \cap \tau_j B_r)| \le \|\chi_B - \chi_B \circ \tau \circ \tau_j\|_{L^1(\partial B(0,\, r), \sigma_r)},$$

and this latter converges to zero as $j \to \infty$. This is due to the fact that the special orthogonal group $SO(d)$ acts continuously on $L^1(S^{d-1},\, \sigma)$. We derive therefore that

$$\sigma_r(A_r \cap \tau_{H_\nu} B_r) = 0 \text{ for every } \nu \in S^{d-1} \cap H_\omega \tag{2.4}$$

for all $r \in (0,\, 1) \setminus N$.

To conclude the argument, choose $r \in (0,\, 1) \setminus N$ such that $\sigma_r(A_r) = \sigma_r(B_r) > 0$. Use Lebesgue's density theorem to select a density point x for A_r lying in A_r, and choose y in B_r similarly. Then $f^*(x) > t \ge f^*(y)$. So there exists $\nu \in S^{d-1} \cap H_\omega$ such that with $\tau = \tau_{H_\nu}$ we have that $\tau y = x$. But this means that

$$\lim_{\varepsilon \downarrow 0} \frac{\sigma_r(A_r \cap \tau B_r \cap B(x,\, \varepsilon))}{\sigma_r(\partial B(0,\, r) \cap B(x,\, \varepsilon))} = 1,$$

so that, in fact, $\sigma_r(A_r \cap \tau B_r) > 0$, contradicting (2.4). $\square$

Proof of Theorem 1.2. Assume for a contradiction that $f \neq C_\omega f$ for each $\omega \in S^{d-1}$. Then there exists $\omega \in S^{d-1}$ such that

$$\delta := \inf_{\nu \in S^{d-1}} \| f - C_\nu f \|_{L^1(D,\, m)} = \| f - C_\omega f \|_{L^1(D,\, m)} > 0.$$

By Lemma 2.3 there exists $H \in \mathcal{H}_\omega$ such that

$$\| f_H - C_\omega f \|_{L^1(D,\, m)} < \| f - C_\omega f \|_{L^1(D,\, m)}.$$

It is plain that $f \neq f_H$. But also $f \circ \tau_H \neq f_H$, for otherwise,

$$\| f - C_{\sigma\omega} f \|_{L^1(D,\, m)} = \| f_H - C_\omega f \|_{L^1(D,\, m)} < \| f - C_\omega f \|_{L^1(D,\, m)},$$

contradicting optimality of ω. It follows by Theorem 2.1 that $J(f) < J(f_H)$ and this contradicts the optimality of f in the expression for β_t. $\square$

3 Preliminaries for the one-dimensional problem

In the remainder of the article we work in the one-dimensional setting where $D = (-1,\, 1)$. In this context, the corresponding Green operator G has kernel given by

$$G(x, y) = \begin{cases} \frac{1}{2}(1-y)(1+x) & \text{for} \quad x \leq y, \\ \frac{1}{2}(1+y)(1-x) & \text{for} \quad x > y, \end{cases} \tag{3.1}$$

for $x, y \in D$. We record the useful inequality

$$\big| G(x,\, y) - G(x,\, x) \big| \leq \big| y - x \big| \text{ for all } x,\, y \in D, \tag{3.2}$$

for future use. As noted above, the torsion function $\psi := G\chi_D$ is given explicitly by $\psi(x) = (1/2)\,(1 - x^2\,)$ for $x \in D$, and

$$(1,\, \psi) = 2/3. \tag{3.3}$$

The Green kernel may be bounded in terms of ψ; that is,

$$G(x,\, y) \leq \psi(x) \text{ for all } y \in D, \tag{3.4}$$

with fixed $x \in D$.

For $t \in (0,\, 2/3)$ introduce the shape space

$$U_t := \Big\{ \ f = \chi_A : \ A \subseteq D \text{ is open and } (f,\, \psi) \leq t \ \Big\}.$$

We may then write

$$\alpha_t = \sup \Big\{ \ J(f) : \ f \in U_t \ \Big\}. \tag{3.5}$$

For each $t \in (0,\, 2/3)$ and $m \in \mathbb{N}$ define $U_t^{(m)}$ to be the collection of all functions of the form $f = \chi_A$ where A is a union of at most m disjoint open intervals in D with the additional requirement that $(f,\, \psi) \leq t$. We occassionally refer to the condition

$$\operatorname{int} \overline{A} = A. \tag{3.6}$$

We also introduce the variational problem

$$\alpha_t^{(m)} := \sup \Big\{ \ J(f) : \ f \in U_t^{(m)} \ \Big\}. \tag{3.7}$$

We now derive the crucial property that (3.7) attains its optimum.

Theorem 3.1. *For each $t \in (0,\, 2/3)$ and $m \in \mathbb{N}$ there exists $f \in U_t^{(m)}$ with $(f,\, \psi) = t$ such that $\alpha_t^{(m)} = J(f)$.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a maximising sequence for $\alpha_t^{(m)}$. Each f_n may be written in the form $f_n = \sum_{j=1}^{k_n} \chi_{A_{nj}}$ for some $1 \leq k_n \leq m$ where $A_{nj} = (a_{nj},\, b_{nj})$ and

$$-1 \leq a_{n1} < b_{n1} \leq a_{n2} < b_{n2} \leq \cdots \leq a_{nk_n} < b_{nk_n} \leq 1.$$

After selecting a subsequence if necessary we may suppose that k_n takes a fixed value k for some k between 1 and m. On appeal to the Bolzano-Weierstrass theorem, we may assume (perhaps after discarding a subsequence) that $a_{nj} \to a_j$ and $b_{nj} \to b_j$ as $n \to \infty$ where

$$-1 \leq a_1 \leq b_1 \leq a_2 \leq b_2 \leq \cdots \leq a_k \leq b_k \leq 1. \tag{3.8}$$

Set $f := \sum_{j=1}^{k} \chi_{A_j}$ where $A_j = (a_j,\, b_j)$. By the dominated convergence theorem, (f_n) converges weakly to f in $\mathscr{H}$.

Put $u_n := Gf_n$ as before. Then the sequence (u_n) is bounded in $W_0^{1,2}(D)$ as in the proof of Theorem 1.1. By Morrey's inequality (see [3] 5.6.2, for example),

$$\| u_n \|_{C^{0,1/2}(\overline{D})} \leq c \text{ for all } n \in \mathbb{N}$$

for some finite constant c; in particular,

$$|\, u_n(x) - u_n(y)\, | \leq c\, |\, x - y\, |^{1/2} \text{ for any } x, y \in \overline{D}$$

and any $n \in \mathbb{N}$. Thus, (u_n) forms a bounded and equicontinuous sequence in $C(\overline{D})$. By the Arzela-Ascoli compactness criterion, we may assume that (u_n) converges uniformly to some $u \in C(\overline{D})$ as $n \to \infty$ after extracting a subsequence if necessary. Now continue the argument as in the proof of Theorem 1.1 to conclude that $\alpha_t^{(m)} = J(f)$.

We now show that $(f,\, \psi) = t$. First note that $(f,\, \psi) \leq t$; this flows from the fact that f is a weak limit of elements in $U_t^{(m)}$. Suppose for a contradiction that $(f,\, \psi) < t$. As $(f,\, \psi) < 2/3$, in (3.8) there must exist $j = 0, \ldots, k$ such that $b_j < a_{j+1}$ with the understanding that $b_0 := -1$ and $a_{k+1} := 1$. By choosing B to be a suitable (semi-)open interval in $[b_j,\, a_{j+1}]$ we can arrange that the function $f_1 := f + \chi_B$ satisfies the requirement $(f_1,\, \psi) \leq t$ as well as $J(f) < J(f_1)$. This contradicts the optimality of f. □

We now revisit the operation of polarisation in the one-dimensional setting. We use the letter P to signify the polarisation operator with respect to the closed half-space $[0,\, \infty)$. Thus, for $f \in U_t$, the polarisation is defined by

$$Pf(x) := \begin{cases} f(x) \vee f(-x) & \text{if} \quad 0 \leq x < 1, \\ f(x) \wedge f(-x) & \text{if} \quad -1 < x < 0. \end{cases} \tag{3.9}$$

Alternatively, suppose that $f = \chi_A$ where A is an open subset of D. Then $Pf = \chi_{PA}$ where PA denotes the polarisation of the set A; in other words,

$$PA = A \cap \tau A \bigcup \big(A \cup \tau A\big) \cap (0,\, 1) \tag{3.10}$$

where $\tau : D \to D$ stands for the reflection $x \mapsto -x$. We shall sometimes refer to the symmetric resp. non-symmetric parts of PA; that is,

$$\begin{aligned} A_1 &:= A \cap \tau A; \\ A_2 &:= \big(A \cup \tau A\big) \cap (0,\, 1) \setminus A \cap \tau A. \end{aligned} \tag{3.11}$$

Lemma 3.1. *Let $f \in U_t$ for some $t \in (0,\, 2/3)$. Then the following statements are equivalent:*

(i) $f \in PU_t$*;*

(ii) $f = 1$ *on* $S := \{x \in (0, 1) : f(-x) = 1\}$*.*

Proof. Let $f \in PU_t$ so that $f = Pg$ for some $g \in U_t$. Let $x \in (0, 1)$ with $f(-x) = 1$. Then $1 = f(-x) = Pg(-x) = g(x) \wedge g(-x)$. So $g(x) = 1$ and $f(x) = Pg(x) = g(x) \vee g(-x) = 1$. On the other hand, suppose that $f = 1$ on S. For $x \in S$,

$$Pf(x) = 1 \vee f(-x) = 1 = f(x) \text{ while } Pf(-x) = 1 \wedge f(-x) = f(-x),$$

and for $x \in (0, 1) \setminus S$,

$$Pf(x) = f(x) \vee 0 = f(x) \text{ while } Pf(-x) = f(x) \wedge 0 = 0 = f(-x).$$

In other words, $f = Pf$. □

It is sometimes useful to polarise with respect to the closed half-space $(-\infty, 0]$. To distinguish between these two polarisations we use the notations P_+, P_-. In particular,

$$P_- f(x) := \begin{cases} f(x) \wedge f(-x) & \text{if} \quad 0 < x < 1, \\ f(x) \vee f(-x) & \text{if} \quad -1 < x \leq 0. \end{cases} \tag{3.12}$$

Lemma 3.2. *Let* $f = \chi_A \in P_+ U_t^{(m)}$ *for some* $m \in \mathbb{N}$ *and* $t \in (0, 2/3)$ *where* A *satisfies condition (3.6). Put* $g := \chi_B$ *where* $B := D \setminus \overline{A}$*. Then* g *is an* m*-version of* $1 - f$ *and* $g \in P_- U_{3/2-t}$*.*

Proof. We may suppose that $A = \bigcup_{j=1}^k A_j$ for some $1 \leq k \leq m$ and $A_j = (a_j, b_j)$ with

$$-1 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \leq 1.$$

We use the criterion in Lemma 3.1. Let $x \in (-1, 0)$ such that $g(-x) = 1$. We first note that x cannot be a boundary point (that is, $x \notin \{a_1, \ldots, a_k, b_1, \ldots, b_k\}$). For if it is, then either $-x$ is a boundary point or $-x \in A$. This is due to the fact that f is polarised to the right. In either case, we obtain the contradiction that $g(-x) = 0$. We want to show that $g(x) = 1$ so suppose on the contrary that $g(x) = 0$. Then for $y = -x \in (0, 1)$, it holds that $f(-y) = 1$, but $f(y) = 0$. This counters the fact that $f \in P_+ U_t$ by the criterion. □

4 A (non-)optimality criterion

In this section we develop a (non-)optimality criterion for configurations f in $U_t^{(m)}$. Given $f \in U_t$ define $u := Gf$. It is known that $D(-\Delta) = W_0^{1,2}(D) \cap W^{1,2}(D)$. Thus, $u \in W^{2,2}(D)$ and by a Sobolev inequality (see [3] 5.6.3 for example), u belongs to the Hölder space $C^{1,1/2}(\overline{D})$. Define

$$h = h_f := \frac{u}{\psi}.$$

Then $h \in C(D)$ and by l'Hôpital's rule,

$$h(-1) = \lim_{x \downarrow -1} \frac{u'(x)}{-x} = u'(-1), \tag{4.1}$$

and similarly $h(1) = -u'(1)$ at the right-hand end-point. In short, $h \in C(\overline{D})$.

Lemma 4.1. *Suppose that* $f = \chi_A$ *for some open subset* A *in* D*. Let* $u \in C(D)$*. Given* $a \in \overline{A} \cap D$*, put* $A_\eta := [a - \eta, a + \eta]$ *for* $\eta > 0$ *small. Then*

(i) $\lim_{\eta \downarrow 0} \frac{(f\chi_{A_\eta}, u)}{(f\chi_{A_\eta}, \psi)} = h(a)$*;*

(ii) $\lim_{\eta\downarrow 0} \frac{(f\chi_{A_\eta},\, G[f\chi_{A_\eta}])}{(f\chi_{A_\eta},\, \psi)} = 0.$

Proof. *(i)* Notice that $A\cap(a-\eta,\, a+\eta) \neq \emptyset$ for each $\eta > 0$. Consequently, $(f\chi_{A_\eta},\, 1) = m(A\cap A_\eta) > 0$ for each $\eta > 0$ (small) and likewise for $(f\chi_{A_\eta},\, \psi)$. Write

$$\frac{(f\chi_{A_\eta},\, u)}{(f\chi_{A_\eta},\, \psi)} = \frac{u(a)\,(f\chi_{A_\eta},\, 1) + (f\chi_{A_\eta},\, u - u(a))}{\psi(a)\,(f\chi_{A_\eta},\, 1) + (f\chi_{A_\eta},\, \psi - \psi(a))} = h(a) + \frac{\psi(a)\zeta_1 - u(a)\zeta_2}{\psi(a)(\psi(a) + \zeta_2)}$$

where

$$\zeta_1 := \frac{(f\chi_{A_\eta},\, u - u(a))}{(f\chi_{A_\eta},\, 1)},$$
$$\zeta_2 := \frac{(f\chi_{A_\eta},\, \psi - \psi(a))}{(f\chi_{A_\eta},\, 1)}.$$

Both these last vanish in the limit $\eta \downarrow 0$ and this leads to the identity.

(ii) From the estimate (3.4), for $\eta > 0$ small,

$$\psi^{-1}\, G[f\chi_{A_\eta}] \leq \psi^{-1}\, G[\chi_{A_\eta}] \leq 2\,\eta,$$

and this establishes the limit. □

With this preparation in hand we arrive at the crucial (non-)optimality condition.

Theorem 4.1. *Let $t \in (0,\, 2/3)$, $m \in \mathbb{N}$ and $f = \chi_A \in U_t^{(m)}$. Assume that A satisfies condition (3.6). Suppose that $a,\, b \in D$ with $a \neq b$ such that*

(i) $h(a) < h(b)$;

(ii) $a \in \partial A$;

(iii) $b \in \partial A$.

Then there exists $f_1 \in U_t^{(m)}$ such that $J(f) < J(f_1)$.

Proof. Write f in the form $f = \sum_{j=1}^{k} \chi_{A_k}$ for some $1 \leq k \leq m$ where $A_j = (a_j,\, b_j)$ and

$$-1 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \leq 1.$$

Given $\eta > 0$ put $A_\eta := [a - \eta,\, a + \eta]$ and $B_\eta := (b - \eta,\, b + \eta)$. Set $g := 1 - f$. The functions

$$\eta \mapsto (\chi_{A_\eta}\, f,\, \psi) \text{ and } \eta \mapsto (\chi_{B_\zeta}\, g,\, \psi)$$

are strictly increasing at least for $\eta > 0$ small. For $\varepsilon > 0$ sufficiently small, there exist unique $\eta > 0$ and $\zeta > 0$ depending upon ε such that

$$\varepsilon = (\chi_{A_\eta}\, f,\, \psi) = (\chi_{B_\zeta}\, g,\, \psi).$$

Define

$$f_\varepsilon := f - f\,\chi_{A_\eta} + g\,\chi_{B_\zeta}.$$

Then $f_\varepsilon \in U_t^{(m)}$ for $\varepsilon > 0$ small. Now

$$\begin{aligned} J(f_\varepsilon) - J(f) &= (f_\varepsilon,\, G\,f_\varepsilon) - (f,\, G\,f) \\ &= (f_\varepsilon - f,\, G\left[\, f_\varepsilon + f\,\right]) \\ &= (-f\chi_{A_\eta} + g\chi_{B_\zeta},\, G\left[\, 2\,f - f\chi_{A_\eta} + g\chi_{B_\zeta}\,\right]) \\ &= 2\,(g\chi_{B_\zeta},\, u) - 2\,(f\chi_{A_\eta},\, u) + (g\chi_{B_\zeta} - f\chi_{A_\eta},\, G\left[\, g\chi_{B_\zeta} - f\chi_{A_\eta}\,\right]) \end{aligned} \quad (4.2)$$

where $u = G\,f$ as usual. Thus, by Lemma 4.1 (with the help of the Cauchy-Schwarz inequality to deal with the cross-terms),

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-1} \Big\{\, J(f_\varepsilon) - J(f) \Big\} = 2(\, h(b) - h(a)\,) > 0.$$

In particular, there exists $\varepsilon > 0$ (small) such that $f_1 := f_\varepsilon \in U_t^{(m)}$ satisfies $J(f_1) > J(f)$. □

5 More on the (non-)optimality condition

In this section we verify condition *(i)* in Theorem 4.1 for some particular configurations f in U_t.

Lemma 5.1. *Let $f \in U_t$ for some $t \in (0, 2/3)$ and set $u := G f$. Then*

(i) $u'(-1) + u'(1) = -\int_D x\, f\, dm$;

(ii) $u'(-1) - u'(1) = \int_D f\, dm$.

Proof. *(i)* Using $\psi' = -x$ and the integration-by-parts formula,

$$\begin{aligned}\int_D x\, f\, dm &= \int_D \psi'\, u''\, dm \\ &= \psi'(1)\, u'(1) - \psi'(-1)\, u'(-1) - \int_D \psi''\, u'\, dm \\ &= -u'(1) - u'(-1).\end{aligned}$$

(ii) This follows from $\int_D u''\, dm = -\int_D f\, dm = u'(1) - u'(-1)$. □

Assuming that $f \not\equiv 0$, define $a,\, b \in \mathbb{R}$ by

$$\begin{aligned} a &:= \inf\Big\{ x > -1 : (\, \chi_{(-1,\, x]},\, f\,) > 0 \Big\}, \\ 1 - b &:= \inf\Big\{ x > 0 : (\, \chi_{[1-x,\, 1)},\, f\,) > 0 \Big\}; \end{aligned} \tag{5.1}$$

so that $a \in [-1,\, 1)$ and $b \in (-1,\, 1]$.

Lemma 5.2. *For each $f \in U_t$ with $f \not\equiv 0$,*

(i) $h(y) = \frac{1}{1-y} \int_D \big(\, 1 - x\,\big)\, f\, dm$ *for each* $y \in [-1,\, a]$;

(ii) $h(y) = \frac{1}{1+y} \int_D \big(\, 1 + x\,\big)\, f\, dm$ *for each* $y \in [b,\, 1]$.

Proof. *(i)* Suppose that $a = -1$. By (4.1) and Lemma 5.1,

$$h(-1) = u'(-1) = (1/2) \int_D \big(\, 1 - x\,\big)\, f\, dm.$$

Now suppose that $a \in (-1,\, 1)$. Using integration-by-parts,

$$\begin{aligned} u(y) &= \int_{(-1,\, y]} u'\, dm \\ &= -\int_{(-1,\, y]} u'\, \psi'' dm \\ &= -u'(y)\, \psi'(y) + u'(-1)\, \psi'(-1) + \int_{(-1,\, y]} u''\, \psi' dm \\ &= -u'(y)\, \psi'(y) + u'(-1) \end{aligned}$$

as $u'' = -f = 0$ m-a.e. on $(-1,\, a]$. For the same reason,

$$u'(y) - u'(-1) = \int_{(-1,\, y]} u''\, dm = 0.$$

Therefore,

$$u(y) = \big(\, 1 - \psi'(y)\,\big)\, u'(-1) = \frac{1 + y}{2} \int_D \big(\, 1 - x\,\big)\, f\, dm$$

from which the statement is clear. Part *(ii)* follows in a similar fashion. □

Proposition 5.1. *Let $f \in PU_t$ for some $t \in (0,\, 2/3)$. With a, b as in (5.1) assume that*

(i) $-1 < a < 0 < -a < b < 1$;

(ii) $f = 1$ *m-a.e. on* $(-a,\, b)$.

Then $h(a) < h(b)$.

Proof. From Lemma 5.2,

$$h(b) - h(a) = -\frac{a+b}{(1-a)\,(1+b)} \int_D f\, dm + \frac{2-a+b}{(1-a)\,(1+b)} \int_D x\, f\, dm.$$

Put $S := \{x \in (0,\, 1) :\ f(-x) = 1\}$ as in Lemma 3.1. Then

$$\int_D x\, f\, dm = \int_S x\, \{\, f(x) - f(-x)\,\}\; m(dx) + \int_{(0,\,1)\setminus S} x\, f\, dm$$

$$= \int_{(0,\,1)\setminus S} x\, f\, dm \geq \int_{(-a,\,b)} x\, f\, dm = (1/2)(b^2 - a^2)$$

making use of *(ii)*. Thus,

$$(2-a+b) \int_D x\, f\, dm \geq (1/2)\,(2-a+b)\,(b^2-a^2) > b^2 - a^2 \geq (a+b) \int_D f\, dm$$

and hence $h(b) - h(a) > 0$. □

In the next two sections, we show non-optimality of polarised configurations in three broad cases.

6 Two non-symmetric cases

Let $t \in (0,\, 1/3]$ and imagine a configuration polarised to the right that charges the left-hand interval $(-1,\, 0)$ but which is not symmetric under reflection in the origin. We show this is non-optimal.

Lemma 6.1. *Let $m \in \mathbb{N}$ and $t \in (0,\, 1/3]$. Suppose that $f \in U_t^{(m)}$ satisfies the properties*

(i) $f = Pf$;

(ii) $(\, f,\, \chi_{(-1,\,0)}\,) > 0$.

(iii) $(\, f,\, \chi_{(0,\,1)}\,) > (\, f,\, \chi_{(-1,\,0)}\,)$.

Then there exists $f_1 \in U_t^{(m)}$ with the property that $f_1 = P\, f_1$ such that $J(f) < J(f_1)$.

Proof. We may assume that $f = \chi_A$ where A satisfies condition (3.6). We may then write f in the form described at the beginning of the proof of Theorem 4.1. By *(ii)*, $a_1 < 0$; and by *(i)*, $b_k \geq -a_1$.

Case (a): $-1 < a_1$ and $b_k < 1$. Then, in fact, $-1 < a_1 < 0 < -a_1 \leq b_k < 1$. Put

$$k_1 := \min\Big\{\, j = 1, \ldots, k :\ b_j \geq -a_1 \,\Big\}.$$

Suppose first of all that $-a_1 = b_{k_1}$. Decompose A into its symmetric and non-symmetric parts A_1 and A_2 as in (3.11). By *(iii)*, $A_2 \neq \emptyset$. Write $f_1 := \chi_{A_1}$ and $f_2 := \chi_{A_2}$. By symmetry, $h_{f_1}(a_1) = h_{f_1}(-a_1)$. Further, $h_{f_2}(a_1) < h_{f_2}(-a_1)$, this being a consequence of (2.1). Therefore, as $h = h_{f_1} + h_{f_2}$, we obtain $h(a_1) < h(-a_1)$. The conclusion follows with an application of Theorem 4.1.

If $-a_1 \neq b_{k_1}$ then $-a_1 < b_{k_1}$ and $f = 1$ on $(-a_1,\, b_{k_1})$. If $k_1 = k$ then $h(a_1) < h(b_k)$ by Proposition 5.1. On the other hand, if $k_1 < k$ define

$$f_1 := \sum_{j=1}^{k_1} \chi_{A_j} \text{ and } f_2 := \sum_{j=k_1+1}^{k} \chi_{A_j}.$$

By Proposition 5.1, $h_{f_1}(a_1) < h_{f_1}(b_{k_1})$. It can be seen from the representation in Lemma 5.2 that h_{f_2} is increasing on $[-1,\, a_{k_1+1}]$. In sum, then, $h(a_1) < h(b_{k_1})$. Now apply Theorem 4.1 once more.

Case (b): $-1 < a_1$ and $b_k = 1$. In this situation, $-1 < a_1 < 0 < -a_1 < b_k = 1$. Define k_1 as before. The case $k_1 < k$ may be dealt with in a similar way to case (a) above. So assume that $k_1 = k$. As f is polarised to the right, the interval $((-b_1) \vee 0,\, -a_1)$ must sit inside A and so it must hold that $a_k < -a_1$. In case $t \in (0,\, 1/3)$, it must also hold that $0 < a_k$. The situation $t = 1/3$ and $a_k = 0$ forces $(\, f,\, \chi_{(-1,\,0)}\,) = 0$ contradicting *(ii)*. In either case, therefore, $0 < a_k < -a_1$ and $k \geq 2$.

Consider the function $g := \chi_B$ where $B := D \setminus \overline{A}$. By Lemma 3.2, $g \in P_{-}U_{3/2-t}$. Thus,

$$-1 < a_1 < -a_k < 0 < a_k < b_k = 1,$$

and $g = 1$ just to the right of $-a_k$ as g is polarised to the left. This situation corresponds to the one described at the start of the consideration of this case but for g instead of f. Use the fact that $h_g = 1 - h_f$.

Case (c): $a_1 = -1$. Then $b_k = 1$ and $a_k \leq -b_1$ as $f = Pf$. Apply the arguments in case *(a)* to the function g. □

We now take $t \in (0,\, 1/3)$ and imagine a configuration that lies entirely in the right-hand interval $(0,\, 1)$ but that has not yet been pushed rightwards to the maximum extent. We again show non-optimality.

Lemma 6.2. *Let $m \in \mathbb{N}$ and $t \in (0,\, 1/3)$. Suppose that $f \in U_t^{(m)}$ satisfies the properties*

(i) $(\, f,\, \chi_{(-1,\,0)}\,) = 0$*;*

(ii) $(\, f,\, \chi_{(0,\,\xi_t)}\,) > 0$.

Then there exists $f_1 \in U_t^{(m)}$ with the property that $f_1 = P\, f_1$ such that $J(f) < J(f_1)$.

Proof. Again take $f = \chi_A$ where A satisfies condition (3.6) and suppose f takes the form described at the beginning of the proof of Theorem 4.1. By *(i)*, $a_1 \geq 0$ and by *(ii)*, $a_1 < \xi_t$. Therefore $a_1 < b_1 < 1$; for otherwise, if $b_1 = 1$ then

$$(f,\, \psi) \geq (\chi_{(a_1,\,1)},\, \psi) > (\chi_{(\xi_t,\,1)},\, \psi) = t.$$

Again borrowing the notation of Theorem 4.1, put $f_1 := \chi_{A_1}$ and $f_2 := \sum_{j=2}^{k} \chi_{A_j}$. By Lemma 5.2,

$$h_{f_1}(a_1) = \frac{1}{1-a_1} \int_{(a_1,\,b_1)} (\,1-x\,)\, dm = \frac{b_1 - a_1}{1-a_1} \left\{\, 1 - (1/2)\big(a_1 + b_1\big) \right\}$$

and

$$h_{f_1}(b_1) = \frac{1}{1+b_1} \int_{(a_1,\,b_1)} (\,1+x\,)\, dm = \frac{b_1 - a_1}{1+b_1} \left\{\, 1 + (1/2)\big(a_1 + b_1\big) \right\}.$$

A little algebra yields $h_{f_1}(b_1) > h_{f_1}(a_1)$. Lemma 5.2 also indicates that h_{f_2} is monotone increasing on $[-1,\, a_2]$. Therefore, $h(b_1) > h(a_1)$. The conclusion now follows with the help of Theorem 4.1. □

7 The symmetric case

In the last of the three cases, we consider a configuration that is symmetric under reflection in the origin.

Proposition 7.1. *Let $m \in \mathbb{N}$ and $t \in (0,\, 2/3)$. Suppose that $f \in U_t^{(m)}$ satisfies the properties*

(i) $f = Pf$;

(ii) $(\, f,\, \chi_{(-1,\,0)}\,) > 0$;

(iii) $(\, f,\, \chi_{(0,\,1)}\,) = (\, f,\, \chi_{(-1,\,0)}\,)$.

Then there exists $f_1 \in U_t^{(m)}$ with the property that $f_1 = P\, f_1$ such that $J(f) < J(f_1)$.

Before embarking on the proof of Proposition 7.1, we require a number of supplementary results.

Lemma 7.1. *Suppose that $f = \chi_A \in U_t^{(m)}$ for some $m \in \mathbb{N}$ and $t \in (0,\, 2/3)$. Assume that A satisfies condition (3.6). Suppose that $a \in \partial A \cap D$. Then*

$$\lim_{\eta \downarrow 0} \frac{(f\chi_{A_\eta},\, G[f\chi_{A_\eta}])}{(f\chi_{A_\eta},\, \psi)^2} = \psi(a)^{-1},$$

where $A_\eta = [a-\eta,\, a+\eta]$ as before.

Proof. Write

$$G[f\chi_{A_\eta}](x) = \psi(a)\,(f\chi_{A_\eta},\, 1) + \{\, \psi(x) - \psi(a)\,\}\,(f\chi_{A_\eta},\, 1) + r(x)$$

where $r(x) := (\, G(x, \cdot) - \psi(x),\, f\chi_{A_\eta}\,)$ for $x \in D$. Since $\psi(x) = G(x,\, x)$, the estimate (3.2) gives

$$\big|\, (\, f\chi_{A_\eta},\, r)\big| \le 2\,\eta\,(f\chi_{A_\eta},\, 1)^2.$$

Forming the inner product we obtain

$$(f\chi_{A_\eta},\, G[f\chi_{A_\eta}]) = \psi(a)\,(f\chi_{A_\eta},\, 1)^2 + (f\chi_{A_\eta},\, \psi - \psi(a))\,(f\chi_{A_\eta},\, 1) + (\, f\chi_{A_\eta},\, r).$$

It is clear from this that

$$\lim_{\eta \downarrow 0} \frac{(f\chi_{A_\eta},\, G[f\chi_{A_\eta}])}{(f\chi_{A_\eta},\, 1)^2} = \psi(a).$$

A short step leads to the assertion. □

Lemma 7.2. *Suppose that $f = \chi_A \in U_t^{(m)}$ for some $m \in \mathbb{N}$ and $t \in (0,\, 2/3)$ and that A satisfies condition (3.6). Suppose that $a,\, b \in D$ with $a \neq b$ such that both $a \in \partial A$ and $b \in \partial A$. Put $g := 1 - f$. Given $\varepsilon > 0$ sufficiently small there exist unique $\eta > 0$ and $\zeta > 0$ depending upon ε such that $\varepsilon = (f\,\chi_{A_\eta},\, \psi) = (g\,\chi_{B_\zeta},\, \psi)$. Then*

$$\lim_{\varepsilon \downarrow 0} \frac{(f\chi_{A_\eta},\, G[g\chi_{B_\zeta}])}{\varepsilon^2} = \frac{G(a,\, b)}{\psi(a)\,\psi(b)}.$$

Proof. Write

$$G[g\chi_{B_\zeta}](x) = (G(x, \cdot),\, g\chi_{B_\zeta}) = G(a,\, b)\,(g\chi_{B_\zeta},\, 1) + r(x)$$

where $r(x) := (G(x, \cdot) - G(a,\, b),\, g\chi_{B_\zeta})$. For $x \in A_\eta$ and $y \in B_\zeta$,

$$\big|\, G(x,\, y) - G(a,\, b)\,\big| \le \eta + \zeta$$

by (3.2). Consequently,

$$\left|\,(\,f\chi_{A_\eta},\,r)\right| \le \big(\,\eta+\zeta\,\big)\,(f\chi_{A_\eta},\,1)\,(g\chi_{B_\zeta},\,1).$$

Now,

$$(f\chi_{A_\eta},\,G[g\chi_{B_\zeta}]) = G(a,\,b)\,(f\chi_{A_\eta},\,1)\,(g\chi_{B_\zeta},\,1) + (\,f\chi_{A_\eta},\,r),$$

from which we derive

$$\lim_{\varepsilon\downarrow 0}\frac{(f\chi_{A_\eta},\,G[g\chi_{B_\zeta}])}{(f\chi_{A_\eta},\,1)\,(g\chi_{B_\zeta},\,1)} = G(a,\,b),$$

and the conclusion follows straightforwardly.

□

Lemma 7.3. *Let $b \in (0,\,1)$ and $a := -b$. Let $\eta > 0$ small and define $\zeta = \zeta(\eta)$ via the relation*

$$(\chi_{[a,\,a+\eta]},\,\psi) = (\chi_{[b,\,b+\zeta]},\,\psi).$$

Then ζ depends smoothly upon η in a neighbourhood of $\eta = 0$ and

$$\zeta = \eta + \frac{2\,b}{1-b^2}\,\eta^2 + O(\eta^3)$$

in the limit $\eta \downarrow 0$.

Proof. A short computation gives that

$$(\chi_{[a,\,a+\eta]},\,\psi) = (1/2)\left\{\,\big(1-a^2\big)\,\eta - a\,\eta^2 - (1/3)\,\eta^3\,\right\}.$$

Define smooth functions $f,\,g : \mathbb{R}\to\mathbb{R}$ by

$$f(\eta) := \big(1-b^2\big)\,\eta + b\,\eta^2 - (1/3)\,\eta^3 \text{ and } g(\zeta) := \big(1-b^2\big)\,\zeta - b\,\zeta^2 - (1/3)\,\zeta^3.$$

Now $f'(0) = g'(0) = 1-b^2 > 0$. In particular, f is strictly increasing in a neighbourhood of $\eta = 0$ and g possesses a local smooth inverse h in the neighbourhood of $\zeta = 0$ by the inverse function theorem. Note that ζ is characterised by the relation $g(\zeta) = f(\eta)$ for $\eta > 0$ small. Thus $\zeta = (h\circ f)(\eta)$ and depends smoothly upon η for $\eta > 0$ small. Implicit differentiation yields $\zeta'(0) = 1$ and $\zeta''(0) = \frac{4\,b}{1-b^2}$. Taylor's theorem with remainder then yields the expansion. □

Proof of Proposition 7.1. We may suppose that $f = \chi_A$ where A satisfies condition (3.6). Define a as in (5.1). Assume in the first instance that $a \in (-1,\,0)$. Put $b := -a$. Conditions *(i)-(iii)* entail that f is even. In particular, $a,\,b \in \partial A$. As in Lemma 7.2 we write

$$\varepsilon = (f\,\chi_{A_\eta},\,\psi) = (g\,\chi_{B_\zeta},\,\psi)$$

for $\varepsilon > 0$ small. We aim to show that $J(f_\varepsilon) - J(f) > 0$ at least for $\varepsilon > 0$ small as in Theorem 4.1 and we shall borrow notation without comment from its proof. We first claim that (see (4.2))

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\frac{J(f_\varepsilon)-J(f)}{\varepsilon^2} &= \lim_{\varepsilon\downarrow 0}\varepsilon^{-2}\left\{2\,(g\chi_{B_\zeta},\,u) - 2\,(f\chi_{A_\eta},\,u) + (g\chi_{B_\zeta} - f\chi_{A_\eta},\,G\left[\,g\chi_{B_\zeta} - f\chi_{A_\eta}\,\right])\right\}\\ &= \frac{2}{\psi(b)^2}\left\{\,b\,h(b) + u'(b) + b\,(1-b)\,\right\}.\end{aligned} \tag{7.1}$$

As $u \in C^{1,\,1/2}(\overline{D})$ we have that

$$(f\chi_{A_\eta},\,u) = u(a)\,\eta + u'(a)\,(1/2)\eta^2 + O(\eta^{5/2}).$$

Also, with the help of Lemma 7.3,

$$\begin{aligned}(g\,\chi_{B_\zeta},\,u) &= u(b)\,\zeta + u'(b)\,(1/2)\,\zeta^2 + O(\zeta^{5/2})\\ &= u(b)\left\{\eta + \frac{2\,b}{1-b^2}\,\eta^2\right\} + u'(b)\,(1/2)\,\eta^2 + O(\eta^{5/2})\\ &= u(b)\,\eta + \left\{\frac{2\,b}{1-b^2}\,u(b) + \frac{1}{2}u'(b)\right\}\,\eta^2 + O(\eta^{5/2}).\end{aligned}$$

Now $u'(a) = -u'(b)$ because u is even so

$$\lim_{\varepsilon\downarrow 0}\frac{(g\,\chi_{B_\zeta},\,u) - (f\,\chi_{A_\eta},\,u)}{\varepsilon^2} = \lim_{\varepsilon\downarrow 0}\left(\frac{\eta}{\varepsilon}\right)^2\frac{(g\,\chi_{B_\zeta},\,u) - (f\,\chi_{A_\eta},\,u)}{\eta^2} = \frac{1}{\psi(b)^2}\left\{b\,h(b) + u'(b)\right\}.$$

On the other hand, from Lemmas 7.1 and 7.2,

$$\lim_{\varepsilon\downarrow 0}\frac{(g\chi_{B_\zeta} - f\chi_{A_\eta},\,G\left[g\chi_{B_\zeta} - f\chi_{A_\eta}\right])}{\varepsilon^2} = \frac{2}{\psi(a)} - \frac{2\,G(a,\,b)}{\psi(a)\,\psi(b)} = \frac{2\,b\,(1-b)}{\psi(b)^2}$$

as $\psi(b) - G(a,\,b) = b\,(1-b)$. The combination of these identities establishes the claim (7.1).

We now show that the expression in (7.1) is positive. From Lemma 5.2 and the even property of f,

$$h(b) = \frac{1}{1+b}\int_D f\,dm$$

and from Lemma 5.1,

$$u'(b) = u'(1) = -\frac{1}{2}\int_D f\,dm.$$

So we may write

$$\begin{aligned}b\,h(b) + u'(b) + b\,(1-b) &= \left\{\frac{b}{1+b} - \frac{1}{2}\right\}\int_D f\,dm + b\,(1-b)\\ &\geq \left\{\frac{b}{1+b} - \frac{1}{2}\right\}2b + b\,(1-b)\\ &= \frac{b^2(1-b)}{1+b} > 0.\end{aligned}$$

The conclusion now follows by Theorem 4.1.

The case $a = -1$ may be dealt with by applying the above argument (with appropriate modifications) to $g := 1 - f$. □

8 The main result

We are now in a position to prove the main result Theorem 1.3.

Theorem 8.1. *Let $t \in (0,\,1/3]$ and $m \in \mathbb{N}$ and put $A_t := (\xi_t,\,1)$ with ξ_t as in (1.3). Then $\alpha_t^{(m)} = J(f)$ where $f = \chi_{A_t}$.*

Proof. By Theorem 3.1 there exists $f \in U_t^{(m)}$ such that $\beta_t^{(m)} = J(f)$. Now $U_t^{(m)}$ is closed under polarisation. So $P\,f \in U_t^{(m)}$ and $J(f) \leq J(Pf)$ by Theorem **??**. We may assume therefore that $f = Pf$.

Assume that $(f,\,\chi_{(-1,\,0)}) > 0$. Since $f = Pf$ it must be the case that $(f,\,\chi_{(0,\,1)}) \geq (f,\,\chi_{(-1,\,0)})$ (in consequence of Lemma 3.1). By Lemma 6.1 and Proposition 7.1, there exists $f_1 \in U_t^{(m)}$ with the property that $f_1 = Pf_1$ and $J(f) < J(f_1)$. This contradicts the optimality of f. We conclude that $(f,\, \chi_{(-1,\,0)}) = 0$.

If $t = 1/3$ this compels $f = \chi_{(0,\,1)}$ bearing in mind that $f \in U_{1/3}^{(m)}$ and $f = P\,f$. So let us now take $t \in (0,\, 1/3)$. Suppose that $(f,\, \chi_{(0,\,\xi_t)}) > 0$. Then the requirements of Lemma 6.2 are satisfied and hence there exists $f_1 \in U_t^{(m)}$ with the property that $f_1 = P\,f_1$ such that $J(f) < J(f_1)$. Again this contradicts optimality. Hence, $(f,\, \chi_{(0,\,\xi_t)}) = 0$. In fact, $(f,\, \chi_{(-1,\,\xi_t)}) = 0$. As $f \in U_t^{(m)}$ we draw the conclusion that $f = \chi_{A_t}$. □

Corollary 8.1. *Let* $t \in (0,\, 1/3]$. *Then* $\alpha_t = J(f)$ *where* $f = \chi_{A_t}$.

Proof. Let $f \in U_t$. By Lindelöf's theorem, we may write f in the form $f = \chi_A$ where $A = \bigcup_{k=1}^{\infty} A_k$ is a countable union of disjoint open intervals A_k in D. Put $f_n := \sum_{k=1}^{n} \chi_{A_k}$. By the monotone convergence theorem, $J(f) = \lim_{n\to\infty} J(f_n)$. Note that $f_n \in U_t^{(n)}$. By Theorem 8.1, $J(f_n) \le J(\chi_{A_t})$ and $J(f) \le J(\chi_{A_t})$ on taking limits. □

Proof of Theorem 1.3. We only need to deal with the case $t \in (1/3,\, 2/3)$ in view of Corollary 8.1. Let $f \in V_t$ for such a t. Then $g := 1 - f \in V_{2/3-t}$ and

$$J(f) = 2(t - 1/3) + J(g) \le 2(t - 1/3) + J(\chi_{A_{2/3-t}}) = J(1 - \chi_{A_{2/3-t}}) = J(\chi_{A_t}).$$

This clinches the result in the final case. □

9 Application: maximum flux exchange flow

In this section we prove Theorem 1.4.

Proposition 9.1. *It holds that*

(i) $\gamma_\lambda = 2\,\alpha_{\frac{1-\lambda}{3}} - (1/3)(1-\lambda)^2$ *for* $\lambda \in (-1,\, 1)$*;*

(ii) $\gamma = \sup_{\lambda\in(-1,\,1)} \gamma_\lambda$.

Proof. (i). Fix $\lambda \in (-1,\, 1)$. Let A be an open subset in D. Suppose that u satisfies (1.4) along with the flux-balance condition $(u,\, 1) = 0$. Put

$$f = f_{A,\lambda} := \begin{cases} -(\lambda+1) & \text{on} \quad A, \\ -(\lambda-1) & \text{on} \quad D \setminus A, \end{cases}$$

Then $u = Gf$. From the flux-balance condition and symmetry of the Green operator,

$$0 = (1,\, G\,f) = (\psi,\, f) = -(\lambda+1)(\psi,\, \chi_A) - (\lambda-1)(\psi,\, \chi_{D\setminus A});$$

so that $\lambda = (\psi, 1)^{-1}(\psi, \chi_{D\setminus A} - \chi_A)$ and $(\psi,\, \chi_A) = \frac{1-\lambda}{2}(\psi,\, 1)$. Moreover,

$$\begin{aligned}
(\chi_{D\setminus A},\, u) &= (G\,\chi_{D\setminus A},\, f) \\
&= -(\lambda+1)(G\,\chi_{D\setminus A},\, \chi_A) - (\lambda-1)(G\,\chi_{D\setminus A},\, \chi_{D\setminus A}) \\
&= 2(\psi, 1)^{-1}\Big\{ -(\psi,\, \chi_{D\setminus A})(G\,\chi_{D\setminus A},\, \chi_A) + (\psi,\, \chi_A)(G\,\chi_{D\setminus A},\, \chi_{D\setminus A}) \Big\} \\
&= 2(\psi,\, 1)^{-1}\Big\{ (\psi,\, 1)(G\,\chi_A,\, \chi_A) - (\psi,\, \chi_A)^2 \Big\} \\
&= 2\,J(\chi_A) - (1/2)(1-\lambda)^2(\psi,\, 1),
\end{aligned}$$

upon writing $\chi_{D\setminus A} = \chi_D - \chi_A$ on each occurrence in the penultimate line. Now simplify using (3.3). These considerations lead to the reformulation *(i)*. The statement in *(ii)* then follows immediately. □

Proof of Theorem 1.4. Part *(i)* follows from Theorem 1.3. For $t \in (0, 2/3)$,

$$\alpha_t = 2(t - 1/3) + \alpha_{2/3-t},$$

as can be seen from the proof of Theorem 1.3. Therefore, for $\lambda \in (-1, 1)$,

$$\gamma_{-\lambda} = 2\,\alpha_{\frac{1+\lambda}{3}} - (1/3)(1+\lambda)^2 = 2\,\alpha_{\frac{1-\lambda}{3}} - (1/3)(1-\lambda)^2 = \gamma_\lambda.$$

We now show that $\gamma_\lambda < \gamma_0$ for each $\lambda \in (0, 1)$.

Let $\xi \in (-1, 1)$ and $u := G\,f$ where $f := \chi_{(\xi,\,1)}$. Then

$$u(x) = \begin{cases} \frac{1}{4}(\xi-1)^2(x+1) & \text{if} \quad -1 < x \le \xi, \\ -\frac{1}{2}x^2 + \frac{1}{4}(\xi+1)^2\,x + \frac{1}{2} - \frac{1}{4}(\xi+1)^2 & \text{if} \quad \xi \le x < 1. \end{cases}$$

A computation leads to

$$J(f) = -\frac{1}{8}\,\xi^4 + \frac{1}{6}\,\xi^3 + \frac{1}{4}\,\xi^2 - \frac{1}{2}\,\xi + \frac{5}{24}.$$

Also (see (1.3)),

$$\frac{1-\lambda}{3} = (\psi,\, f) = \varphi(\xi) = \frac{1}{6}\left\{ 2 - 3\,\xi + \xi^3 \right\}.$$

Therefore,

$$\begin{aligned} \gamma_\lambda &= 2\,\alpha_{\frac{1-\lambda}{3}} - (1/3)(1-\lambda)^2 \\ &= 2\,J(f) - 3\,\varphi(\xi)^2 \\ &= \frac{1}{12} - \frac{1}{4}\,\xi^2 + \frac{1}{4}\,\xi^4 - \frac{1}{12}\,\xi^6 \\ &=: h(\xi). \end{aligned}$$

Now $h(0) = \frac{1}{12}$ and $h(1) = 0$ and $h'(\xi) = -(1/2)\,\xi\,(1-\xi^2)^2 < 0$ for $\xi \in (0, 1)$. This shows that $\gamma_\lambda < \gamma_0$ for each $\lambda \in (0, 1)$ as desired. The result follows from this and *(i)* of the Theorem. □

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