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Modular Analytical Solutions for Foundation Damping in Soil-Structure Interaction Applications

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Foundation damping incorporates combined effects of energy loss from waves propagating away from a vibrating foundation (radiation damping) and hysteretic action in supporting soil (material damping). Foundation damping appears in analysis and design guidelines for force- and displacement-based analysis of seismic building response (ASCE-7, ASCE-41), typically in graphical form (without predictive equations). We derive closed-form expressions for foundation damping of a flexible-based single degree-of-freedom oscillator from first principles. The expressions are modular in that structure and foundation stiffness terms, along with radiation and hysteretic damping ratios, appear as variables. Assumptions inherent to our derivation have been employed previously, but the present results are differentiated by: (1) the modular nature of the expressions; (2) clearly articulated differences regarding alternate bases for the derivations and their effects on computed damping; and (3) completeness of the derivations. Resulting expressions indicate well-known dependencies of foundation damping on soil-to-structure stiffness ratio, structure aspect ratio, and soil damping. We recommend a preferred expression based on the relative rigor of its derivation.

INTRODUCTION

Following early work by Parmelee (1967), foundation damping as a distinct component of structural system damping was introduced as part of Bielak's (1971) derivation of the replacement (flexible-base) single-degree-of-freedom (SDOF) system and was later refined by Veletsos and Nair (1975), Roesset (1980) and others. The work was predicated on the need to evaluate the effects of soil-structure interaction (SSI) on the seismic response of nuclear power

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plant reactor containment structures, later extended to buildings and similar systems with more significant higher mode effects (e.g., Crouse and McGuire, 2001). Based on that need, and following work by Luco and Westmann (1971), alternative sets of equations were developed to predict foundation damping of a rigid circular foundation resting on a uniform elastic halfspace.

Due in part to the convenience of its application in evaluating seismic demands, foundation damping appears in several seismic design guidelines for buildings (e.g., ASCE-7; ASCE-41; NIST, 2012). In both force-based procedures (ASCE-7) and displacement-based procedures (ASCE-41), foundation damping affects the damping ratio used to compute ordinates on the pseudo-spectral acceleration spectrum representing seismic demands. Early versions of ASCE-7 and ASCE-41 utilized graphical solutions for foundation damping that required specific assumptions of foundation geometry; this has been replaced with equation-based methods that appeared first in NIST (2012). We developed the expressions in the NIST report, which are derived, extended, and more fully explained only in this manuscript. Further details on the effects of foundation damping within seismic design guidelines are given in NIST (2012) and Appendix A of this article.

The principal objective of this paper is to present derivations of foundation damping based on alternative approaches for matching the response of a SDOF equivalent fixed-base oscillator with that of an oscillator founded on a compliant medium. Unlike some prior models, our equations are not specific to particular impedance function equations, but rather are *modular* in the sense that any appropriate set of impedance functions (analytically or numerically-derived) can be utilized (Modularity in this work implies a system constructed of standard components). Our results show some differences from classical solutions, which we re-derive using the underlying assumptions inherent to those models and express in a similarly modular form. Important distinctions between the current and prior models are related to the present use of a generalized damping formulation that allows for both hysteretic and viscous components, increased transparency regarding assumptions made in the derivation and their effects, and the aforementioned modular nature of the resulting equations. This modularity allows the functions to be readily adapted for various practical conditions not considered in classical solutions such as arbitrary foundation shapes, embedded foundations, and non-uniform soil conditions.

Following this introduction, we: present notation related to impedance and oscillators that are used to develop the theory; derive modular equations for foundation damping based on alternate approaches for the matching of flexible-base oscillator response to an equivalent fixed-base response; and compare our results to solutions derived in accordance with classical models from the literature. We conclude with an example and recommendations on the use of the derived equations in engineering practice. Two appendices to this manuscript explain the use of foundation damping in seismic design guidelines and describe the significance of using a generalized damping formulation (as employed here) versus perfectly viscous damping (employed in classical solutions).

PROBLEM DEFINITION AND NOTATION

The concept of foundation damping arises from the analogy of a SDOF oscillator of mass m, height h, stiffness $\tilde{k}$, period $\tilde{T}$, and adjusted damping ratio β_0 (Figure 1a), which replaces an otherwise similar oscillator with structural stiffness $k = m\left(2\pi/T\right)^2$ and damping β_i that is supported by translational and rotational springs (Figure 1b). Period T and damping β_i are oscillator properties for fixed-base conditions in which the base springs have infinite stiffness.

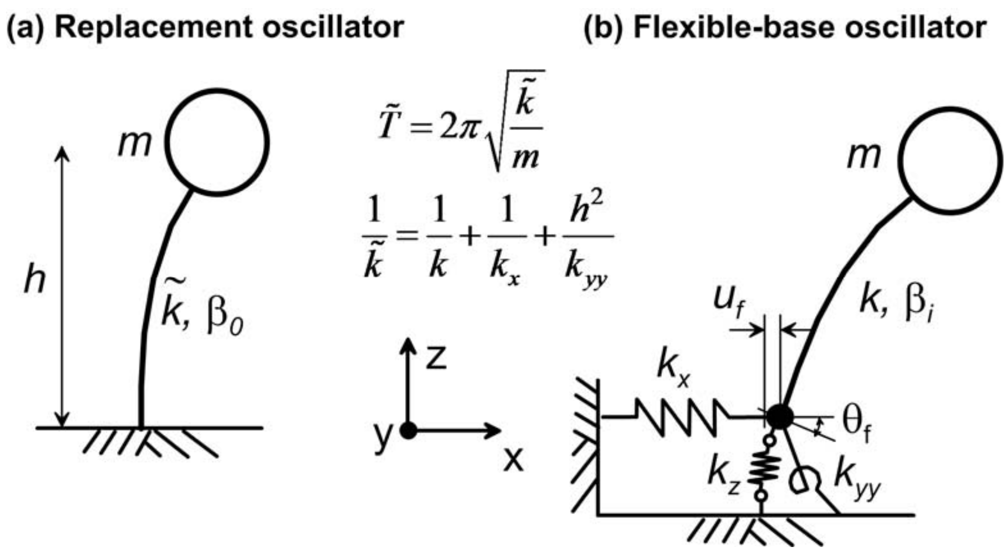

Figure 1. (a) Replacement oscillator used to represent flexible-base system, having stiffness $\tilde{k}$. (b) Flexible-base system with horizontal, vertical, and rotational foundation springs (k_x, k_z, and k_{yy}, respectively) having deflections of u_f (horizontal) and θ_f (rotation) – the structural elements have stiffness and damping of k and β_i, respectively. Both systems have identical fundamental-mode lateral periods of $\tilde{T}$ and damping of β_0.

The distinction between fixed- and flexible-base oscillator properties are evaluated from period lengthening ($\tilde{T}/T$) and foundation damping (β_f) as follows (Veletsos and Meek, 1974):

$$\frac{\tilde{T}}{T} = \sqrt{1 + \frac{k}{k_x} + \frac{kh^2}{k_{yy}}}, \tag{1}$$

$$\beta_f = \beta_o - \frac{1}{\left(\tilde{T}/T\right)^n}\beta_i \tag{2}$$

where k_x and k_{yy} represent foundation spring stiffnesses for horizontal translation and rotational vibration modes, and n is an exponent that should be taken as 2 when the damping is of a general frequency-dependent form and not necessarily perfectly viscous (details in Appendix B). We assume throughout this paper that horizontal translation is along the x-axis (x) and rotation is in the x-z plane, as indicated by the axes in Figure 1.

Whereas the analysis of period lengthening is relatively straightforward (Eq. 1), formulating an analytical solution for foundation damping is more complex, as it requires assessing the relative contributions of hysteretic and radiation damping in multiple modes of foundation vibration. Essential to this process is the parameterization of foundation stiffness and damping using complex-valued impedance functions ($\bar{k}_j$), for which we adopt the following notation (consistent with NIST, 2012):

$$\bar{k}_j = k_j + i\omega c_j \tag{3}$$

where k_j=frequency-dependent foundation stiffness, c_j=dashpot coefficient, ω=circular frequency (rad/s), and subscript j denotes either the translational (x) or rotational (yy) vibration mode. Imaginary unit i indicates a 90 degree phase difference between the viscous component (ωc_j) and the elastic one (k_j) (the same applies to forces derived from complex stiffnesses).

An alternative form for Eq. (3) is

$$\bar{k}_j = k_j\left(1 + 2i\beta_j\right) \tag{4}$$

where

$$\beta_j = \frac{\omega\, c_j}{2k_j} \text{ (defined for } k_j \neq 0\text{)} \tag{5}$$

Dimensionless number β_j can be interpreted as a percentage of critical damping in the classical sense at resonance of the system in Figure 1 (Clough and Penzien, 1993). Stiffness coefficient (k_j) is a function of the soil shear modulus (G), Poisson's ratio (ν), dynamic stiffness modifier (α_j) and foundation dimensions:

$$k_j = K_j \cdot \alpha_j \tag{6}$$

$$K_j = GB^m f_1\left(B/L, \nu\right) \tag{7}$$

$$\alpha_j = f_2(a_0, B/L) \tag{8}$$

$$a_0 = \frac{\omega B}{V_s} \tag{9}$$

where K_j is the static foundation stiffness at zero frequency for mode j, a_0 is dimensionless frequency, exponent m is 1 for translation (x) and 3 for rotation (yy), and B and L are foundation plan half-dimensions, as indicated in Figure 2. The aforementioned equations are described for rectangular foundations; the notation for circular foundations is identical except that radius r is substituted for half-width B in Eqs. (7) and (9) and $B/L = 1$.

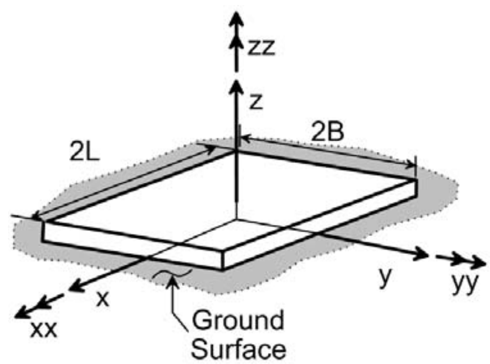

Figure 2. Geometry of rectangular foundations adopted for impedance function equations ($L \geq B$).

Approximate impedance equations for rigid circular foundations resting on a visco-elastic halfspace were presented by Veletsos and Verbic (1973), which were based on earlier solutions by Veletsos and Wei (1971). Solutions for rectangular foundations by Pais and Kausel (1988) and Mylonakis et al (2006) form the basis for recommendations presented in Tables 2.2 to 2.3 in NIST (2012). The modular nature of the foundation damping solutions in this paper allow these or any other impedance solutions to be used to represent soil-foundation interaction.

The formulation of the foundation impedance in Eq. (3) to (8) does not explicitly include coupling terms between translational and rotational vibration modes, which are important for embedded foundations (e.g., Assimaki and Gazetas, 2009). However, the present formulation without coupling terms can be readily adapted to embedded foundations through the use of an eccentricity (computed from the ratio of coupling/translational stiffness) that is added to the structural height, as described by Maravas et al. (2014).

FOUNDATION DAMPING DERIVATIONS

As shown in Figure 1, the stiffness of the replacement oscillator $\tilde{k}$ can be related to the stiffness of the individual components of the SSI system as:

$$\frac{1}{\tilde{k}} = \frac{1}{k} + \frac{1}{k_x} + \frac{h^2}{k_{yy}} \tag{10}$$

In this section we present derivations for foundation damping that begin with a more general form of Eq. (10), in which each term is generalized for dynamic loading by introducing complex-valued stiffnesses (indicated by an overbar) as follows:

$$\frac{1}{\bar{\tilde{k}}} = \frac{1}{\bar{k}} + \frac{1}{\bar{k}_x} + \frac{h^2}{\bar{k}_{yy}} \tag{11}$$

We present two approaches for using Eq. (11) to derive expressions for foundation damping. The first approach, which is similar in some respects to prior work by Bielak (1971), Roesset (1980), and Wolf (1985), separates Eq. (11) into its real and complex parts, then operates exclusively on the imaginary part to evaluate the effective damping of the replacement oscillator. The foundation damping is then readily derived from the system damping. The second approach, which is similar in some respect to prior work by Veletsos and Nair (1975) and Maravas et al. (2014), retains both the real and complex parts of Eq. (11) in the evaluation of the dynamic properties of the replacement oscillator. Subsequent sections describe differences between foundation damping derived from the two approaches and compare the present solutions to prior results.

DERIVATION FROM IMAGINARY COMPONENT

The first approach proceeds from Eq. (11) by expanding each complex stiffness term according to Eq. (4):

$$\frac{1}{\tilde{k}\left(1+2i\beta_0\right)} = \frac{1}{k\left(1+2i\beta_i\right)} + \frac{1}{k_x\left(1+2i\beta_x\right)} + \frac{h^2}{k_{yy}\left(1+2i\beta_{yy}\right)} \tag{12}$$

Note that hysteretic soil damping effects are not considered at this stage, but are accounted for subsequently. Multiplying and dividing each term by its complex conjugate, neglecting the higher-order damping terms (i.e., $\beta^2 \sim 0$), and multiplying both sides by *k*, we obtain:

$$\frac{k}{\tilde{k}}\left(1-2i\beta_0\right)=\left(1-2i\beta_i\right)+\frac{k}{k_x}\left(1-2i\beta_x\right)+\frac{kh^2}{k_{yy}}\left(1-2i\beta_{yy}\right) \tag{13}$$

The equality in Eq. (13) requires that both the real and imaginary parts of the expressions on the right and left sides of the equal sign be equal. In this section, we consider the equality of the imaginary parts as follows:

$$\beta_0=\frac{\tilde{k}}{k}\beta_i+\frac{\tilde{k}}{k_x}\beta_x+\frac{\tilde{k}h^2}{k_{yy}}\beta_{yy}=\frac{\tilde{k}}{k}\beta_i+\frac{\tilde{k}}{k}\left(\frac{k}{k_x}\beta_x+\frac{kh^2}{k_{yy}}\beta_{yy}\right) \tag{14}$$

Eq. (14) is convenient because the flexible-base system damping components are proportional to the stiffness ratio for flexible-base and fixed-base oscillators ($\tilde{k}/k$), which can be related to the period lengthening (Eq. 1) as follows when foundation mass and rotational moments of inertia in the foundation and superstructure are ignored:

$$\frac{\tilde{k}}{k}=\frac{\tilde{k}}{m}\cdot\frac{m}{k}=\frac{\tilde{\omega}_n^2}{\omega_n^2}=\frac{1}{\left(\tilde{T}/T\right)^2} \tag{15}$$

Using Eq. (15), the flexible-base system damping in Eq. (14) can be presented as a function of period lengthening:

$$\beta_0=\frac{1}{\left(\tilde{T}/T\right)^2}\beta_i+\frac{1}{\left(\tilde{T}/T\right)^2}\left(\frac{k}{k_x}\beta_x+\frac{kh^2}{k_{yy}}\beta_{yy}\right) \tag{16}$$

Eq. (16) was developed based on the general impedance functions (Eq. 3-4) that consider the damping terms to be non-viscous. Frequency-independent (i.e., hysteretic) soil damping (β_s) can be included in the system by simply adding it to the translational and rotational damping terms (Roesset, 1980, and Wolf, 1985). When applied to the damping formulation in Eq. (16), and upon re-arrangement, we obtain:

$$\beta_0=\frac{1}{\left(\tilde{T}/T\right)^2}\left[\beta_i+\frac{k}{k_x}\beta_x+\frac{kh^2}{k_{yy}}\beta_{yy}+\left(\frac{k}{k_x}+\frac{kh^2}{k_{yy}}\right)\beta_s\right] \tag{17}$$

We remove the stiffness ratios before the β_x and β_{yy} terms by introducing fictitious vibration periods for foundation vibration (these would represent actual system period if the superstructure were rigid and the respective foundation vibrations were the only available degree of freedom of a fictitious SDOF system):

$$T_x = 2\pi\sqrt{\frac{m}{k_x}} \quad T_{yy} = 2\pi\sqrt{\frac{mh^2}{k_{yy}}} \tag{18}$$

We remove the stiffness ratio before β_s by recognizing from Eq. (1) that it is equivalent to $\left(\tilde{T}/T\right)^2 - 1$. Moreover, using Eq. (18) and recalling $T = 2\pi\sqrt{m/k}$, term $k/k_x=(T_x/T)^2$ and $kh^2/k_{yy}=(T_{yy}/T)^2$, Eq. (17) can be written as:

$$\beta_0 = \frac{1}{\left(\tilde{T}/T\right)^2}\beta_i + \left[1 - \frac{1}{\left(\tilde{T}/T\right)^2}\right]\beta_s + \frac{1}{\left(\tilde{T}/T_x\right)^2}\beta_x + \frac{1}{\left(\tilde{T}/T_{yy}\right)^2}\beta_{yy} \tag{19}$$

Per Eq. (2) with exponent n=2, the foundation damping becomes

$$\beta_f = \left[1 - \frac{1}{\left(\tilde{T}/T\right)^2}\right]\beta_s + \frac{1}{\left(\tilde{T}/T_x\right)^2}\beta_x + \frac{1}{\left(\tilde{T}/T_{yy}\right)^2}\beta_{yy} \tag{20}$$

The advantage of Eqs. (19) and (20) over earlier formulations lies in the nature of the dimensionless multipliers of damping terms, which can be interpreted as weight factors. The sum of the two factors multiplying β_i and β_s terms is unity. Eq. (20) was developed by the authors for NIST (2012), although the derivation appears here for the first time. Previous solutions developed using a comparable set of assumptions to those applied here are described further in subsections below detailing the approaches of Bielak (1971), Roesset (1980), and Wolf (1985).

DERIVATION FROM COMPLEX-VALUED IMPEDANCE EXPRESSIONS

Our second derivation of foundation damping retains the complex-valued form of Eq. (11), but enforces equality of both the real and imaginary parts. Equality of the real-valued terms is given in Eq. (10). We re-arrange Eqs. (10-11) to isolate foundation stiffnesses on the right side as:

$$\frac{1}{\tilde{k}} - \frac{1}{k} \simeq \frac{1}{k_x} + \frac{h^2}{k_{yy}} \tag{21}$$

$$\frac{1}{\tilde{\bar{k}}} - \frac{1}{\bar{k}} = \frac{1}{\bar{k}_x} + \frac{h^2}{\bar{k}_{yy}} \tag{22}$$

Note that the second equation is exact, while the first is approximate for conditions other than static, since higher-order damping terms have been neglected. The left-side of Eq. (22) can be expanded to include the real and imaginary terms (per Eq. 4), and then multiplied by the complex conjugate, to produce:

$$\frac{1}{\tilde{\bar{k}}}-\frac{1}{\bar{k}}=\frac{1-2i\beta_0}{\tilde{k}}-\frac{1-2i\beta_i}{k} \tag{23}$$

in which higher-order damping terms have been omitted. Eq. (23) is re-written by isolating the complex terms on the right side and using the relations in Eq. (21-22) as:

$$-2i\left(\frac{\beta_0}{\tilde{k}}-\frac{\beta_i}{k}\right)=\left(\frac{1}{\bar{k}_x}-\frac{1}{k_x}\right)+\left(\frac{1}{\bar{k}_{yy}}-\frac{1}{k_{yy}}\right)h^2 \tag{24}$$

Reducing the right side of Eq. (24) for common denominators provides:

$$\left(\frac{1}{\bar{k}_x}-\frac{1}{k_x}\right)+\left(\frac{1}{\bar{k}_{yy}}-\frac{1}{k_{yy}}\right)h^2=\left(\frac{k_x-\bar{k}_x}{\bar{k}_x k_x}\right)+\left(\frac{k_{yy}-\bar{k}_{yy}}{\bar{k}_{yy}k_{yy}}\right)h^2 \tag{25}$$

The right side of Eq. (25) can be re-written by expanding the complex-valued impedance terms in the numerator to include their real and complex parts per Eq. (4) as:

$$\begin{aligned}&\left(\frac{k_x-\bar{k}_x}{\bar{k}_x k_x}\right)+\left(\frac{k_{yy}-\bar{k}_{yy}}{\bar{k}_{yy}k_{yy}}\right)h^2=\\&\left(\frac{k_x-k_x\left(1+2i\beta_x\right)}{\bar{k}_x k_x}\right)+\left(\frac{k_{yy}-k_{yy}\left(1+2i\beta_{yy}\right)}{\bar{k}_{yy}k_{yy}}\right)h^2=\frac{-2i\beta_x}{\bar{k}_x}+\frac{-2i\beta_{yy}h^2}{\bar{k}_{yy}}\end{aligned} \tag{26}$$

Equating the left side of Eq. (24) to the right side of Eq. (26) produces:

$$-2i\left(\frac{\beta_0}{\tilde{k}}-\frac{\beta_i}{k}\right)=-2i\left(\frac{\beta_x}{\bar{k}_x}+\frac{\beta_{yy}h^2}{\bar{k}_{yy}}\right) \tag{27}$$

Dividing through by -2*i*, multiplying through by $\tilde{k}$, and moving the β_i term to the right side produces:

$$\beta_0=\frac{\tilde{k}}{k}\beta_i+\frac{\tilde{k}}{k}\left(\frac{k}{\bar{k}_x}\beta_x+\frac{kh^2}{\bar{k}_{yy}}\beta_{yy}\right) \tag{28}$$

We note that Eq. (28) is impossible, because by definition β_0 is real-valued while the right-hand side is complex-valued. As shown subsequently, this is a result of having ignored β^2

damping terms in various places. We explain below alternative ways of overcoming this problem.

Recalling the relationship between $\tilde{k}/k$ and $\left(\tilde{T}/T\right)^2$ in Eq. (15), we obtain:

$$\beta_0 = \frac{1}{\left(\tilde{T}/T\right)^2}\beta_i + \frac{1}{\left(\tilde{T}/T\right)^2}\left(\frac{k}{\bar{k}_x}\beta_x + \frac{kh^2}{\bar{k}_{yy}}\beta_{yy}\right) \tag{29}$$

Eq. (29) matches Eq. (16) with the exception of the horizontal and rotational impedance terms within the brackets having gone from real- to complex-valued. As before, we introduce soil hysteretic damping at this stage to obtain:

$$\beta_0 = \frac{1}{\left(\tilde{T}/T\right)^2}\beta_i + \frac{1}{\left(\tilde{T}/T\right)^2}\left(\frac{k}{\bar{k}_x}\left(\beta_x + \beta_s\right) + \frac{kh^2}{\bar{k}_{yy}}\left(\beta_{yy} + \beta_s\right)\right) \tag{30}$$

As before, soil hysteretic damping can be approximately accounted for by adding β_s to the respective radiation damping ratios β_x and β_{yy}. This addition should also be performed within the imaginary term of the complex-valued impedance functions as follows:

$$\bar{k}_j = k_j\left[1 + 2i\left(\beta_j + \beta_s\right)\right] \tag{31}$$

We introduce fictitious vibration periods, now complex-valued, which can be understood as phase differences in damped natural periods for the hypothetical fixed superstructure (Veletsos and Ventura, 1986):

$$\bar{T}_x = 2\pi\sqrt{\frac{m}{\bar{k}_x}} \quad \bar{T}_{yy} = 2\pi\sqrt{\frac{m}{\bar{k}_{yy}}} \tag{32}$$

In an analogous manner to the previous section (below Eq. 18), the stiffness ratios on the right side of Eqs. (29-30) can be written as $k/\bar{k}_x = \left(\bar{T}_x/T\right)^2$ and $kh^2/\bar{k}_{yy} = \left(\bar{T}_{yy}/T\right)^2$. With these substitutions, Eq. (30) becomes:

$$\beta_0 = \frac{1}{\left(\tilde{T}/T\right)^2}\beta_i + \frac{1}{\left(\tilde{T}/\bar{T}_x\right)^2}\left(\beta_x + \beta_s\right) + \frac{1}{\left(\tilde{T}/\bar{T}_{yy}\right)^2}\left(\beta_{yy} + \beta_s\right) \tag{33}$$

To avoid the use of complex numbers, and to balance the error associated with ignoring β^2 terms, we replace $\bar{k}_j$ with its amplitude in Eq. (32), with the resulting periods denoted $\left|\bar{T}_x\right|$ and

$\left|\bar{T}_{yy}\right|$, which are real-valued. A general expression for foundation damping can then be written as:

$$\beta_f = \frac{1}{\left(\tilde{T}/\left|\bar{T}_x\right|\right)^2}\left(\beta_x + \beta_s\right) + \frac{1}{\left(\tilde{T}/\left|\bar{T}_{yy}\right|\right)^2}\left(\beta_{yy} + \beta_s\right) \tag{34}$$

Other ways of avoiding the use of complex numbers are to take the absolute value of the right-side of Eq. (33) or of the right-side minus the β_i term. We have investigated these options and found no significant difference; the approach in Eq. (34) is adopted due to its ease of application (minimizing the manipulation of complex numbers in the calculations). In a subsection below entitled *Veletsos and Nair (1975) Solution*, we show that their solution was developed using a similar set of assumptions. From results presented thus far, it is clear that when higher-order damping terms are omitted, there is no unique solution for foundation damping.

COMPARISON OF ALTERNATE SOLUTIONS FOR FOUNDATION DAMPING

On theoretical grounds, there is no clear preference for one of the aforementioned foundation damping solutions over the other (both were derived using certain approximations, so neither is exact). The two expressions for foundation damping are given in Eqs. (20) and (34). A practical benefit of the first solution is that it is expressed entirely in terms of real-valued variables, whereas the second includes complex variables that produce a complex-valued foundation damping that is difficult to understand.

In Figure 3, we plot foundation damping derived from the two solutions against the ratio $h/(V_S T)$ (h and T are height and fixed-base period of SDOF structure in Figure 1, V_S is soil shear-wave velocity), which is often called the wave parameter (Veletsos, 1977). The wave parameter represents a structure-to-soil stiffness ratio, because h/T quantifies the stiffness of a structure's lateral force resisting system in velocity units whereas V_S is related to the soil shear stiffness. For nonlinear problems, the value of V_S should be reduced in an equivalent-linear sense (details in NIST, 2012). In Figure 3, foundation damping solutions are given for square foundations and various structure height aspect ratios (h/B) for the case of radiation damping only (Figure 3a) and $\beta_s = 0.1$ (Figure 3b). Pais and Kausel (1988) fitted impedance functions were used in the calculations. The calculations were performed using a ratio of structure mass to mass of soil in the volume $4B^2h$ of 0.15 (which is a typical value, per Veletsos 1977).

The solution from the first approach produces higher damping, particularly for small height aspect ratios. These damping differences result from the dropping of β^2 terms in the derivations, the effects of which differ somewhat due to the different assumptions in the two derivations. Otherwise, the solutions show well-known patterns of behavior, in particular:

- As $h/(V_sT)$ increases, the significance of inertial SSI increases, causing increased foundation damping;
- As h/B increases, rotational modes of foundation vibration become more dominant, which reduces foundation damping because foundation rotation produces less radiation damping than foundation translation;
- The effects of hysteretic soil damping scale with the significance of inertial SSI, as measured for example by $h/(V_sT)$. For low $h/(V_sT)$, hysteretic damping has little effect (zero at $h/(V_sT) = 0$), whereas at high $h/(V_sT)$ the foundation damping is nearly the sum of foundation damping from radiation damping and β_s.

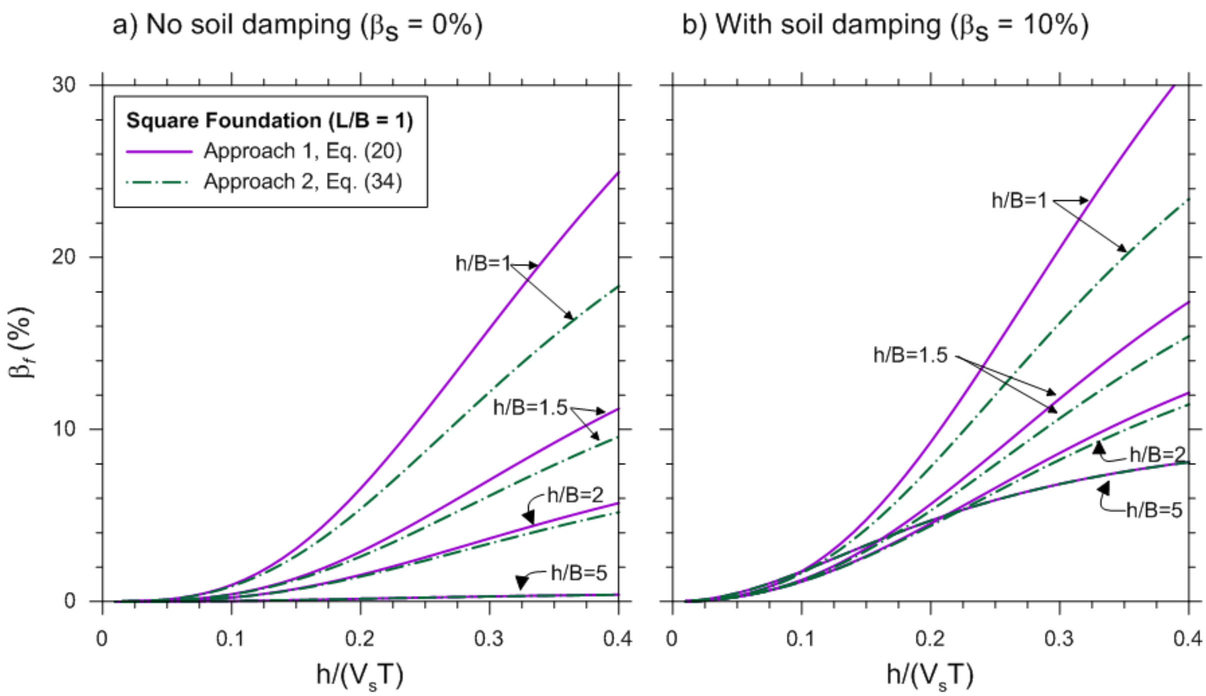

Figure 3. Comparison of foundation damping solutions based on Approaches 1 and 2, plotted against structure-to-soil stiffness ratio $h/(V_ST)$ [$h/(V_ST) = 0$ to 0.4 encompasses the range of practical conditions typically encountered for building structures; Stewart et al., 1999]. Conditions for the plots are a rigid, massless, square foundation supported on an homogeneous isotropic halfspace with Poisson's ratio ν = 0.33 and hysteretic soil damping of (a) $\beta_s = 0\%$ and (b) $\beta_s = 10\%$. Impedance functions from Pais and Kausel (1988) used to derive the foundation damping. Per text, structure to soil mass ratio is 0.15.

COMPARISON TO FOUNDATION DAMPING SOLUTIONS IN LITERATURE

In this section we compare results from the expressions derived above to previous solutions for foundation damping of circular foundations by Veletsos and Nair (1975), Bielak (1971), Roesset (1980), Wolf (1985), and Maravas et al. (2014). The original expressions for circular foundations are re-derived here in a more general form.

BIELAK (1971) SOLUTION

Bielak (1971) (also Jennings and Bielak 1973 and Bielak 1975) derived an expression for foundation damping by identifying the dynamic properties of a replacement fixed-based oscillator to match those of a flexible-base oscillator (Figure 1b). In the derivation, the foundation mass and mass moment of inertia were taken as negligible (as above), the structural damping was taken as viscous, impedance functions for circular foundations were used, and higher-order damping terms were neglected. The viscous damping assumption for the structure was motivated by computational efficiency.

Bielak's damping derivation is similar to the approach presented in the section entitled *Derivation from Imaginary Component* in which the imaginary parts of stiffness terms in the replacement oscillator and flexible-base system are equated. The present derivation mirrors the prior one up to Eq. (16). However, the assumption of viscous structural damping requires modification of the structural damping ratio as described in Appendix B (Eq. B.5-B.6). With the substitution of β_i^{vis} for β_i in Eq. (16), we obtain:

$$\beta_0 = \frac{1}{\left(\tilde{T}/T\right)^3}\beta_i^{vis} + \frac{1}{\left(\tilde{T}/T\right)^2}\left(\frac{k}{k_x}\beta_x + \frac{kh^2}{k_{yy}}\beta_{yy}\right) \tag{35}$$

Note that the period lengthening term before the viscous structural damping now has an exponent of 3. Eq. (35) matches Eq. (3.66c) in Bielak (1971), except the nomenclature has been adapted to be consistent with this paper and periods instead of frequencies are used. Per Eq. (2), foundation damping β_f is the second term in the sum in Eq. (35). To the extent that actual structural damping is not purely viscous, the expression in Eq. (35) is an approximation to the actual flexible-base system damping.

VELETSOS AND NAIR (1975) SOLUTION

Veletsos and Nair (1975) derived an expression for foundation damping by equating amplitudes of responses between the real parts of the flexible-base system and those of the replacement oscillator. Their derivation utilized the full complex form of stiffness terms in equating the stiffness of the replacement oscillator to that of the flexible-base oscillator, which is similar to the second approach described above in the section entitled *Derivation from Complex-Valued Impedance Expressions*. Assumptions made regarding the properties of the replacement oscillator include viscous damping and the presence or absence of foundation mass. The Veletsos and Nair damping terms can be derived using a process matching that used in Approach 2 with two exceptions.

The first exception is that Veletsos and Nair (1975) used viscous structural damping. As in the previous section, in the equation of system damping (e.g. Eq. 35), this converts to 3 the exponent on the period lengthening applied to structural damping.

The second exception concerns the incorporation of hysteretic soil damping into the solution. The simple addition of β_s to radiation damping terms applied in the derivations for Approaches 1 and 2 represents an approximate solution to the mathematically complex problem of how these damping terms interact. For example, a numerical solution (integral equation approach) to this problem is provided by Apsel and Luco (1987). Veletsos and Nair (1975) use an approximate solution to this problem by Veletsos and Verbic (1973) (the approximation is in the fitting of the dynamic impedance coefficients for the case of zero soil damping with simple closed-form expressions, along with an assumption of real-valued Poisson's ratio). In the Veletsos and Verbic solution, the soil damping appears as a term in a series of equations used to derive dynamic modifiers in the general equations for foundation impedance (α_j and β_j in Eq. 6 and 5, respectively).

These two deviations have little impact on the damping solution from Approach 2, and the derived system damping, given below, is very similar to that in Eq. (29):

$$\beta_0 = \frac{1}{\left(\tilde{T}/T\right)^3}\beta_i^{vis} + \frac{1}{\left(\tilde{T}/T\right)^2}\left|\left(\frac{k}{\bar{k}_x}\beta_x + \frac{kh^2}{\bar{k}_{yy}}\beta_{yy}\right)\right| \tag{36}$$

The only differences between Eq. (36) and Eq. (29) are in the structural damping terms (due to the use of viscous damping), the form of impedance function terms $\bar{k}_x$ and $\bar{k}_{yy}$ (which Veletsos and Nair wrote for circular foundations), and in application of absolute values on the right side to avoid complex-valued foundation damping (matching the approach of Veletsos and Nair). Note that Eq. (36) is used with or without soil damping; as discussed in the section entitled

Derivation from Imaginary Component, the effect of β_s can be introduced by simple addition to the radiation damping terms that are also contained within the complex-valued impedance terms.

In Figure 4 we show the effect of the different approaches for incorporating β_s into the solution. As a baseline case, we show the predicted foundation damping for β_s = 0.1 using Approach 2 (Eq. 34). In this calculation, we use radiation damping terms computed from closed-form expressions by Veletsos and Verbic (1973) for the elastic medium (i.e., radiation damping only, or β_s = 0). Also shown in Figure 4 is foundation damping from Eq. (36) (second term to the right of equal sign), using the same Veletsos and Verbic (1973) impedance solution. The two sets of results are similar, diverging only slightly as $h/(V_S T)$ increases for $h/r < 5$.

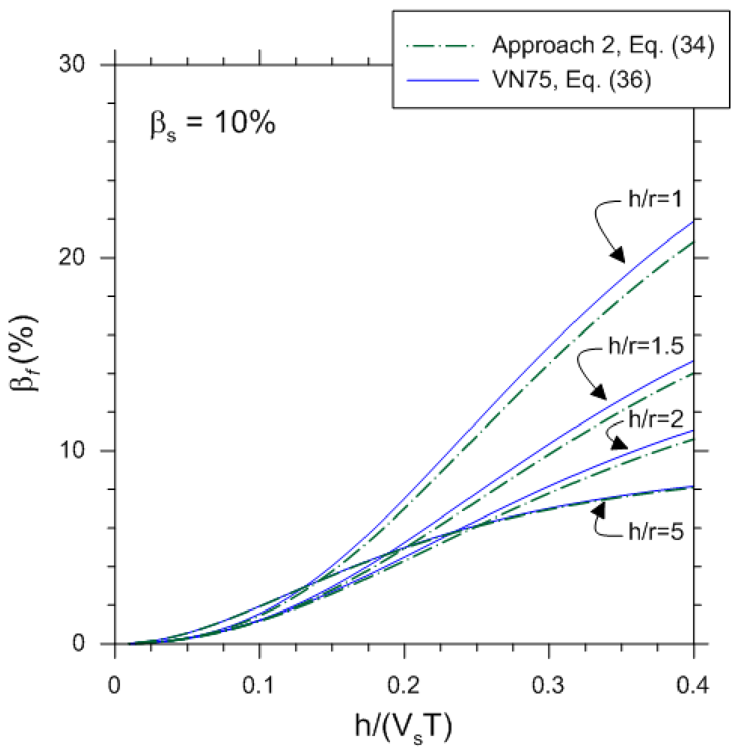

Figure 4. Comparison of foundation damping models (Eq. 34 and Eq. 36) accounting for hysteretic soil damping differently, plotted against structure-to-soil stiffness ratio $h/(V_S T)$. Conditions for the plot are a rigid, massless, circular disc supported on an elastic homogeneous isotropic halfspace with hysteretic soil damping β_s=0.1 and ν = 0.33. Impedance functions are from Veletsos and Verbic (1973). Eq. (34) used Veletsos and Verbic (1973) elastic impedance solution with additive soil damping; Eq. (36) uses similar solution from Veletsos and Nair 1975 (VN75) in which soil damping for a visco-elastic medium is incorporated into the impedance function. Structure to soil mass ratio is 0.15.

ROESSET (1980) AND WOLF (1985) SOLUTIONS

Roesset (1980) presented a foundation damping solution in which the imaginary component of the replacement oscillator stiffness is matched to that of the flexible-base system.

He also used a general (non-viscous) damping formulation for the structural stiffness and similar assumptions to other investigators, so the derivation matches that of Approach 1. Using our nomenclature, Roesset's expression for foundation damping was given as:

$$\beta_f = \left[1 - \frac{1}{\left(\tilde{T}/T\right)^2}\right]\beta_s + \frac{1}{\left(\tilde{T}/T\right)^2}\left(\frac{k}{k_x}\beta_x + \frac{kh^2}{k_{yy}}\beta_{yy}\right) \tag{37}$$

This expression essentially matches Eq. (20) except that the scaling terms in front of the radiation damping coefficients remain as stiffnesses and have not been converted to periods.

Wolf (1985) presented a solution that essentially matches that of Roesset (1980), except that the fictitious vibration periods given in Eq. (18) are introduced to re-write the foundation damping in the form given in Eq. (20).

MARAVAS ET AL. (2014) SOLUTION

Recognizing the previous solutions as approximate, Maravas et al. (2014) built upon previous work by Avilés and Pérez-Rocha (1996) to develop an exact solution for damping of rigid circular foundations. The derivation begins with Eq. (12), which includes the complex-valued stiffnesses of each element in the SSI system. Multiplying each term by the complex-conjugate, without ignoring the higher-order damping terms, results in:

$$\frac{(1-2i\beta_0)}{\tilde{k}\left(1+4\beta_0^2\right)} = \frac{(1-2i\beta_i)}{k\left(1+4\beta_i^2\right)} + \frac{(1-2i\beta_x)}{k_x\left(1+4\beta_x^2\right)} + \frac{h^2\left(1-2i\beta_{yy}\right)}{k_{yy}\left(1+4\beta_{yy}^2\right)} \tag{38}$$

Eq. (38) can be separated into real and imaginary parts as follows:

$$\frac{1}{\tilde{k}\left(1+4\beta_0^2\right)} = \frac{1}{k\left(1+4\beta_i^2\right)} + \frac{1}{k_x\left(1+4\beta_x^2\right)} + \frac{h^2}{k_{yy}\left(1+4\beta_{yy}^2\right)} \tag{39}$$

$$\frac{\beta_0}{\tilde{k}\left(1+4\beta_0^2\right)} = \frac{\beta_i}{k\left(1+4\beta_i^2\right)} + \frac{\beta_x}{k_x\left(1+4\beta_x^2\right)} + \frac{h^2\beta_{yy}}{k_{yy}\left(1+4\beta_{yy}^2\right)} \tag{40}$$

Recognizing that the term $\tilde{k}\left(1+4\beta_0^2\right)$ exists in both the real and imaginary part of the solution, the system damping can be established by first rearranging the real part (Eq. 39) as

$$\tilde{k}\left(1+4\beta_0^2\right)=\frac{1}{\dfrac{1}{k\left(1+4\beta_i^2\right)}+\dfrac{1}{k_x\left(1+4\beta_x^2\right)}+\dfrac{h^2}{k_{yy}\left(1+4\beta_{yy}^2\right)}} \tag{41}$$

Secondly, both sides of the imaginary part (Eq. 40) are multiplied by $\tilde{k}\left(1+4\beta_0^2\right)$ to obtain an expression for the flexible-base system damping:

$$\beta_0=\tilde{k}\left(1+4\beta_0^2\right)\left[\frac{\beta_i}{k\left(1+4\beta_i^2\right)}+\frac{\beta_x}{k_x\left(1+4\beta_x^2\right)}+\frac{h^2\beta_{yy}}{k_{yy}\left(1+4\beta_{yy}^2\right)}\right] \tag{42}$$

The flexible-base system damping is then formulated by inserting the right side of Eq. (41) into Eq. (42) and multiplying the numerator and denominator by *k*:

$$\beta_0=\frac{\dfrac{\beta_i}{\left(1+4\beta_i^2\right)}+\dfrac{k}{k_x}\dfrac{\beta_x}{\left(1+4\beta_x^2\right)}+\dfrac{kh^2}{k_{yy}}\dfrac{\beta_{yy}}{\left(1+4\beta_{yy}^2\right)}}{\dfrac{1}{\left(1+4\beta_i^2\right)}+\dfrac{k}{k_x}\dfrac{1}{\left(1+4\beta_x^2\right)}+\dfrac{kh^2}{k_{yy}}\dfrac{1}{\left(1+4\beta_{yy}^2\right)}} \tag{43}$$

The exact expression in Eq. (43) can be reduced by simultaneously substituting $\hat{k}_j=k_j\left(1+4\beta_j^2\right)/\left(1+4\beta_i^2\right)$ (where $j=x$ or yy) for k_x and k_{yy} and multiplying both the numerator and denominator terms by $\left(1+4\beta_i^2\right)$. It should be noted that $\hat{k}_j>k_j$ for the common case of β_j terms being larger than β_i. With the substitutions, we obtain:

$$\beta_0=\frac{\beta_i+\beta_x\dfrac{k}{\hat{k}_x}+\beta_{yy}\dfrac{kh^2}{\hat{k}_{yy}}}{1+\dfrac{k}{\hat{k}_x}+\dfrac{kh^2}{\hat{k}_{yy}}} \tag{44}$$

It should be noted that soil hysteretic damping β_s can be directly added into the radiation damping terms β_x and β_{yy} in Eqs. (43) or (44). Foundation damping can be back-calculated from the system damping (using Eqs. 43 or 44) by setting β_i = 0 in the numerator, while maintaining as finite these terms in the denominator of Eq. (43) or in the $\hat{k}_j$ terms of Eq. (44). Unlike Approaches 1 and 2, Eqs. (43 and 44) involves weight factors that include squared damping terms from all oscillation modes. These more complex solutions reveal foundation damping to arise from nonlinear coupling of damping resulting from linear material behavior in both the soil and structure. Moreover, as mentioned above, this result shows foundation damping to depend on structural damping, which is unique to the present solution.

Interestingly, Eqs. (43 or 44) reduces to Eq. (16) if β^2 terms are ignored (or if all damping terms are equivalent, which causes $\hat{k}_j = k_j$). Hence, despite the fact that real and complex parts are included in the derivation, which would seem to make the Maravas et al. (2014) solution conceptually similar to Approach 2, it nonetheless matches the solution from Approach 1 if β^2 terms are set to zero or if they are all equal. For this reason, the need to use absolute values in the Approach 2 solution equations is caused by ignoring β^2 terms in the derivation. As shown by Givens (2013), foundation damping results obtained using the Maravas et al. (2014) approach are nearly identical to those from Approach 2 over the range of wave parameter $h/(V_ST) = 0 - 0.3$ (Approach 2 produces slightly larger damping for $h/(V_ST) > 0.3$, as required by the inequality $\hat{k}_j > k_j$). That similarity suggests that taking the absolute value of the complex-valued impedance terms as in Eq. (34) largely compensates for the error associated with ignoring β^2 terms in Approach 2.

Although an exact solution such as Eqs. (43) or (44) could be coded into spreadsheets and applied, we do not recommend it for routine application because (1) its complex form (Eq. 43) or unfamiliar nomenclature (e.g. $\hat{k}_j$ terms in Eq. 44) convey less clearly the physical sources of foundation damping than do Approaches 1 and 2, and (2) the aforementioned negligible differences from the recommended equations for practical situations (the differences only become appreciable when radiation damping is exceptionally high, such as squat structures on uniform soils subjected to very high frequency excitation, which are rarely encountered conditions).

SUMMARY OF PRIOR WORK AND ITS RELATION TO PRESENT RESULTS

Both Approaches 1 and 2 for computation of foundation damping β_f utilize logical progressions in the derivation process that have been employed previously, as explained earlier. The main disadvantages of the prior derivations, that we sought to address here, are:

1. In their originally published form, foundation damping was not expressed in a modular form allowing any impedance formulation to be used, but were connected with equations for the impedance for a particular (usually circular) foundation shape

and were available only in graphical form. Accordingly, the present expressions (Eq. 20 and 34) are more amenable to practical application.

2. Individual prior studies take one of the fundamental approaches described here, and the similarities and differences of results obtained using alternate approaches are not illustrated. The present approach illustrates directly these differences.

3. The documents presenting the original equations or graphical representations are, in most cases, incomplete with respect to explaining the steps and logic of the derivation process. We derived Eq. (20) and (34) from first principles and explain the logic of the solution process.

EXAMPLE APPLICATION

As an example application, we evaluate foundation damping for the Garner Valley, California test site (nees.ucsb.edu/facilities/gvda) using a hypothetical 10×50 m surface foundation. Figures 5a-b shows the soil shear wave velocity profile (from measurements, as compiled by Star et al., 2015) and foundation geometry. The foundation geometry selected for analysis does not match the dimensions of an actual foundation at the site. The analyzed foundation has a higher aspect ratio (in plan view) for compatibility with plane strain analysis and larger dimensions than actual foundation systems at the site to mobilize responses of relatively deep portions of the profile to enhance effects of soil heterogeneity. Hence, we seek to illustrate through this example how a site-specific impedance function can be employed with the modular foundation damping solutions developed in this paper, and to do so for a case where site-specific complexities in the soil layering would be expected to significantly influence the impedance functions and hence potentially the foundation damping.

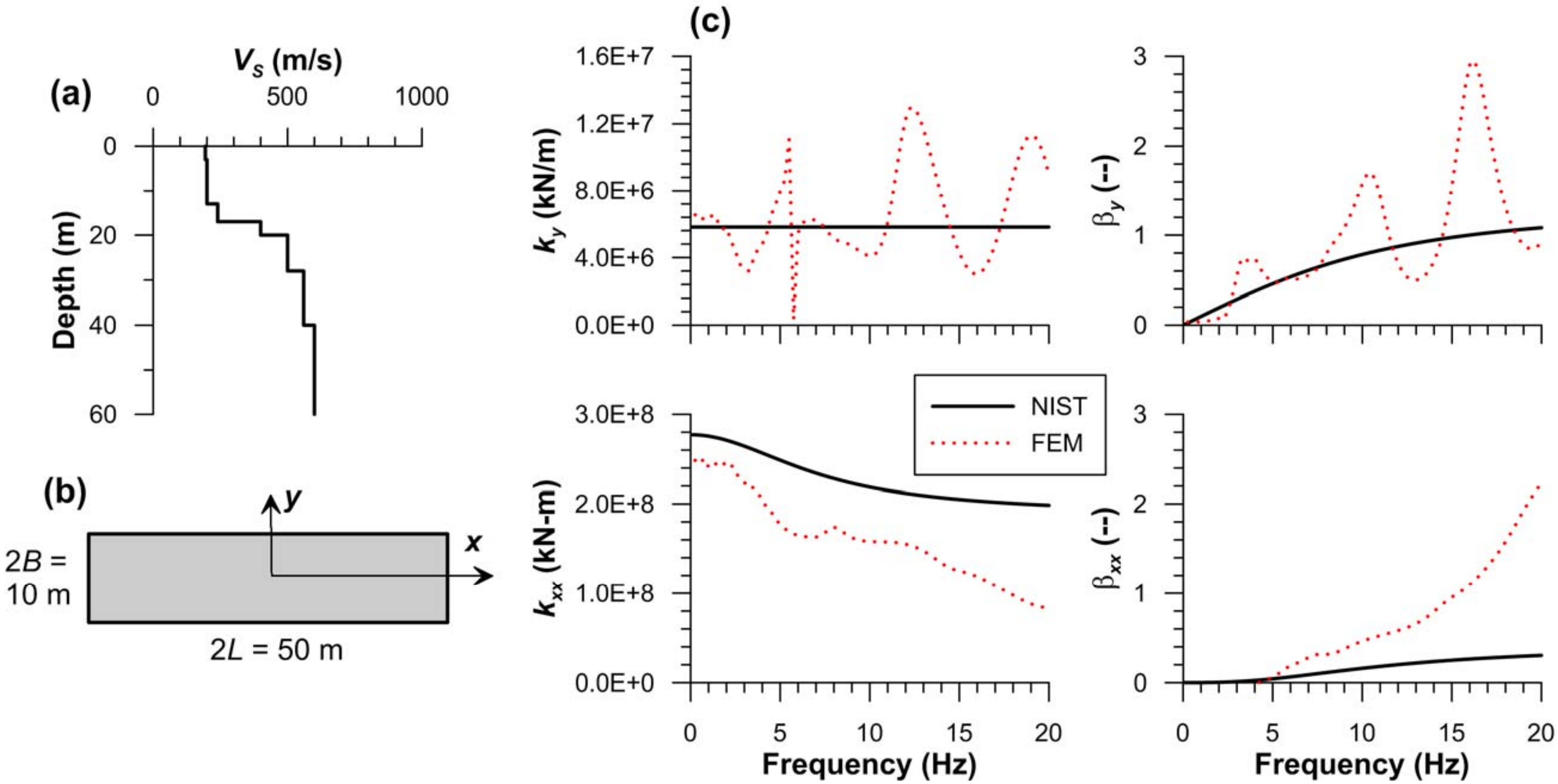

Figure 5. Conditions employed for example computations of foundation damping. (a) V_S profile, reflecting actual conditions at the Garner Valley, CA test site (nees.ucsb.edu/facilities/gvda); (b) plan view of assumed foundation geometry; (c) frequency-dependent foundation stiffnesses for translation (k_y), rotation (k_{xx}), and associated radiation damping terms (β_y and β_{xx}) for *y*-component excitation. Foundation stiffness and damping results are shown for finite element simulations using zero soil hysteretic damping (FEM) and Poisson's ratio = 0.45 and from closed form expressions for a soil halfspace adapted to the present conditions following guidelines in NIST (2012).

Using a methodology for plane-strain finite element analysis of foundation-soil systems (Esmaeilzadeh et al., 2015), E. Esmaeilzadeh (*personal communication*, June, 2015) evaluated frequency-dependent and complex-valued impedances for y-component translation and xx-component rotation of the foundation-soil system, with the results shown in Figure 5c. The soil was modelled as elastic (no hysteretic damping) and meshing procedures given in Esmaeilzadeh et al. (2015) were adhered to. The impedance ordinates were obtained from the software by applying unit-amplitude cyclic displacements or rotations, computing the resulting nodal forces on the foundation that develop, and integrating those nodal forces into shear forces and moments at the foundation centroid (which comprise the desired impedance quantities). The plane strain analyses are for excitation in the y-direction; both the real and complex parts were multiplied by $2L$ to obtain the results in Figure 5c labelled as 'FEM'. Radiation damping coefficients were computed from the ratio of complex/real components using Eq. (5) (the derivations earlier in this paper were for x-component translation and yy-component rotation, they can be applied to the present case by simply changing subscripts x to y and yy to xx in the equations). Also shown in Figure 5c are stiffness and damping predictions using Pais and Kausel (1988) halfspace equations adapted for non-uniform soil profiles following recommendations in NIST (2012).

Using the impedance ordinates in Figure 5c, we compute foundation damping using Approach 2 (Eq. 34) for three fixed-base structure periods (T=0.1, 0.2, and 0.4 sec), a single structure height h=5 m (h/B = 1), and excitation in the y-direction. We use a structure mass m of 6.9×10^5 kg (15% of the soil mass in a volume equivalent to the foundation footprint times structure height). The computation process proceeds as follows:

1. Preliminary estimates of foundation stiffnesses k_y and k_{xx} are obtained by entering Figure 5c at f=1/T. Calculations are performed using both sets of impedance ordinates.
2. The lengthened building period $\tilde{T}$ is computed using Eq. (1).
3. Updated foundation stiffnesses are obtained using $\tilde{f} = 1/\tilde{T}$. Lengthened period is computed again and the process continues until period lengthening is no longer changing between iterations (usually 2-3 are sufficient).
4. Radiation damping coefficients β_y and β_{xx} are obtained by entering Figure 5c at $\tilde{f} = 1/\tilde{T}$.

5. Fictitious periods are computed using Eq. (32) with the amplitude of the corresponding complex-valued stiffness from Eq. (31) (i.e., for the *y*-direction, $\left|\bar{k}_y\right|$ is used in the expression for $\left|\bar{T}_y\right|$).

6. Foundation damping is computed using Eq. (34), with the results in Table 1 (the tabulated results are derived from the site-specific impedance ordinates).

Table 1. Results of example foundation damping computations for site and foundation conditions shown in Figure 5 with structure height *h*=*B*=5m and fixed-base periods indicated below. Results are shown for the case of site-specific impedance from FEM.

T (sec)	h/(V$_S$T)	$\tilde{T}/T$	$\tilde{f}$ (Hz)	k_y (kN/m)	β_y	k_{xx} (kN-m)	β_{xx}	$\left\|\bar{T}_y\right\|$ (sec)	$\left\|\bar{T}_{xx}\right\|$ (sec)	β_f
0.1	0.25	1.36	7.4	6.1×10^6	0.58	1.7×10^8	0.30	0.054	0.059	0.16
0.2	0.13	1.09	4.6	6.5×10^6	0.54	1.9×10^8	0.02	0.053	0.060	0.036
0.4	0.06	1.03	2.4	4.4×10^6	0.16	2.4×10^8	0	0.077	0.054	0.007

As expected, foundation damping varies strongly with wave parameter $h/(V_S T)$, as shown previously in Figures 3 and 4. For comparison, the foundation damping results obtained using the equivalent-halfspace impedance solutions (NIST, 2012) are 0.14, 0.038, and 0.007 (for T = 0.1, 0.2, and 0.4 sec, respectively). For these calculations, the equivalent halfspace velocity was taken as 198 m/s based on the ratio of effective depth of influence below foundation (7.5 m) to shear wave travel time as given in NIST (2012). These results are close to those obtained using the site-specific impedance ordinates in Figure 5.

CONCLUSIONS

We initially presented the first of our modular equations for foundation damping (from Approach 1) in NIST (2012), to overcome limitations of graphical methods for evaluating foundation damping that appeared in earlier seismic analysis and design guidelines documents (ASCE-7, 2010; ASCE-41, 2006). Those guidelines have since been updated using our solutions as presented in NIST (2012). In this paper, we have presented the basis for Approach 1 and extended the derivation using a different set of assumptions for matching the real and complex parts of the response of an equivalent fixed-based oscillator to that of a flexible-base oscillator (Approach 2). The underlying assumptions and logic behind Approaches 1 and 2 are not original, but the derivations here have unique and useful elements relative to prior work as explained in the section entitled *Summary of Prior Work and its Relation to Present Results*.

Given the presence of two sets of equations for foundation damping, a natural question is which solution is preferred for application? As illustrated in Figure 3, the two solutions are not significantly different and other factors (such as the modeling of heterogeneous soil conditions in the impedance) are likely to exert more influence on results than the choice of equations. Nonetheless, our view at the present time is that Approach 2 is preferred, principally because it is more complete in its assessment of the equivalent oscillator response (by considering real and complex parts). The foundation damping from Approach 2 is given by Eq. (34) and an example application is given in the previous section. Because the foundation damping expressions are derived to match SDOF oscillator responses, they are applicable strictly to analysis of SSI effects on the first-mode response of structures.

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