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Lyapunov exponents, one-dimensional Anderson localisation and products of random matrices

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Abstract. The concept of Lyapunov exponent has long occupied a central place in the theory of Anderson localisation; its interest in this particular context is that it provides a reasonable measure of the localisation length. The Lyapunov exponent also features prominently in the theory of products of random matrices pioneered by Furstenberg. After a brief historical survey, we describe some recent work that exploits the close connections between these topics. We review the known solvable cases of disordered quantum mechanics involving random point scatterers and discuss a new solvable case. Finally, we point out some limitations of the Lyapunov exponent as a means of studying localisation properties.

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1. Introduction

Anderson localisation is the term used to describe a generic phenomenon, discovered in the late fifties by P. W. Anderson, whereby the addition of a certain amount of disorder or randomness in an otherwise deterministic medium causes the waves propagating in the medium to become localised in space [2]. For quantum systems, the understanding of transport properties requires a thorough study of how the presence of disorder affects the nature of quantum states— a question first addressed in an earlier paper of Landauer and Helland [40]. Since that time, the one-dimensional case has been discussed extensively and has led to a better understanding of the physical mechanisms that are responsible for localisation. A remarkable feature of one-dimensional systems is that almost all states become localised as soon as there is *any* disorder. This result, first conjectured by Mott and Twose [47], was made more rigorous by Borland [6] who considered an infinite chain of identical localised potentials separated by regions of zero potential. Assuming that the lengths of these regions are independent random variables with the same probability distribution, Borland studied the growth rate of the wave function on a semi-infinite chain with prescribed boundary conditions at one end. He then argued, by using a “matching argument”, that the positivity of the growth rate implies the exponential localisation of the wave functions. A purely mathematical proof using the properties of transfer matrices was given by Matsuda and Ishii [44].

These physical arguments can in fact be made completely rigorous. The exponential growth of the solutions of the Cauchy problem is a crucial feature of the proof that the spectrum has no absolutely continuous component [53], and that it is pure-point [31]. If $\psi(x, E)$ denotes a solution of the Cauchy problem (i.e. a solution of the Schrödinger equation on the positive half-line subject to boundary conditions at $x = 0$) then the quantity

$$\gamma(E) := \lim_{x \rightarrow \infty} \frac{\ln |\psi(x, E)|}{x} \quad (1)$$

is a self-averaging quantity called the *Lyapunov exponent* of the disordered system. A rigorous demonstration of the fact that, under certain hypotheses, γ also quantifies the exponential decay of the *eigenfunctions*— and therefore that its reciprocal can serve as a definition of the localisation length— appears for the first time in Ref. [10]. It should be borne in mind, however, that this definition of the localisation length is only useful if certain conditions are fulfilled; cases where the definition is inappropriate will be considered briefly in §4.

In order to illustrate the localisation phenomenon, Anderson made use of a model in which the wave function solves a difference equation. The general solution of this “tight-binding” model takes the form of a product of random matrices, say,

$$\Pi_n := M_n M_{n-1} \cdots M_1 \quad (2)$$

where the M_j are independent and identically-distributed square matrices with a

common probability measure $\mu(dM)$. The quantity

$$\gamma_\mu := \lim_{n \rightarrow \infty} \frac{\mathbb{E}(\ln |\Pi_n|)}{n} \quad (3)$$

is called the Lyapunov exponent of the product of random matrices and, as we shall soon see, is effectively the same as γ .

Anderson localisation has been a powerful motivation for the study of products of random matrices. For this reason, the search for precise conditions on the measure $\mu(dM)$ that would guarantee the existence and the positivity of the Lyapunov exponent γ_μ has been of particular interest. The main result in this respect is due to Furstenberg [26]— a result published the same year as Borland’s paper; see also Oseledec [51] whose work can be considered as an extension to dynamical systems of the work of Furstenberg and Kesten [27]. Further developments of these results, and their application to the study of localisation, are described in the works of Ishii [35], Bougerol and Lacroix [8], Carmona and Lacroix [11], Lifshits *et al.* [41], Luck [42], Pastur and Figotin [54], Crisanti *et al.* [19] and the references therein.

An important milestone in the development of the theory of localisation was the discovery of the relationship between the Lyapunov exponent and the integrated density of states $N(E)$. It turns out that the characteristic function

$$\Omega(E) := \gamma(E) - i\pi N(E), \quad (4)$$

viewed as a complex-valued function on \mathbb{R} , is *analytic* in the upper half of the complex plane [34, 62]. Interestingly, this analyticity property of the characteristic function was in fact exploited much earlier by Dyson in his famous paper on the dynamics of a disordered chain [21].

Products of random matrices appear naturally in several other problems related to the physics of disordered systems— see the monographs [19, 42]— and is still the subject of active research [24]. The concept of Lyapunov exponent is a very useful tool for analysing a large class of systems with quenched disorder, such as magnetic systems. A well-known prototype is the Ising model in a random magnetic field, where the calculation of the free energy reduces to analysing an infinite product of 2×2 matrices; in this context, the Lyapunov exponent is proportional to the free energy per spin.

The group from which the matrices M_n in the product (2) are drawn varies not only with the physical context but also with the choice of vector basis. In the analysis of the Schrödinger equation in a random potential, a standard choice is to consider the vector formed by the wave function and its derivative $(\psi'(x), \psi(x))$. As we shall see later on, the evolution of this vector is governed by matrices belonging to the group $SL(2, \mathbb{R})$. This is the formulation chosen for example in Refs. [15, 17] and in the present article. If, instead, one chooses to focus on the scattering aspects of the problem then the wave function, in a region where $V(x) = 0$, is a combination of incoming and outgoing waves $\psi(x) = A e^{ikx} + B e^{-ikx}$. In this setting, it is natural to consider transfer matrices T connecting pairs of complex amplitudes (A, B) . Current conservation then implies that such transfer matrices belong to the group $U(1, 1)$; see the appendix of Ref. [15]

for further details, and Ref. [55] for a pedagogical presentation and a review of the literature.

Interestingly, this scattering formulation provides a natural way of generalising the one-dimensional model— which is associated with products of 2×2 matrices— to a quasi-one-dimensional model where the matrices in the product are $2m \times 2m$, where m is the number of conducting channels. The localisation problem then involves— not just one— but rather m (counting multiplicity) Lyapunov exponents $\gamma_1 \leq \dots \leq \gamma_m$. Since the pioneering work of Dorokhov [20], whose results were later rediscovered independently by Mello, Pereyra and Kumar [45], this topic has attracted a lot of attention owing to its relevance in the description of weakly disordered metallic wires; see the review [3]. These early works relied on the so-called *isotropy assumption*, namely that each elementary slice of disordered metal redistributes the current uniformly amongst the m conducting channels. This assumption produces a set of Lyapunov exponents with the behaviour $\gamma_j = \gamma_1 [1 + \beta(j-1)]$, where $\beta \in \{1, 2, 4\}$ is the Dyson index [3]. The smallest Lyapunov exponent, which scales with the number of channels like $\gamma_1 \propto 1/m$, is usually interpreted as the reciprocal of the localisation length. The ideas of Dorokhov and Mello *et al.* have since been extended to other symmetry classes of disordered models; see the review [23]. Another line of research stimulated by these ideas is to look for models where the isotropy hypothesis may be relaxed [12, 46, 48, 49], with the aim of studying the passage from one-dimensional to higher-dimensional localisation. To close these brief remarks on the multichannel case, we point out that the Lyapunov exponent is at the heart of numerical studies of localisation that rely on the scaling approach [38].

The present paper revisits the interplay between products of random matrices of $\text{SL}(2, \mathbb{R})$ and one-dimensional Anderson localisation. Whereas most of the works cited above emphasise *discrete* models obtained by making the tight-binding approximation [19, 42], we shall focus here on one-dimensional continuous models that make use of the notion of *point scatterer*. One familiar example is the Kronig–Penney model [39] based on delta-scatterers. We shall see that, by considering a natural generalisation of the concept of point scatterer, we arrive at a useful interpretation of a general product of random matrices as a model of disorder. We illustrate the fruitfulness of this interpretation by exhibiting concrete instances of the probability measure $\mu(dM)$ of the matrices M_j for which the Lyapunov exponent may be expressed in terms of special functions. One of these models is new and has interesting connections with Sinai’s study of diffusion in a random environment [7, 59]. Another justification for focusing on models involving generalised random point scatterers is that— far from being special— they can on the contrary be mapped onto the *most general product* of random matrices in $\text{SL}(2, \mathbb{R})$. Hence such models can in principle describe the whole of one-dimensional disordered quantum mechanics.

The remainder of the paper is as follows: §2 describes the concept of generalised point scatterer, explains how models of disorder may be constructed from them, and how completely general products of matrices may be associated with such models. We then list some known solvable cases. §3 is devoted to the analysis of a *new* example where

the Lyapunov exponent may be expressed in terms of the generalised hypergeometric function. The paper ends in §4 with a discussion of the limitations of the Lyapunov exponent as a measure of localisation in disordered systems.

2. Generalised point scatterers and products of random matrices

2.1. Point scatterers and transfer matrices

A *point scatterer* is the idealised limit of a potential whose action is highly localised. The most familiar example is the delta-scatterer at a point, say x_j . In the context of the Schrödinger equation

$$-\psi''(x) + V(x)\psi(x) = E\psi(x), \quad x > 0, \quad (5)$$

this potential vanishes for every $x \neq x_j$ and is defined at $x = x_j$ via the boundary condition

$$\begin{pmatrix} \psi'(x_{j+}) \\ \psi(x_{j+}) \end{pmatrix} = \begin{pmatrix} 1 & u_j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi'(x_{j-}) \\ \psi(x_{j-}) \end{pmatrix} \quad (6)$$

where u_j is the strength of the scatterer, i.e. $V(x) = \sum_j u_j \delta(x - x_j)$. This delta-scatterer is generalised by replacing the 2×2 matrix on the right-hand side by an arbitrary matrix, say B_j . Thus

$$\begin{pmatrix} \psi'(x_{j+}) \\ \psi(x_{j+}) \end{pmatrix} = B_j \begin{pmatrix} \psi'(x_{j-}) \\ \psi(x_{j-}) \end{pmatrix}. \quad (7)$$

Conservation of the probability current requires $B_j \in \text{SL}(2, \mathbb{R})$. If, instead of a single scatterer, we consider a sequence of scatterers placed at the points

$$x_1 < x_2 < \dots$$

then, assuming $E = k^2 > 0$, the solution of the Schrödinger equation may be expressed in the form

$$\begin{pmatrix} \psi'(x_{n+}) \\ \psi(x_{n+}) \end{pmatrix} = \Pi_n \begin{pmatrix} \psi'(x_{1-}) \\ \psi(x_{1-}) \end{pmatrix}. \quad (8)$$

where Π_n is the product (2) with

$$M_j = \begin{pmatrix} \sqrt{k} & 0 \\ 0 & \frac{1}{\sqrt{k}} \end{pmatrix} \begin{pmatrix} \cos \theta_j & -\sin \theta_j \\ \sin \theta_j & \cos \theta_j \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{k}} & 0 \\ 0 & \sqrt{k} \end{pmatrix} B_j \quad (9)$$

and

$$\theta_j = k \ell_j, \quad \ell_j := x_{j+1} - x_j.$$

Now, by applying the Gram–Schmidt algorithm to the columns, every 2×2 matrix M with unit determinant may be expressed in the form

$$M = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} e^w & 0 \\ 0 & e^{-w} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \quad (10)$$

for some real parameters θ , w and u . This is the Iwasawa decomposition of $\mathrm{SL}(2, \mathbb{R})$ into compact, Abelian and nilpotent subgroups. Therefore, by working with point scatterers of the form

$$B_j := \begin{pmatrix} e^{w_j} & 0 \\ 0 & e^{-w_j} \end{pmatrix} \begin{pmatrix} 1 & u_j \\ 0 & 1 \end{pmatrix} \quad (11)$$

and taking $E = k^2 = 1$, we obtain a correspondence between general products of matrices Π_n in $\mathrm{SL}(2, \mathbb{R})$ and the Schrödinger equation. The parameter k may be easily reintroduced by simple dimensional analysis.

2.2. Product of random matrices—The Riccati variable

There are several options in defining a disordered quantum system with point scatterers. The randomness may be in the strength of the scatterer, i.e. in the matrix B_j , or in the position of the scatterer, i.e. in the coordinate x_j or in *both* the strength and the position; see [41] where various models are reviewed.

We shall henceforth confine our attention to the particular case where the spacing between consecutive scatterers, i.e. the angle $\theta_j = k\ell_j$, is exponentially distributed:

$$\mathbb{P}(\ell_j \in S) = p \int_{S \cap \mathbb{R}_+} e^{-p\ell} d\ell \quad (12)$$

where $1/p > 0$ is the mean spacing. This is the situation that arises when impurities are dropped uniformly on \mathbb{R} with a mean density p .

The equation satisfied by the Riccati variable

$$Z(x) := \frac{\psi'(x)}{\psi(x)} \quad (13)$$

is

$$Z'(x) = -E - Z^2(x) \quad \text{for } x \notin \{x_j\} \quad (14)$$

and

$$Z(x_j+) = \mathcal{B}_j(Z(x_j-)) \quad \text{for } j \in \mathbb{N} \quad (15)$$

where \mathcal{B}_j is the linear fractional transformation associated with the matrix B_j , i.e.

$$\mathcal{B}_j(Z) := e^{2w_j} (Z + u_j) . \quad (16)$$

Because the spacing between consecutive scatterers is exponentially distributed, Z is a Markov process, and it was shown by Frisch and Lloyd [25] (see also [15]) that the stationary density $f(Z)$ satisfies

$$\frac{d}{dZ} [(Z^2 + E)f(Z)] + p \int_{\mathrm{SL}(2, \mathbb{R})} \mu_B(dB) \left[f(\mathcal{B}^{-1}(Z)) \frac{d\mathcal{B}^{-1}(Z)}{dZ} - f(Z) \right] = 0 \quad (17)$$

where μ_B is the probability measure of the random matrix B .

The relationship between the Riccati variable and the Lyapunov exponent is easy to establish— at least heuristically. A completely rigorous treatment would follow the

lines of Kotani's work [37]. Given the particular form of the upper-triangular matrix B_j , we have

$$\begin{aligned} \ln |\psi(x_{n+})| &= -w_n + \ln |\psi(x_n-)| \\ &= -w_n + \int_{x_{n-1}}^{x_n} Z(x) dx + \ln |\psi(x_{n-1+})| \\ &= \dots = -\sum_{j=1}^n w_j + \int_{x_1}^{x_n} Z(x) dx + \ln |\psi(x_1-)|. \end{aligned} \quad (18)$$

Dividing by x_n and letting $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} \frac{\ln |\psi(x_n)|}{x_n} = -\lim_{n \rightarrow \infty} \frac{1}{x_n} \sum_{j=1}^n w_j + \lim_{n \rightarrow \infty} \frac{1}{x_n} \int_{x_1}^{x_n} Z(x) dx. \quad (19)$$

Since

$$x_n - x_1 = \sum_{j=1}^{n-1} \ell_j$$

where the ℓ_j are independent and identically distributed with mean $1/p$, the law of large numbers implies that

$$x_n \sim n/p \quad \text{almost surely, as } n \rightarrow \infty. \quad (20)$$

Then, by the ergodic theorem,

$$\gamma = -p \mathbb{E}(w) + \int_{\mathbb{R}} Z f(Z) dZ \quad (21)$$

almost surely, where f is the density of the stationary distribution of the Riccati variable. The upshot is that the Lyapunov exponent of the system may in principle be computed by solving the Frisch–Lloyd equation (17) for the stationary density, and then performing the integral in Formula (21).

2.3. Correspondence with the Furstenberg theory

It is instructive to compare this method of calculation with that based on Furstenberg's theory [26, 30]. The concept of direction, through the projective space $P(\mathbb{R}^2)$, plays a prominent part in that theory. This is due to the fact that the product of matrices grows if and only if the columns of the product tend to align along a common direction. In \mathbb{R}^2 , a direction may be parametrised by a single number; let us choose the reciprocal $z \in \mathbb{R} \cup \{\infty\}$ of the slope. The columns of the product Π_n have directions that are random. When a vector of random direction, say z , is multiplied by a random matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (22)$$

a new vector is produced, whose direction is the random variable

$$\mathcal{M}(z) := \frac{az + b}{cz + d}. \quad (23)$$

We say that the distribution, say $\nu_\mu(dz)$, of z is *invariant* under the action of matrices drawn from μ if the distributions of the old and the new directions are identical, i.e.

$$\mathcal{M}(z) \stackrel{\text{law}}{=} z. \quad (24)$$

In particular, if $\nu_\mu(dz)$ has a density, say $f_\mu(z)$, then this equality in law translates into the following equation for f_μ :

$$f_\mu(z) = \int_{\text{SL}(2, \mathbb{R})} \mu(dM) (f_\mu \circ \mathcal{M}^{-1})(z) \frac{d\mathcal{M}^{-1}}{dz}(z). \quad (25)$$

where, with some abuse of notation, z is no longer random. Knowing f_μ , the Lyapunov exponent γ_μ of the product Π_n may then be computed by the formula

$$\gamma_\mu = \int_{\mathbb{R}} dz f_\mu(z) \int_{\text{SL}(2, \mathbb{R})} \mu(dM) \ln \frac{\left| M \begin{pmatrix} z \\ 1 \end{pmatrix} \right|}{\left| \begin{pmatrix} z \\ 1 \end{pmatrix} \right|}. \quad (26)$$

The calculation of γ_μ when M is of the form (10) and θ is exponentially distributed with mean $1/p$, independent of w and u , may then be related to the calculation of γ in the previous subsection as follows:

- (i) The projective variable z is the Riccati variable Z with $E = k^2 = 1$.
- (ii) The Furstenberg equation (25) for the invariant density reduces to the Frisch–Lloyd equation (17) for the stationary density.
- (iii) The Furstenberg formula (26) for γ_μ reduces to Formula (21) for γ and

$$\gamma_\mu = \frac{1}{p} \gamma. \quad (27)$$

The factor $1/p$ is due to the difference between the definitions: whereas (1) involves the distance variable x , the growth of the matrix product is measured with respect to the index n in (3)— x and n being related by (20).

2.4. Known solvable cases involving random point-scatterers

There is no systematic method for solving the integro-differential equation (17), but some solvable cases have been found and are listed in Table 1. All but the first of these cases have in common that one or both the parameters w and u in the expression for the matrix B are exponentially or gamma distributed. The density of the exponential and gamma distributions satisfies a differential equation with *constant* coefficients, and this leads to a trick for reducing (17) to a purely differential form. This trick will be illustrated in the next section.

Here, we consider the simplest example and merely write down the result. This example consists of taking

$$B = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \quad (28)$$

where u is exponentially distributed with mean $1/q$. Then the stationary density solves

$$\frac{d}{dz} [(z^2 + k^2) f(z)] - p f(z) + q (z^2 + k^2) f(z) = qN. \quad (29)$$

The constant $N = N(E)$ appearing on the right-hand side is the *integrated density of states* per unit length of the quantum model; the *Rice formula*

$$N(E) = \lim_{|z| \rightarrow \infty} z^2 f(z) \quad (30)$$

expresses its relationship to the tail of the stationary density. After integration, we obtain

$$f(z) = \frac{qN}{z^2 + k^2} \exp \left[-qz + \frac{p}{k} \arctan \frac{z}{k} \right] \int_{-\infty}^z \exp \left[qt - \frac{p}{k} \arctan \frac{t}{k} \right] dt.$$

The integrated density of states is then determined by the requirement that f be a probability density. The final formulae for N and γ may be expressed neatly via the characteristic function (4) of the disordered system: for $E = k^2 > 0$,

$$\Omega(E) = 2ik \frac{W'_{\frac{-ip}{2k}, \frac{1}{2}}(-2ikq)}{W_{\frac{-ip}{2k}, \frac{1}{2}}(-2ikq)} \quad (31)$$

where $W_{\alpha, \beta}$ is a Whittaker function [32]. This formula, which was discovered by Nieuwenhuizen [50] using a different approach, corresponds here to the case of “repulsive” scatterers ($u > 0$); it is easily adapted to the “attractive” case $u < 0$ by simply changing the sign of q in Eq. (31) [15].

Density of ℓ	Density of u	Density of w	Reference	Special function
$\delta(\ell - 1/p)$	$\frac{q/\pi}{q^2 + u^2}$	$\delta(w)$	[35]	
$p e^{-p\ell} \mathbf{1}_{\mathbb{R}_+}(\ell)$	$q e^{-qu} \mathbf{1}_{\mathbb{R}_+}(u)$	$\delta(w)$	[50, 15]	Whittaker
$p e^{-p\ell} \mathbf{1}_{\mathbb{R}_+}(\ell)$	$q^2 u e^{-qu} \mathbf{1}_{\mathbb{R}_+}(u)$	$\delta(w)$	[50, 15]	Whittaker
$p e^{-p\ell} \mathbf{1}_{\mathbb{R}_+}(\ell)$	$\delta(u)$	$q e^{-qw} \mathbf{1}_{\mathbb{R}_+}(w)$	[15]	Hypergeometric
$p e^{-p\ell} \mathbf{1}_{\mathbb{R}_+}(\ell)$	$\delta(u)$	$q e^{-2q w }$	this article	generalized Hypergeometric

Table 1. Known solvable cases involving random point scatterers. The notation $\mathbf{1}_A(x)$ is used for the function that equals 1 if $x \in A$ and 0 otherwise.

Besides the Lyapunov exponent, the low-energy behaviour of the density of states is also of interest; it has been analysed for point scatterers with random positions in the following cases: (i) when $w = 0$ and the probability density of u is supported on the positive half-line but is otherwise arbitrary [37, 41]; (ii) in the converse situation where $u = 0$ and it is $w > 0$ that is random [16].

2.5. Continuum limit and models involving Gaussian white noises or Lévy noises

No exact solution is known in the case where *all three* parameters $\ell = \theta$, w and u are random. Some insight into the general case may however be gained by considering

the continuum limit of an infinitely dense set of point scatterers with vanishingly weak strengths, i.e.

$$\theta, w, u \rightarrow 0.$$

This limit was studied very recently in Ref. [17] in the most general case where the three parameters of the Iwasawa decomposition (10) are correlated—the matrices M_j being still mutually independent. This recent work corresponds to a generalisation of the study of the Schrödinger equation with a potential $V(x) = \eta(x) + W(x)^2 - W'(x)$ that combines two (possibly correlated) Gaussian white noises η and W [33]; see Ref. [17] for a detailed discussion.

Finally let us mention a generalisation of the model in another direction: by letting the density of scatterers tend to infinity, it is possible to study certain singular limiting measures for the strength of the scatterers. Such cases may be described in terms of Lévy noises and were studied in [16, 37].

3. Supersymmetry: a new solvable case

In this section, we shall restrict our attention to the particular case where the point scatterers are described by transfer matrices of the form

$$B_j = \begin{pmatrix} e^{w_j} & 0 \\ 0 & e^{-w_j} \end{pmatrix}. \quad (32)$$

As shown in Ref. [15], this corresponds to considering the potential

$$V = W^2 - W' \quad (33)$$

where W is a superposition of delta-scatterers

$$W(x) = \sum_j w_j \delta(x - x_j).$$

A potential of the form (33) is said to be *supersymmetric* with *superpotential* W . Hence we shall refer to the point scatterer associated with (32) as “supersymmetric”. This case has many interesting features: the corresponding Hamiltonian is factorisable and, when the superpotential is deterministic and has the so-called “shape-invariance” property, the discrete spectrum may be obtained exactly [18].

The disordered case is of interest in several physical contexts [14]: in particular, the Schrödinger equation can be mapped onto the Dirac equation with a random mass—a model relevant in several contexts of condensed matter physics; see the reviews [14, 29, 41] and the introduction in Ref. [61]. The model is also closely connected to diffusion in a random environment [7]; indeed, if we set

$$\psi(x) = \exp \left[- \int W(x) dx \right] U(x)$$

then

$$\frac{1}{2} U''(x) - W(x) U'(x) = \lambda U(x) \quad \text{with} \quad \lambda = -\frac{E}{2}. \quad (34)$$

When $E < 0$, this is the equation satisfied by the Laplace transform of a *hitting time* of the diffusion in the environment $\int W(x) dx$; see for instance [9].

After integrating with respect to z , the Frisch–Lloyd equation (17) for the stationary density becomes

$$(z^2 + E) f(z) + p \int_{\mathbb{R}} dw \varrho(w) \int_z^{ze^{-2w}} f(t) dt = N(E) \quad (35)$$

where ϱ is the density of the random variable w . Our strategy for computing the Lyapunov exponent will make use of three simple observations: firstly, the equivalence between the supersymmetric Schrödinger equation and Equation (34) implies that the spectrum is necessarily contained in \mathbb{R}_+ . It follows that $N(E) = 0$ for $E < 0$ and that the stationary density f is supported on the positive half-line; this facilitates the calculation of the Lyapunov exponent. The second observation is that the characteristic function (4) is an *analytic function of E* , except for a branch cut along the positive half-axis. Therefore, if we find an analytical formula for the Lyapunov exponent when $E < 0$, analytic continuation will furnish a formula for the case $E > 0$. The third observation—which we alluded to earlier and whose exploitation goes back to the works of Nieuwenhuizen [50] and Gjessing and Paulsen [28]—is that the integral term appearing in (35) may be eliminated if ϱ solves a linear differential equation with (piecewise) constant coefficients.

With these points in mind, let us set

$$E = -k^2 < 0$$

and take

$$\varrho(w) = q e^{-2q|w|}. \quad (36)$$

If we write $f_{p,q,k}$ to indicate the dependence of the stationary density f on the parameters p , q and k , then we have the following identity:

$$f_{p,q,k}(z) = \frac{1}{k} f_{k,q,1}\left(\frac{z}{k}\right). \quad (37)$$

It is therefore sufficient to consider the case $k = 1$, i.e. $E = -1$. It is then easily deduced from the evenness of ϱ that

$$f(z) = \frac{1}{z^2} f(1/z). \quad (38)$$

As a consequence, to know $f(z)$ for $z \geq 1$ is to know $f(z)$ for $0 < z < 1$, and vice-versa; we shall at times implicitly make use of this helpful property.

3.1. Reduction to a differential equation

Following the recipe outlined in [15], we shall now deduce from the Frisch–Lloyd equation (35) a *differential* equation for the density f . To this end, we introduce the kernel

$$K(y) = -\text{sign}(y) \frac{1}{2} e^{-2q|y|} \quad (39)$$

and note, for future reference, that K satisfies

$$K'(y) = -\text{sign}(y) 2q K(y), \quad y \neq 0, \quad (40)$$

subject to

$$K(0\pm) = \mp \frac{1}{2}. \quad (41)$$

After permuting the order of integration in the Frisch–Lloyd equation, we find

$$(z^2 - 1)f(z) + p \int_0^\infty K\left(\frac{1}{2} \ln \frac{z}{t}\right) f(t) dt = 0. \quad (42)$$

Set

$$\varphi(z) := (z^2 - 1)f(z)$$

and write

$$\int_0^\infty K\left(\frac{1}{2} \ln \frac{z}{t}\right) f(t) dt = \int_0^z K\left(\frac{1}{2} \ln \frac{z}{t}\right) f(t) dt + \int_z^\infty K\left(\frac{1}{2} \ln \frac{z}{t}\right) f(t) dt.$$

Differentiation of (42) with respect to z then yields:

$$\begin{aligned} \varphi'(z) + p K(0+)f(z) - p \frac{q}{z} \int_0^z K\left(\frac{1}{2} \ln \frac{z}{t}\right) f(t) dt \\ - p K(0-)f(z) + p \frac{q}{z} \int_z^\infty K\left(\frac{1}{2} \ln \frac{z}{t}\right) f(t) dt = 0. \end{aligned} \quad (43)$$

Hence, in view of (41),

$$z [\varphi'(z) - p f(z)] - p q \int_0^z K\left(\frac{1}{2} \ln \frac{z}{t}\right) f(t) dt + p q \int_z^\infty K\left(\frac{1}{2} \ln \frac{z}{t}\right) f(t) dt = 0.$$

Another differentiation leads, after use of (41), to

$$\frac{d}{dz} \{z [\varphi'(z) - p f(z)]\} + p \frac{q^2}{z} \int_0^\infty K\left(\frac{1}{2} \ln \frac{z}{t}\right) f(t) dt = 0. \quad (44)$$

The integral term may then be eliminated by making use of Equation (42); the result is the second-order linear differential equation

$$z \frac{d}{dz} \left[z \left(\varphi' - p \frac{\varphi}{z^2 - 1} \right) \right] - q^2 \varphi = 0. \quad (45)$$

3.2. The case $q = 1$

It may be verified by direct substitution that, when $q = 1$, the differential equation (45), expressed in terms of the unknown f , has the solution

$$f_{p,1,1}(z) = \frac{A}{p-2} \frac{1}{z} \left[1 - \frac{1 + pz + z^2}{|z-1|^{1-\frac{p}{2}} (z+1)^{1+\frac{p}{2}}} \right] \quad (46)$$

for $p \neq 2$. The normalisation constant A satisfies

$$1 = \frac{2A}{p-2} \int_0^1 \left[1 - \left(\frac{1-z}{1+z} \right)^{\frac{p}{2}} \frac{1+pz+z^2}{1-z^2} \right] \frac{dz}{z}$$

$$\stackrel{z=\frac{1-x}{1+x}}{=} \frac{2A}{p-2} \int_0^1 \left[2 \frac{1-x^{\frac{p}{2}-1}}{1-x^2} + \left(1 - \frac{p}{2} \right) x^{\frac{p}{2}-1} \right] dx \quad (47)$$

$$= \frac{2A}{p-2} \left\{ 2 \int_0^1 \frac{1-x^{\frac{p}{2}-1}}{1-x^2} dx + \frac{2}{p} \left(1 - \frac{p}{2} \right) \right\}. \quad (48)$$

The value of this last integral is given explicitly by [32], §3.269, in terms of the digamma function Ψ . Hence

$$\frac{1}{A} = \frac{1}{2} \frac{\Psi(p/4) - \Psi(1/2)}{p/4 - 1/2} - \frac{2}{p}. \quad (49)$$

The case $p = 2$ may be studied by letting $p \rightarrow 2$ in Equations (46,49). We get

$$f_{2,1,1}(z) = A \left[\frac{1}{2z} \ln \left| \frac{z+1}{z-1} \right| - \frac{1}{(z+1)^2} \right] \quad \text{and} \quad \frac{1}{A} = \frac{\pi^2}{4} - 1. \quad (50)$$

Plots of the density (46) for several values of p are shown in Figure 1. The integral on

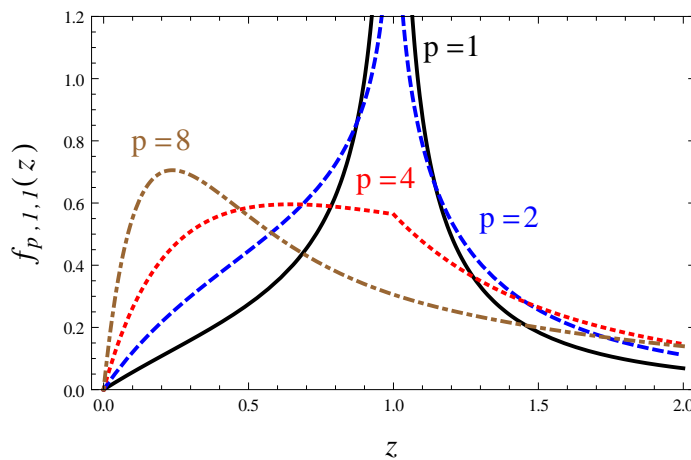


Figure 1. Plots of the stationary probability density $f(z)$ for $q = 1$ and energy $E = -1$.

the right-hand side of Formula (21) may also be computed exactly; the end result is

$$\Omega(-k^2) = \frac{p}{2} \frac{1 - \frac{4k^2}{p^2}}{\Psi\left(\frac{p}{4k}\right) - \Psi\left(\frac{1}{2}\right) + \frac{2k}{p} - 1}. \quad (51)$$

Analytic continuation to positive energies $E = k^2 > 0$ consists of replacing k by $-ik$; this yields

$$\Omega(E + i0+) = \frac{p}{2} \frac{1 + \frac{4k^2}{p^2}}{\Psi\left(\frac{ip}{4k}\right) - \Psi\left(\frac{1}{2}\right) - \frac{2ik}{p} - 1}. \quad (52)$$

Various limits may be analysed with the help of these expressions. Using

$$\Psi(z) \underset{z \rightarrow 0}{=} -\frac{1}{z} - \mathbf{C} + \frac{\pi^2}{6}z + \mathcal{O}(z^2)$$

where $\mathbf{C} = 0.577\dots$ is the Euler-Mascheroni constant, we deduce from (51)

$$\Omega(-k^2) \underset{k \rightarrow \infty}{=} k + p \left(\ln 2 - \frac{1}{2} \right) + \mathcal{O}(k^{-1}). \quad (53)$$

Analytic continuation then shows that $N(k^2) \sim k/\pi$ in the limit $k \rightarrow \infty$, as expected from the free case. Also, the high-energy Lyapunov exponent tends to a constant: $\gamma(k^2) \sim p(\ln 2 - 1/2)$; see Fig. 2.

In the low-energy limit, by using

$$\Psi(z) \underset{z \rightarrow \infty}{=} \ln z - \frac{1}{2z} + \mathcal{O}(z^{-2}),$$

we obtain

$$\Omega(-k^2) \underset{k \rightarrow 0}{=} \frac{p/2}{\ln(\frac{p}{k}) + \mathbf{C} - 1} + \mathcal{O}\left(\frac{k^2}{\ln k}\right). \quad (54)$$

Analytic continuation to positive energies allows one to recover the Dyson singularity of the integrated density of states and the corresponding vanishing of the Lyapunov exponent:

$$N(k^2) \sim \frac{g}{2 \ln^2 k} \quad \text{and} \quad \gamma(k^2) \sim \frac{g}{|\ln k|}$$

with $g = p/2$. In this $k \rightarrow 0$ limit, we thus recover as expected the same results as in the case where the superpotential W is a Gaussian white noise; the characteristic function for this case is recalled below. The fact that $\gamma \rightarrow 0+$ as $E \rightarrow 0+$ means that the eigenstates become delocalised (or extended) in that limit; see the discussion in Section 4. Plots of γ and N for positive energy are shown in Figure 2.

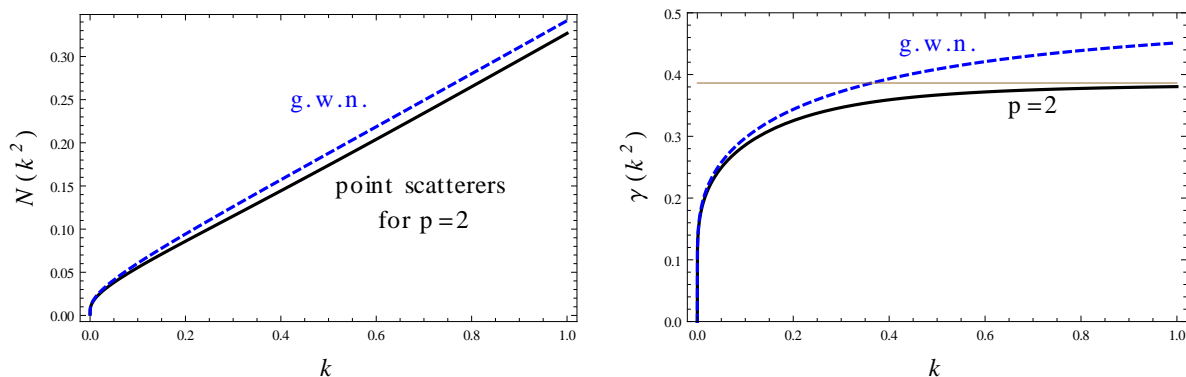


Figure 2. Plots of N and γ against $k = \sqrt{E}$ for $p = 2$ and $q = 1$. The integrated density of states exhibits the Dyson singularity. For comparison, the results corresponding to the case where W is a Gaussian white noise of strength $g = p/2q^2 = 1$ are also shown as blue dashed curves. The thin line is the asymptote of the Lyapunov exponent.

3.3. The characteristic function for arbitrary $q > 0$

Although the calculations are somewhat tedious, the more general case $q > 0$ is also tractable. The results are summarised in the following

Theorem 1. *The invariant probability density is given by*

$$f_{p,q,1}(z) = \frac{C}{z} \int_0^\infty e^{-\frac{x}{4}(z+1/z)} M_{\frac{p}{4},q}(x) \frac{dx}{\sqrt{x}}.$$

where C is a normalisation constant and $M_{\alpha,\beta}$ is a Whittaker function.

Corollary 3.1. *For $E = k^2 > 0$.*

$$\Omega(E + i0+) = -ik \frac{q+1}{q} \frac{{}_3F_2\left(q + \frac{1}{2} - \frac{ip}{4k}, q+2, q; 2q+1, q + \frac{3}{2}; 1\right)}{{}_3F_2\left(q + \frac{1}{2} - \frac{ip}{4k}, q+1, q+1; 2q+1, q + \frac{3}{2}; 1\right)}.$$

Proof. See Appendix A. □

Although this formula for Ω in terms of a known special function is pleasing, the task of extracting from it concrete information is not entirely straightforward. We proceed to discuss various limits.

The integrated density of states behaves in the limit $E \rightarrow +\infty$ as in the free case. We may obtain the behaviour of γ by using the perturbative approach described in Ref. [4]. When the superpotential W is given by a superposition of delta-functions, the high energy Lyapunov exponent of the supersymmetric Hamiltonian tends to a finite limit given by $\gamma_\infty = p \mathbb{E}(\ln \cosh w)$. Therefore we obtain

$$\gamma(E) \sim \frac{p}{2} \left[\frac{1}{q} - \Psi\left(\frac{q}{2} + 1\right) + \Psi\left(\frac{q+1}{2}\right) \right] \quad \text{as } E \rightarrow \infty. \quad (55)$$

For example, in the case $q = 1$, we recover the limit $\gamma \sim p(\ln 2 - 1/2)$ discussed earlier.

The low-energy limit may be conveniently studied by using the integral representation of the characteristic function

$$\Omega(-k^2) = k \frac{\int_0^\infty \frac{dx}{\sqrt{x}} M_{\frac{p}{4k},q}(x) K_1(x/2)}{\int_0^\infty \frac{dx}{\sqrt{x}} M_{\frac{p}{4k},q}(x) K_0(x/2)} \quad (56)$$

following from (A.25). We set $E = -k^2 = -1$ for simplicity. The $p \rightarrow \infty$ limit provides the $E \rightarrow 0$ behaviour, after reintroducing k . We now analyse the integrals in this limit. There holds [32]

$$M_{\frac{p}{4},q}(x) \sim x^{q+1/2} \quad \text{as } x \rightarrow 0 \quad (57)$$

and, since the Whittaker function solves the differential equation

$$w''(x) = \frac{x^2 - px + (4q^2 - 1)}{4x^2} w(x),$$

we expect the asymptotic form (57) to hold for x smaller than $(4q^2 - 1)/p$ (this reasoning assumes that $4q^2 - 1 > 0$). Moreover, we see from the differential equation that

the Whittaker function is a rapidly oscillating function in the interval $[x_-, x_+]$ where $x_{\pm} = \frac{p}{2} \pm \sqrt{(\frac{p}{2})^2 - 4q^2 + 1}$; in the limit of interest here we have $x_- \sim (4q^2 - 1)/p$ and $x_+ \sim p$. Thus we expect that the contribution to the integrals of the interval $[x_-, x_+]$ is negligible. For x exceeding x_+ we have

$$\frac{1}{\sqrt{x}} M_{\frac{p}{4}, q}(x) K_s(x/2) \sim x^{-\frac{p}{4}-1}.$$

Finally both integrals in (56) are dominated by the interval $[0, x_-]$, from which we get

$$\Omega(-1) \underset{p \rightarrow \infty}{\sim} \frac{q+1}{q} \frac{2}{x_- \ln(1/x_-)} \sim \frac{p}{\ln p}. \quad (58)$$

Re-introducing k , we indeed get the expected behaviour $\Omega(-k^2) \sim -1/\ln k$ for $k \rightarrow 0$.

The limit $p \rightarrow \infty$ and $q \rightarrow \infty$ keeping p/q^2 fixed corresponds to a case where $W(x)$ converges to a Gaussian white noise of strength $g = p \mathbb{E}(w^2) = p/2q^2$ — a model that has been solved in Ref. [7, 52] (see also [41]). Therefore we expect that the characteristic function (56), i.e. Eq. (A.30), converges in this limit towards the expression

$$\Omega(-k^2) \underset{p \rightarrow \infty, q = \sqrt{p/2g}}{\longrightarrow} k \frac{K_1(k/g)}{K_0(k/g)}. \quad (59)$$

4. Concluding remarks: limitations of the Lyapunov exponent for the study of localisation

To close this short review, we point out the limitations of the Lyapunov exponent as a means of characterising the localised nature of the spectrum in a disordered system. A first observation is that the definition (1) relies on the assumption of a linear increase of $\ln |\psi(x; E)|$, i.e. on the application of the central limit theorem. It is however possible to consider models of disorder such that $\ln |\psi(x; E)|$ scales as x^α , leading to the phenomenon of sub-localisation (for $\alpha < 1$) or super-localisation (for $\alpha > 1$); this has been studied in, for example, Refs. [4, 5, 43, 58] (see also references therein). However, this limitation is easily overcome by extending the notion of the Lyapunov exponent so that it characterises the typical length over which eigenstates are localised.

Another, more serious, limitation of the usefulness of the Lyapunov exponent pertains to the possible existence of extended states in one dimension. This might occur when the disordered potential is correlated [36]— as happens for instance in the dimer model. It might occur also for symmetry reasons, as is the case for the supersymmetric Hamiltonians studied in Refs. [7, 13, 14, 33, 61] and in section 3; there, the Lyapunov exponent $\gamma(E)$ decays to zero in the limit of zero energy. In this respect, it must first be pointed out that the vanishing of the Lyapunov exponent is a necessary but not a sufficient condition for delocalisation; sometimes it only signals sub-localisation. Secondly, several works on disordered supersymmetric quantum mechanics have shown that the Lyapunov exponent, when it vanishes at the bottom of the spectrum does not provide the relevant length scale characterising the localisation of eigenstates at low non-zero energies. As mentioned already in section 3, when $\mathbb{E}(W) = 0$, the Lyapunov

exponent vanishes like $\gamma(E) \sim g/|\ln E|$ as $E \rightarrow 0$ [7, 13, 14]. This suggests that low-energy eigenstates are localised on a scale commensurate with $|\ln E|/g$. However, the study of the ordered statistics problem [60, 61], as well as the analysis of other physical quantities such as the averaged Green's function at non coinciding points [29, 7] or the boundary-sensitive averaged density of states [61], show that eigenstates are localised on a much larger scale, namely $|\ln E|^2/g$.

The inability of the Lyapunov exponent to capture the localisation property for energies $E \sim E_c$ where $\gamma(E_c) = 0$ may be related to the fact that it is defined, via Eq. (1), in terms of a solution $\psi(x; E)$ of the Cauchy problem that vanishes at just *one* boundary. This definition may indeed fail to describe the interesting properties of the real eigenstates— which are solutions of the Schrödinger equation vanishing at the *two* boundaries $x = 0$ and $x = L$ — as these eigenstates become less localised.

Acknowledgements

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Appendix A. Proofs of Theorem 1 and its corollary

Let us look for a solution of the differential equation (45) of the form

$$\varphi(z) := z^q \left(\frac{1-z}{1+z} \right)^{\frac{p}{2}} y(z).$$

Then

$$y'' + \left[\frac{2q+1}{z} + \frac{p/2}{z-1} + \frac{-p/2}{z+1} \right] y' + \frac{pq}{z(z-1)(z+1)} y = 0. \quad (\text{A.1})$$

The solution we require has various representations in terms of special functions.

Appendix A.1. Heun's function

Heun's differential equation is [57]

$$y'' + \left[\frac{\eta}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a} \right] y' + \frac{\alpha\beta z - q}{z(z-1)(z-a)} y = 0 \quad (\text{A.2})$$

where the parameters satisfy the relation

$$\alpha + \beta + 1 = \eta + \delta + \varepsilon.$$

Heun's function is the particular solution analytic inside the unit disk:

$$Hl(a, q; \alpha, \beta, \eta, \delta; z) := \sum_{n=0}^{\infty} y_n z^n, \quad |z| < 1, \quad (\text{A.3})$$

where the y_n satisfy the recurrence relation

$$\begin{aligned} & a(n+2)(n+1+\eta)y_{n+2} \\ & - [(n+1)(n+\eta+\delta)a + (n+1)(n+\eta+\varepsilon) + q] y_{n+1} \\ & + (n+\alpha)(n+\beta)y_n = 0 \end{aligned} \quad (\text{A.4})$$

with the starting values

$$y_0 = 1 \quad \text{and} \quad y_1 = \frac{q}{a\eta}. \quad (\text{A.5})$$

Comparing Equation (A.1) with Heun's equation (A.2), we find the particular solution

$$y = Hl(-1, -pq; 0, 2q, 2q+1, p/2; z).$$

The recurrence relation for the coefficients y_n of this solution is

$$(n+2)(n+2q+2)y_{n+2} = p(n+q+1)y_{n+1} + n(n+2q)y_n \quad (\text{A.6})$$

with the starting values

$$y_0 = 1 \quad \text{and} \quad y_1 = \frac{pq}{2q+1}. \quad (\text{A.7})$$

It follows in particular that y is strictly positive for $z > 0$. Hence, for $z \in [0, 1)$,

$$f_{p,q,1}(z) = \frac{Cz^q}{1-z^2} \left(\frac{1-z}{1+z} \right)^{\frac{p}{2}} Hl(-1, -pq; 0, 2q, 2q+1, p/2; z) \quad (\text{A.8})$$

where C is a normalisation constant. The property (38) provides the obvious formula in the interval $z > 1$.

Appendix A.2. Gauss' hypergeometric function

Set

$$a := \frac{p}{4} + q - \frac{1}{2}, \quad b := \frac{p}{4} \quad \text{and} \quad c := q + \frac{1}{2}.$$

Then, by comparing the Taylor series, we find that

$$\begin{aligned} & Hl(-1, -pq; 0, 2q, 2q+1, p/2; z) \\ & = \frac{(1+z)^{2b}}{c(c+1)} \left\{ (a+1)(b+1)z^2(1-z^2) {}_2F_1(a+2, b+2; c+2; z^2) \right. \\ & \left. + (c+1)[c(1-z) - bz](1+z) {}_2F_1(a+1, b+1; c+1; z^2) \right\}. \end{aligned} \quad (\text{A.9})$$

The contiguity relation

$$\begin{aligned} & t(1-t)(a+1)(b+1) {}_2F_1(a+2, b+2; c+2; t) \\ & + [c - (a+b+1)t](c+1) {}_2F_1(a+1, b+1; c+1; t) \\ & - c(c+1) {}_2F_1(a, b; c; t) = 0 \end{aligned} \quad (\text{A.10})$$

leads to the simpler expression

$$\begin{aligned} & Hl(-1, -pq; 0, 2q, 2q+1, p/2; z) \\ & = (1+z)^{2b} \left\{ {}_2F_1(a, b; c; z^2) - \frac{b}{c}z(1-z) {}_2F_1(a+1, b+1; c+1; z^2) \right\}. \end{aligned} \quad (\text{A.11})$$

The differentiation formula

$${}_2F_1'(a, b; c; t) = \frac{ab}{c} {}_2F_1(a+1, b+1; c+1; t)$$

then yields

$$\begin{aligned} & Hl(-1, -pq; 0, 2q, 2q+1, p/2; z) \\ &= (1+z)^{2b} \left\{ {}_2F_1(a, b; c; z^2) - \frac{z}{a}(1-z) {}_2F_1'(a, b; c; z^2) \right\} \end{aligned} \quad (\text{A.12})$$

$$= (1+z)^{2b} \left\{ {}_2F_1(a, b; c; z^2) - \frac{1}{2a}(1-z) \frac{d}{dz} {}_2F_1(a, b; c; z^2) \right\} \quad (\text{A.13})$$

$$\begin{aligned} &= \frac{-1}{2a} \frac{(1+z)^{2b}}{(1-z)^{2a-1}} \left\{ -2a(1-z)^{2a-1} {}_2F_1(a, b; c; z^2) \right. \\ &\quad \left. + (1-z)^{2a} \frac{d}{dz} {}_2F_1(a, b; c; z^2) \right\}. \end{aligned} \quad (\text{A.14})$$

So we obtain the ‘‘compact’’ representation

$$Hl(-1, -pq; 0, 2q, 2q+1, p/2; z) = -(1-z)^{2(1-q)} \left(\frac{1+z}{1-z} \right)^{\frac{p}{2}} F'_{p,q}(z), \quad (\text{A.15})$$

where

$$F_{p,q}(z) := \frac{(1-z)^{\frac{p}{2}+2q-1}}{\frac{p}{2}+2q-1} {}_2F_1\left(\frac{p}{4}+q-\frac{1}{2}, \frac{p}{4}; q+\frac{1}{2}; z^2\right). \quad (\text{A.16})$$

Appendix A.3. The associated Legendre function

Formula (6) in [22], §3.13, expresses the hypergeometric function in terms of the associated Legendre function:

$$\begin{aligned} & \Gamma\left(q + \frac{1}{2}\right) Q_{q-1}^{\frac{p}{4}-\frac{1}{2}}(\cosh \eta) = 2\sqrt{\pi} e^{i(\frac{p}{4}-\frac{1}{2})\pi} \Gamma\left(\frac{p}{4} + q + \frac{1}{2}\right) \\ & \times \left[\frac{e^{-\eta}}{(1-e^{-\eta})^2} \right]^q \left(\frac{1+e^{-\eta}}{1-e^{-\eta}} \right)^{\frac{p}{4}-\frac{1}{2}} F_{p,q}(e^{-\eta}). \end{aligned} \quad (\text{A.17})$$

Equation (8) in [32], §7.621, gives a useful integral representation of the Legendre function in terms of a Whittaker function:

$$\begin{aligned} & \int_0^\infty e^{-\frac{x}{4}(z+1/z)} M_{\frac{p}{4}-\frac{1}{2}, q-\frac{1}{2}}(x) \frac{dx}{x} \\ &= 2 \frac{\Gamma(2q)}{\Gamma(q + \frac{p}{4} - \frac{1}{2})} e^{-i\frac{p}{4}-\frac{1}{2}\pi} \left(\frac{1-z}{1+z} \right)^{\frac{p}{4}-\frac{1}{2}} Q_{q-1}^{\frac{p}{4}-\frac{1}{2}}\left(\frac{z+1/z}{2}\right). \end{aligned} \quad (\text{A.18})$$

Then, by Equation (A.17), we obtain

$$\begin{aligned} & - \left[\frac{z}{(1-z)^2} \right]^q \frac{1-z}{1+z} F'_{p,q}(z) = \frac{1}{4\sqrt{\pi}(\frac{p}{4} - \frac{1}{2} + q)} \frac{\Gamma(q+1/2)}{\Gamma(2q)} \\ & \times \int_0^\infty e^{-\frac{x}{4}(z+1/z)} M_{\frac{p}{4}-\frac{1}{2}, q-\frac{1}{2}}(x) \left[\frac{q}{xz} - \left(\frac{1-z}{2z} \right)^2 \right] dx. \end{aligned} \quad (\text{A.19})$$

Now,

$$\begin{aligned} M_{\frac{p}{4}-\frac{1}{2},q-\frac{1}{2}}(x) \left[\frac{q}{xz} - \left(\frac{1-z}{2z} \right)^2 \right] &= M_{\frac{p}{4}-\frac{1}{2},q-\frac{1}{2}}(x) \left[\left(\frac{q}{x} + \frac{1}{2} \right) \frac{1}{z} - \frac{1+z^2}{4z^2} \right] \\ &= \frac{\frac{p}{4} + q - \frac{1}{2}}{2q\sqrt{x}} \frac{1}{z} M_{\frac{p}{4},q}(x) + \frac{1}{z} M'_{\frac{p}{4}-\frac{1}{2},q-\frac{1}{2}}(x) - \frac{1+z^2}{4z^2} M_{\frac{p}{4}-\frac{1}{2},q-\frac{1}{2}}(x) \end{aligned} \quad (\text{A.20})$$

where we have used Formulae (13.4.28) and (13.4.32) from [1] to obtain the last equality. The proof of Theorem 1 follows after reporting this in Equation (A.19) and using integration by parts for the term involving the derivative of the Whittaker function.

Remark. Formula (5) in [22], §2.8, expresses $F_{p,0}$ in simple terms:

$$F_{p,0}(z) = \frac{1}{p-2} \left[\left(\frac{1-z}{1+z} \right)^{\frac{p}{2}-1} + 1 \right]. \quad (\text{A.21})$$

It then follows from Equation (A.17) and from the recurrence relation for the associated Legendre function (see Formula (18) in [22], §3.8) that, for $q \in \mathbb{N}$, $F_{p,q}$ may be expressed in terms of elementary functions. For example, we have

$$F_{p,1}(z) = \frac{1/8}{\left(\frac{p}{4}-\frac{1}{2}\right)\left(\frac{p}{4}+\frac{1}{2}\right)} \frac{(1-z)^2}{z} \left[1 - \left(\frac{1-z}{1+z} \right)^{\frac{p}{2}-1} \right] \quad (\text{A.22})$$

and

$$\begin{aligned} F_{p,2}(z) &= \frac{3/32}{\left(\frac{p}{4}-\frac{3}{2}\right)\left(\frac{p}{4}-\frac{1}{2}\right)\left(\frac{p}{4}+\frac{1}{2}\right)\left(\frac{p}{4}+\frac{3}{2}\right)} \\ &\times \frac{(1-z)^4}{z^3} \left\{ \left(\frac{1-z}{1+z} \right)^{\frac{p}{2}-1} \left[z^2 + \left(\frac{p}{2} - 1 \right) z + 1 \right] - \left[z^2 - \left(\frac{p}{2} - 1 \right) z + 1 \right] \right\}. \end{aligned} \quad (\text{A.23})$$

Formulae for $f_{p,1,1}$ and $f_{p,2,1}$ are easily deduced.

Appendix A.4. The characteristic function

Define

$$\hat{f}_{p,q,k}(s) := \int_0^\infty z^{-s} f_{p,q,k}(z) dz. \quad (\text{A.24})$$

Using Theorem 1 and changing the order of integration, we find

$$\hat{f}_{p,q,1}(s) = 2C \int_0^\infty M_{\frac{p}{4},q}(x) K_s \left(\frac{x}{2} \right) \frac{dx}{\sqrt{x}} = 2\sqrt{\pi} C \int_0^\infty M_{\frac{p}{4},q}(x) W_{0,s}(x) \frac{dx}{x}. \quad (\text{A.25})$$

Next, Formula (7) in [22], §6.9, says

$$M_{\frac{p}{4},q}(x) = (-1)^{q+\frac{1}{2}} M_{-\frac{p}{4},q}(-x).$$

Hence

$$\hat{f}_{p,q,1}(s) = 2\sqrt{\pi} C (-1)^{q+\frac{1}{2}} \int_0^\infty M_{-\frac{p}{4},q}(-x) W_{0,s}(x) \frac{dx}{x}. \quad (\text{A.26})$$

The integral on the right-hand side is of the same form as that in [32], §7.625, Formula (1)—albeit outside the range indicated since in our case $\alpha = -1$ and $\beta = 1$.

Nevertheless, direct numerical verification indicates that the formula does remain valid and so

$$\begin{aligned} \hat{f}_{p,q,1}(s) &= 2\sqrt{\pi} C \frac{\Gamma(q+1+s)\Gamma(q+1-s)}{\Gamma\left(q+\frac{3}{2}\right)} \\ &\times {}_3F_2\left(q+\frac{1}{2}-\frac{p}{4}, q+1+s, q+1-s; 2q+1, q+\frac{3}{2}; 1\right). \end{aligned} \quad (\text{A.27})$$

The normalisation condition

$$\hat{f}_{p,q,1}(0) = 1$$

yields

$$C = \frac{\Gamma\left(q+\frac{3}{2}\right)}{2\sqrt{\pi}\Gamma(q+1)^2 {}_3F_2\left(q+\frac{1}{2}-\frac{p}{4}, q+1, q+1; 2q+1, q+\frac{3}{2}; 1\right)} \quad (\text{A.28})$$

and so we deduce the formula:

$$\begin{aligned} \hat{f}_{p,q,1}(s) &= \frac{\Gamma(q+1+s)\Gamma(q+1-s)}{\Gamma(q+1)^2} \\ &\times \frac{{}_3F_2\left(q+\frac{1}{2}-\frac{p}{4}, q+1+s, q+1-s; 2q+1, q+\frac{3}{2}; 1\right)}{{}_3F_2\left(q+\frac{1}{2}-\frac{p}{4}, q+1, q+1; 2q+1, q+\frac{3}{2}; 1\right)} \end{aligned} \quad (\text{A.29})$$

for $-1 \leq \text{Re } s \leq 1$. Then, using Formula (21) for the Lyapunov exponent, together with the identity (37) and the fact that $N(E) = 0$ for $E < 0$, we eventually find

$$\Omega(-k^2) = k \frac{q+1}{q} \frac{{}_3F_2\left(q+\frac{1}{2}-\frac{p}{4k}, q+2, q; 2q+1, q+\frac{3}{2}; 1\right)}{{}_3F_2\left(q+\frac{1}{2}-\frac{p}{4k}, q+1, q+1; 2q+1, q+\frac{3}{2}; 1\right)}. \quad (\text{A.30})$$

The analytic continuation from negative to positive energy consists of replacing k by $-ik$. This completes the proof of Corollary 3.1.

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