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NORMAL FORMS *À LA MOSER* FOR APERIODICALLY TIME-DEPENDENT HAMILTONIANS IN THE VICINITY OF A HYPERBOLIC EQUILIBRIUM.

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ABSTRACT. The classical theorem of Moser, on the existence of a normal form in the neighbourhood of a hyperbolic equilibrium, is extended to a class of real-analytic Hamiltonians with aperiodically time-dependent perturbations. A stronger result is obtained in the case in which the perturbing function exhibits a time decay.

1. INTRODUCTION

The classical theorem of Moser, proven in [Mos56], establishes the existence of a (convergent) normal form in a neighbourhood of a hyperbolic equilibrium of an area preserving map, either autonomous or periodically dependent on time. A result contained in [CG94], extends this result to the the flow of a priori unstable system in a neighbourhood of a partially hyperbolic torus, including in this way the quasiperiodic case. A concise description of the latter case can be found in [Gal97].

The aim of this paper is to show the existence of a normal form for Hamiltonians in the form (1), i.e. real-analytic and non-autonomous perturbations of a hyperbolic equilibrium, for which the time dependence is not required to be periodic or quasiperiodic i.e. *aperiodic*.

In the same spirit of the aperiodic version of the Kolmogorov theorem of [FW14a], which we use as a guideline (see also [Giob]), the proof consists on the extension of the KAM approach of [CG94] and [Gal97]. Even in the original problem of Moser, despite the absence of "genuine" small divisors[1], the well known property of *superconvergence* of the KAM schemes, turns out to be of crucial importance in order to compensate the accumulation of "artificial" divisors generated by the Cauchy estimates. This feature is profitably used also in our case.

The treatment of the class of time-dependent homological equations, naturally arising in the normalization algorithm, has been improved with respect to [FW14a]. Basically, the canonical transformation on which the single step of the mentioned algorithm is based, has the property to leave the time unchanged[2]. Hence, this can be interpreted as a family of canonical maps for which the time plays the role of "parameter". This allows to weaken the analyticity hypothesis for the time dependence leading to a remarkable simplification of the quantitative estimates.

The proof is carried out by using the formalism of the Lie series method developed by Giorgilli et al. (see e.g. [Gio03] and references therein).

2. PRELIMINARIES AND STATEMENT OF THE RESULT

Let us consider the following Hamiltonian

$$H(p,q,\eta,t) = \omega pq + \eta + F(p,q,t), \tag{1}$$

where $\omega \in (0,1]$, $(p,q,\eta) \in [-r,r]^2 \times \mathbb{R} =: \mathcal{D}$ with $r > 0$ and $t \in \mathbb{R}^+ := [0,\infty)$. As usual, Hamiltonian (1) is equivalent to the non-autonomous Hamiltonian $\mathcal{H}(p,q,t) = \omega pq + F(p,q,t)$ (which represents

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[1]This is a common feature with the "non-purely hyperbolic" case treated in [Gioa].

[2]This class of transformations was initially considered in [GZ92].

our original problem), by defining as η the conjugate variable to t.
The function F will be supposed to be real-analytic in p and q and such that, denoted as $f_{\underline{\alpha}}(t)$ its Taylor coefficients, one has $f_{\underline{\alpha}}(t) = 0$ for all[3] $|\underline{\alpha}| \leq 2$, and all $t \in \mathbb{R}^+$. Namely, the Taylor expansion of F starts from the terms of degree 3.
The standard framework for the analysis, features the complexification of the domain $\mathcal{D}$ as follows.
Let $R \in (0, 1/2]$ and define

$$\mathcal{Q}_R := \{(p,q) \in \mathbb{C}^2 : |p|, |q| \leq R\}, \qquad \mathcal{S}_R := \{\eta \in \mathbb{C} : |\Im \eta| \leq R\},$$

then set $\mathcal{D}_R := \mathcal{Q}_R \times \mathcal{S}_R$. The perturbation F will be supposed continuous on $\mathcal{Q}_R$ and holomorphic in the interior for all $t \in \mathbb{R}^+$ (then H is on $\mathcal{D}_R$) for some R. It will be sufficient to suppose that the real and imaginary parts of the complex valued functions $f_{\underline{\alpha}}(t)$ belong to $\mathcal{C}^1(\mathbb{R}^+)$ for all $\underline{\alpha}$.
Given a function $G : \mathcal{Q}_R \times \mathbb{R}^+ \to \mathbb{C}$, we consider the *Taylor norm*

$$\|G(p,q,t)\|_R := \sum_{\underline{\alpha}} |g_{\underline{\alpha}}(t)|_+ R^{|\underline{\alpha}|}, \tag{2}$$

where $|\cdot|_+ := \sup_{t \in \mathbb{R}^+} |\cdot|$. Clearly $|G|_R := \sup_{\mathcal{Q}_R} |G|_+ \leq \|G\|_R$. We briefly recall the following standard result (which motivates the above described assumptions on F): if a function G is continuous on $\mathcal{Q}_R$ and holomorphic in the interior, for all $t \in \mathbb{R}^+$, one has $|g_{\underline{\alpha}}(t)|_+ \leq |G|_R R^{-|\underline{\alpha}|}$. In particular, $\|G\|_{R'} < +\infty$ for all $R' < R$.
In the described setting the main result can be stated as follows

Theorem 2.1 (Aperiodic Moser '56). *Suppose that* $1 + \|F(p,q,t)\|_R =: M_F < \infty$. *Then there exist* R_*, R_0 *with* $0 < R_* < R_0 \leq R^4$ *and a family of canonical changes* $\mathcal{M} : \mathcal{D}_{R_*} \to \mathcal{D}_{R_0}$, *analytic on* $\mathcal{D}_{R_*}$ *for all* $t \in \mathbb{R}^+$, *casting the Hamiltonian (1) in the* time-dependent Moser normal form

$$H^{(\infty)}(p^{(\infty)}, q^{(\infty)}, \eta^{(\infty)}, t) = J^{(\infty)}(x^{(\infty)}, t) + \eta^{(\infty)}, \tag{3}$$

where $x := pq$, $J^{(\infty)}(0,t) = 0$ *and* $\partial_x J^{(\infty)}(0,t) = \omega$ *for all* $t \in \mathbb{R}^+$.

Exactly as in the classical Moser theorem, the quantity $x^{(\infty)}$ is a first integral, hence the flow associated to Hamiltonian (3) can be reduced to quadratures. In particular, one has

$$p^{(\infty)}(t) = p^{(\infty)}(0) \exp(-\mathcal{A}(x^{(\infty)}(0), t)), \qquad q^{(\infty)}(t) = q^{(\infty)}(0) \exp(\mathcal{A}(x^{(\infty)}(0), t)),$$

where $\mathcal{A}(x,t) := \int_0^t \partial_x J^{(\infty)}(x,s) ds$.
The use of an additional ingredient leads to an even stronger result. Given $G : \mathcal{Q}_R \times \mathbb{R}^+ \to \mathbb{C}$ we define as the "time-dependent" Taylor norm of G, the quantity $\|G\|_{R;\mathbb{R}^+} := \sum_{\underline{\alpha}} |g_{\underline{\alpha}}(t)| R^{|\underline{\alpha}|}$, i.e. (2) in which $|\cdot|_+$ is replaced with $|\cdot|$. Now we introduce the next

Hypothesis 2.2. (Slow decay) Suppose that there exist $M_F \in [1, +\infty)$ and $a > 0$ such that

$$\|F(p,q,t)\|_{R;\mathbb{R}^+} \leq M_F e^{-at}, \tag{4}$$

for all $(p,q,t) \in \mathcal{Q}_R \times \mathbb{R}^+$.

In this way we are able to prove the following

Theorem 2.3 (Strong Aperiodic Moser '56). *Under Hypothesis 2.2 it is possible to determine* $0 < \hat{R}_* < \hat{R}_0 \leq R^4$ *and a family of canonical transformations* $\mathcal{M}_{\mathcal{S}}$, *analytic on* $\mathcal{D}_{\hat{R}_*}$ *for all* $t \in \mathbb{R}^+$, *for which the Hamiltonian (1) is transformed into the* strong Moser normal form

$$\hat{H}^{(\infty)}(\hat{p}^{(\infty)}, \hat{q}^{(\infty)}, \hat{\eta}^{(\infty)}, t) = \omega \hat{x}^{(\infty)} + \hat{\eta}^{(\infty)}. \tag{5}$$

[3]It will be understood throughout the paper $\underline{\alpha} \in \mathbb{N}^2$, denoting $|\underline{\alpha}| := \alpha_1 + \alpha_2$.

The Hypothesis 2.2, already used in [FW14a], turns out to be necessary in order to ensure the existence of certain improper integrals, which appear when dealing with time-dependent homological equations. As in the latter paper, this particular rate of decay is assumed only for simplicity of discussion. Similarly, we stress that no lower bounds are imposed on a (except zero), in this way the time decay can be arbitrarily slow. The natural side-effect is that the estimates on the convergence radius of the normal form worsen as a is smaller and smaller.
It should be stressed that, in both cases, the choice of ω in the interval $(0,1]$ is discussed as the "interesting" case. On the other hand, it is clear that the contribution of the time perturbation is smaller as ω increases[4]. That is why, the case $\omega \geq 1$ can be treated with the same tools leading, in general, to easier estimates.
The proof of Theorem 2.1 is (traditionally) achieved in two steps. In the first one (Sec. 3), a suitable normalization algorithm is constructed and discussed at a formal level. In the second part (Sec. 5) the problem of its convergence is addressed, after having stated some tools of a technical nature (Sec. 4). Proof of Theorem 2.3 is just a *variazione sul tema*. The necessary modifications are outlined in Sec. 6.

3. The formal perturbative setting

The formal perturbative algorithm has the typical inductive structure. To start, we shall suppose that Hamiltonian (1) can be written at the $j-$th stage of the normalization process as

$$H^{(j)}(p,q,\eta,t)=\tilde{J}^{(j)}(x,t)+\eta+\tilde{F}^{(j)}(p,q,t), \tag{6}$$

with $\tilde{F}^{(j)}$ at least of degree 3 in p,q. It is immediate to realize that (1) is in the form (6) so that we can set $H^{(0)}:=H$. Our aim is to construct a class of canonical transformations $\mathcal{M}_j$, parametrised by t, such that $H^{(j+1)}:=H^{(j)}\circ\mathcal{M}_j$ is still of the form (6). Roughly, the transformations $\mathcal{M}_j$ will be determined in such a way the "mixed" terms, i.e. of the form $p^{\alpha_1}q^{\alpha_2}$ with $\alpha_1\neq\alpha_2$ contained in the perturbation, are "gradually" removed as j increases, while the terms of the form $(pq)^n$ are progressively stored in $\tilde{J}^{(j)}$. This effect will be quantified in the next section, showing that the size of the "residual" perturbation is asymptotic to zero, as $j\to\infty$. Hence one sets

$$\mathcal{M}:=\lim_{j\to\infty}\mathcal{M}_j\circ\mathcal{M}_{j-1}\circ\ldots\circ\mathcal{M}_0, \tag{7}$$

so that, at least formally, $H^{(\infty)}=H\circ\mathcal{M}$.
First of all we write

$$\tilde{F}^{(j)}(p,q,t)=\sum_{\substack{|\underline{\alpha}|\geq 3\\ \alpha_1\neq\alpha_2}}\tilde{f}^{(j)}_{\underline{\alpha}}(t)p^{\alpha_1}q^{\alpha_2}+\sum_{k\geq 2}\tilde{f}^{j}_{\underline{k}}(pq)^k=:F^{(j)}(p,q,t)+\Delta^{(j)}(x,t), \tag{8}$$

where $\underline{k}:=(k,k)$, then setting $J^{(j)}(x,t):=\tilde{J}^{(j)}(x,t)+\Delta^{(j)}(x,t)$, in such a way

$$H^{(j)}(p,q,t)=J^{(j)}(x,t)+\eta+F^{(j)}(p,q,t), \tag{9}$$

where $F^{(j)}$ contains only "mixed" terms.
Now we consider the action on $H^{(j)}$ of the transformation $\mathcal{M}_j$, which is defined by the the Lie series operator $\exp(\mathcal{L}_{\chi^{(j)}})=\mathrm{Id}+\mathcal{L}_{\chi^{(j)}}+\sum_{s\geq 2}(1/s!)\mathcal{L}^s_{\chi^{(j)}}$. We recall that $\mathcal{L}_G F=\{F,G\}=F_qG_p+F_tG_\eta-F_pG_q-F_\eta G_t$, while $\chi^{(j)}=\chi^{(j)}(p,q,t)$ is the (unknown) generating function. Supposing that it is possible to determine it in such a way

$$\mathcal{L}_{\chi^{(j)}}(J^{(j)}(x,t)+\eta)+F^{(j)}(p,q,t)=0, \tag{10}$$

[4]Namely, let $\mu:=O(\omega^{-1})$ and set $\hat{\omega}:=\mu\omega=O(1)$. Via a time rescaling $t=\mu\tau$, problem (1) is equivalent to the "slowly time-dependent" Hamiltonian $\hat{H}=\hat{\omega}pq+\mu\eta+\mu F(q,u,\mu\tau)$.

one has that, by setting $\tilde{J}^{(j+1)} := J^{(j)}$, and

$$\tilde{F}^{(j+1)} := \sum_{s\geq 1} \frac{1}{s!}\mathcal{L}^s_{\chi^{(j)}} F^{(j)} + \sum_{s\geq 2} \frac{1}{s!}\mathcal{L}^s_{\chi^{(j)}} (J^{(j)} + \eta), \tag{11}$$

the transformed Hamiltonian $H^{(j+1)} := \exp(\mathcal{L}_{\chi^{(j)}})H^{(j)}$ has exactly the form (6).
Note that by (10) and (11)

$$\begin{aligned}\tilde{F}^{(j+1)} &= \sum_{s\geq 1} \frac{1}{s!}\mathcal{L}^s_{\chi^{(j)}} \left[F^{(j)} + \frac{1}{s+1}\mathcal{L}_{\chi^{(j)}}(J^{(j)} + \eta)\right] \\ &= \sum_{s\geq 1} \frac{s}{(s+1)!}\mathcal{L}^s_{\chi^{(j)}} F^{(j)}. \end{aligned} \tag{12}$$

Defining $g^{(j)}(x,t) := \partial_x J^{(j)}(x,t)$ one has that equation (10) reads as

$$[g^{(j)}(x,t)\eth + \partial_t]\chi^{(j)}(p,q,t) = F^{(j)}(p,q,t), \tag{13}$$

having denoted $\eth := q\partial_q - p\partial_p$. Taking into account of the expansion $F^{(j)} =: \sum_{\substack{|\underline{\alpha}|\geq 3 \\ \alpha_1 \neq \alpha_2}} f^{(j)}_{\underline{\alpha}}(t)p^{\alpha_1}q^{\alpha_2}$, the solution of equation (13) reads as

$$\chi^{(j)}(p,q,t) = \sum_{\underline{\alpha}} \mathcal{F}_{\underline{\alpha}}(x,t)p^{\alpha_1}q^{\alpha_2}, \qquad \mathcal{F}^{(j)}_{\underline{\alpha}}(x,t) := e^{-\lambda A^{(j)}(x,t)}\left[\mathcal{F}^{(j)}_{\underline{\alpha},0}(x) + \int_0^t e^{\lambda A^{(j)}(x,s)} f^{(j)}_{\underline{\alpha}}(s)ds\right]. \tag{14}$$

where $A^{(j)}(x,t) := \int_0^t g^{(j)}(x,s)ds$, $\lambda := \alpha_2 - \alpha_1 \geq 1$ by hypothesis on $F^{(j)}$ and $\mathcal{F}^{(j)}_{\underline{\alpha},0}(x)$ are functions to be determined. Clearly, we shall set $\mathcal{F}^{(j)}_{\underline{\alpha},0}(x) \equiv 0$ for all $\underline{\alpha}$ such that $\alpha_1 = \alpha_2$ and such that $f^{(j)}_{\underline{\alpha}}(s) \equiv 0$ in such a way $\mathcal{F}^{(j)}_{\underline{\alpha}}(x,t)$ are identically zero for those values.
It is evident that as $|\underline{\alpha}| \geq 3$ for by hypothesis on $F^{(j)}$, the generating function $\chi^{(j)}$ will be at least of degree 3. This implies that, by (12), $\tilde{F}^{(j+1)}$ will be at least of degree 4, in particular it will not contain terms of degree 2. By hypothesis on $F \equiv F^{(0)}$ and by induction, this is true for all j, implying that $g^{(j)}(0,t) = \omega$ for all $t \geq 0$, i.e. $g^{(j)}$ has a strictly positive real part (by hypothesis on ω), in a suitable neighbourhood of the origin and more precisely via a suitable choice of R_0. This will play a crucial role in our later arguments. The formal part is complete.

Remark 3.1. It is immediate to recognize the similarity between equation (13) and those found in [FW14a] and [FW14b]. The main difference is the presence of the function $g^{(j)}(x,t)$ which requires a careful analysis about its variation on time, as anticipated above.

4. Some preliminary results

4.1. **Bounds on the solutions of the homological equation.** First of all let us recall the following elementary equality, valid for all $\lambda \in [0,1)$, which will be repeatedly used in the follow

$$\sum_{\underline{\alpha}} \lambda^{|\underline{\alpha}|} = \sum_{l\geq 0}(l+1)\lambda^l = (1-\lambda)^{-2}. \tag{15}$$

Then we state the next

Proposition 4.1. *Suppose the existence of a positive constant $M^{(j)}$ such that*

$$\left\|F^{(j)}(p,q,t)\right\|_{R_j} \leq M^{(j)}, \tag{16}$$

and that, for all $(x,t) \in \mathcal{Q}_{R_j} \times \mathbb{R}^+$ one has

$$\Re g^{(j)}(x,t) \geq \omega/2, \tag{17a}$$
$$|g^{(j)}(x,t)| \leq (3/2)\omega. \tag{17b}$$

Then for all $\delta \in (0,1)$ the solution of (13) satisfies

$$\left\|\chi^{(j)}(p,q,t)\right\|_{(1-\delta)R_j}, \left\|\partial_t\chi^{(j)}(p,q,t)\right\|_{(1-\delta)R_j} \leq \frac{4M^{(j)}}{\omega\delta^2}. \tag{18}$$

Remark 4.2. Note that hypothesis (17a) is essential as it is easy to find $g^{(j)}(x,t)$ satisfying (17b) for which the solution of (14) is unbounded on $\mathbb{R}^+$.

The proof goes along the lines of a similar result contained in [FW14a], with the remarkable simplification due to the fact that now t is purely real. The very minor drawback with respect to the "analytic" case treated in [FW14a], is that, in this case, the estimate of the time derivative does not follow directly from a Cauchy estimate.

Proof. Recall that by hypothesis on F one has $|f_{\underline{\alpha}}^{(j)}(s)|_+ \leq M^{(j)}R_j^{-|\underline{\alpha}|}$. If $\underline{\alpha}$ is such that $\lambda > 0$, we shall set $\mathcal{F}_{\underline{\alpha},0}^{(j)}(x) \equiv 0$. By (17a), we have that $\Re(A^{(j)}(x,t) - A^{(j)}(x,s)) \geq \omega(t-s)/2$ on $\mathcal{Q}_{R_j}$, yielding

$$|\mathcal{F}_{\underline{\alpha}}^{(j)}(x,t)| \leq M^{(j)}R_j^{-|\underline{\alpha}|}e^{-\frac{\lambda\omega t}{2}}\int_0^t e^{-\frac{\lambda\omega s}{2}}ds \leq \frac{2M^{(j)}}{\lambda\omega}R_j^{-|\underline{\alpha}|}. \tag{19}$$

In the case $\lambda < 0$, set $\lambda \to -\lambda$ with $\lambda > 0$, then we shall choose $\mathcal{F}_{\underline{\alpha},0}^{(j)}(x) := -\int_{\mathbb{R}^+}\exp(-\lambda A^{(j)}(x,s))f_{\underline{\alpha}}(s)ds$. It is immediate to check that $|\mathcal{F}_{\underline{\alpha},0}^{(j)}| < +\infty$ as, in particular, $\Re(A^{(j)}(s)) > \omega s/2$ by hypothesis. In such a way we get

$$|\mathcal{F}_{\underline{\alpha}}^{(j)}(x,t)| \leq M^{(j)}R_j^{-|\underline{\alpha}|}e^{\frac{\lambda\omega t}{2}}\int_t^{+\infty} e^{-\frac{\lambda\omega s}{2}}ds \leq \frac{2M^{(j)}}{\lambda\omega}R_j^{-|\underline{\alpha}|}. \tag{20}$$

Hence, by definition (2) and by (19) and (20), for all $\lambda \in \mathbb{Z}\setminus\{0\}$

$$\left\|\chi^{(j)}(p,q,t)\right\|_{(1-\delta)R_j} \leq \frac{2M^{(j)}}{|\lambda|\omega}\sum_{\underline{\alpha}}(1-\delta)^{|\underline{\alpha}|} \overset{(15)}{\leq} \frac{2M^{(j)}}{\omega\delta^2}, \tag{21}$$

which implies the first part of (18).
The second part of (18) is straightforward from (14), bounds (16), (19), (20), and hypothesis (17b) then proceeding as in (21). □

4.2. **An estimate on the Lie operator.** This is a standard result in the works of A. Giorgilli et al., see e.g. [Gioa]. The statement recalled below, is adapted to the notational setting at hand

Lemma 4.3. *Suppose that $\|\chi\|_{(1-\delta)R}$ and $\|G\|_{(1-\delta)R}$ are bounded for some $\delta \in (0,1/2)$. Then*

$$\left\|\mathcal{L}_\chi^s G\right\|_{(1-2\delta)R} \leq s!(e^2\delta^{-2}\|\chi\|_{(1-\delta)R})^s\|G\|_{(1-\delta)R}, \qquad \forall s \geq 1. \tag{22}$$

We shall also consider the case of bounded $\|G\|_R$, for which (22) clearly holds with the obvious replacement. It is evident that a sufficient condition for the convergence of the Lie operator $\exp(\mathcal{L}_\chi)$ is that $e^2\delta^{-2}\|\chi\|_{(1-\delta)R} \leq 1/2$.

5. Quantitative estimates

5.1. **The iterative lemma.** Let us consider a sequence $\{\underline{u}^{(j)}\}_{j\in\mathbb{N}} \in [0,1]^5$ with $\underline{u}^{(0)}$ to be determined, where $\underline{u}^{(j)} := (d_j, \varepsilon_j, R_j, \tilde{m}_j, \tilde{M}_j)$. Let $\underline{u}_* := (0, 0, R_*, \tilde{m}_*, \tilde{M}_*)$ with $\omega/2 \leq \tilde{m}_* < \tilde{M}_* \leq (3/2)\omega$ and $R_* > 0$ to be determined as well. Our aim is now to prove the next

Lemma 5.1. *Suppose that for some* $j \in \mathbb{N}$, *there exists* $\underline{u}^{(j)}$ *with* $u_l^{(j)} > (u_*)_l$ *for* $l = 1, \ldots, 4$ *and* $\tilde{M}_j < \tilde{M}_*$, *satisfying*

$$\left\| \tilde{F}^{(j)}(p,q,t) \right\|_{R_j} \leq \varepsilon_j, \tag{23}$$

$$\Re \tilde{g}^{(j)}(x,t) \geq \tilde{m}_j, \tag{24}$$

$$|\tilde{g}^{(j)}(x,t)| \leq \tilde{M}_j. \tag{25}$$

for all $(x,t) \in \mathcal{Q}_{R_j} \times \mathbb{R}^+$. *Then, under the condition*

$$\frac{4e^2\varepsilon_j}{\omega R_*^2 d_j^6} \leq \frac{1}{2}, \tag{26}$$

it is possible to determine $u_l^{(j+1)} \in [(u_*)_l, u_l^{(j)})$ *for* $l = 1, \ldots, 4$ *and* $\tilde{M}_{j+1} \in (\tilde{M}_j, \tilde{M}_*]$ *such that conditions (23), (24) and (25) are satisfied by* $\tilde{F}^{(j+1)}$ *and* $\tilde{g}^{(j+1)}$ *as defined in Sec 3.*

The validity of (24) and (25) (compare with (17a) and (17b)) with the above mentioned bounds on $\tilde{m}_*$ and on $\tilde{M}_*$, is clearly related to the possibility of using Prop 4.1 for all j .

Proof. First of all, immediately from (8) and (23), it follows $\left\| F^{(j)} \right\|_{R_j} \leq \varepsilon_j$. On the other hand, recall $g^{(j)}(x,t) = \tilde{g}^{(j)}(x,t) + \partial_x \Delta^{(j)}(x,t)$, where $\partial_x \Delta^{(j)}(x,t) \equiv \sum_{k\geq 2} k f_{\underline{k}}^{(j)}(t) x^{k-1}$, which implies

$$\left\| \partial_x \Delta^{(j)}(x,t) \right\|_{(1-2d_j)R_j} \leq \varepsilon_j [(1-2d_j)R_j]^{-2} \sum_{k\geq 2} k(1-2d_j)^{2k} \leq \varepsilon_j (R_* d_j)^{-2},$$

hence on $\mathcal{Q}_{(1-2d_j)R_j} \times \mathbb{R}^+$

$$\Re g^{(j)}(x,t) \overset{(24)}{\geq} \tilde{m}_j - \varepsilon_j (R_* d_j)^{-2} =: m_j. \tag{27}$$

The last quantity is well defined as a consequence of the (stronger) condition (26), being $\tilde{m}_j > \tilde{m}_* \geq \omega/2$. Similarly, $|g^{(j)}(x,t)| \leq \tilde{M}_j + \varepsilon_j (R_* d_j)^{-2} =: M_j$.
From Lemma 4.3 with $\delta = d_j$, (12), (18) and (23), under the *convergence condition* guaranteed by (26) we get

$$\left\| F^{(j+1)}(p,q,t) \right\|_{(1-2d_j)R_j} \leq \varepsilon_j \sum_{s\geq 1} \left(\frac{4e^2\varepsilon_j}{\omega d_j^4} \right)^s \leq \frac{8e^2\varepsilon_j^2}{\omega d_j^4}. \tag{28}$$

Hence we shall set

$$\varepsilon_{j+1} := 8e^2\omega^{-1}\varepsilon_j^2 R_*^{-2} d_j^{-6}, \qquad R_{j+1} := (1-2d_j)R_j, \qquad \tilde{m}_{j+1} := m_j, \qquad \tilde{M}_{j+1} := M_j, \tag{29}$$

in order to obtain the validity of (23), (24) and (25) at the $j+1$-th step. The first of (29) is the well known "heart" of the quadratic method.

□

5.2. **Determination of the bounding sequences.** Our aim is now to construct the sequence $\underline{u}^{(j)}$ for all j under the constraints (29) and show that $\lim_{j\to\infty}\underline{u}^{(j)}=\underline{u}_*$. The last step will be the determination of $\underline{u}_0$, completed in the next section. The procedure is analogous to [FW14a]. We start by choosing, for all $j\geq 1$

$$\varepsilon_j := \varepsilon_0(j+1)^{-12}. \tag{30}$$

By substituting the latter into the first of (29) we get $4e^2\varepsilon_j/(\omega R_*^2 d_j^6) = 2^{-1}[(j+1)/(j+2)]^{12} \leq 1/2$, hence condition (26) holds for all $j\geq 0$. Similarly we get

$$d_j = \left(\frac{8e^2\varepsilon_0}{R_*^2\omega}\right)^{\frac{1}{6}} \frac{(j+2)^2}{(j+1)^4}. \tag{31}$$

By supposing

$$\varepsilon_0 \leq 3^6 8^{-7}\pi^{-12}e^{-2}\omega R_*^2, \tag{32}$$

it is easy to see that

$$\sum_{j\geq 0} d_j \leq 4\left[8e^2\varepsilon_0(R_*\omega)^{-1}\right]^{\frac{1}{6}} \sum_{j\geq 0}(j+1)^{-2} \leq 1/4. \tag{33}$$

which implies, in particular, $d_j \leq 1/4$ for all $j\geq 0$ (essential for the correct definition of R_{j+1}). Condition (32) will be obtained via a suitable choice of R_0 that will be addressed in Sec. 5.3.
By (30), (31), then by (32),

$$(R_*)^{-2}\sum_{j\geq 0}\varepsilon_j d_j^{-2} \leq [8^{-1}\varepsilon_0^2\omega(R_*^2 e)^{-2}]^{\frac{1}{3}}(\pi^2/6) < \omega/4.$$

Hence, comparing (27) with (29), $\lim_{j\to\infty}\tilde{m}_{j+1} = \tilde{m}_j - \varepsilon_j(R_* d_j)^{-2} \geq \tilde{m}_0 - (\omega/4)$. This implies that it is sufficient to set $\tilde{m}_0 = (3/4)\omega$ and $\tilde{m}_* := \omega/2$. Similarly we have $\lim_{j\to\infty}\tilde{M}_j \leq \tilde{M}_* := (3/2)\omega$ if $\tilde{M}_0 := (5/4)\omega$ is chosen.
As for R_* we have $R_j := R_0\prod_{l=0}^{j-1}(1-2d_l)$. By writing $\log\prod_l(1-2d_l) = \sum_l \log(1-2d_l)$ and using (31) under condition (26), we obtain[5] $\lim_{j\to\infty} R_j \geq R_0/2 =: R_*$. By replacing this value in (31) and (32), we see that ε_0 and d_0 are determined once R_0 will be chosen.

5.3. **Transformation of variables and convergence of the scheme.** For all $j\geq 0$ the transformation $\mathcal{M}_j : \mathcal{D}_{R_{j+1}} \to \mathcal{D}_{R_j}$ acts on the variables as follows $(p^{(j)}, q^{(j)}, \eta^{(j)}) = \mathcal{L}_{\chi^{(j)}}(p^{(j+1)}, q^{(j+1)}, \eta^{j+1})$, while t is unchanged (as $\chi^{(j)}$ does not depend on η). Hence, by Lemma 4.3, then by the first of (18) and condition (26), we get

$$|p^{(j+1)} - p^{(j)}| \leq \sum_{s\geq 1}(1/s!)\left\|\mathcal{L}_{\chi^{(j)}} p^{(j+1)}\right\|_{(1-2d_j)R_j} \leq R_0^3 d_j^2/4, \tag{34}$$

analogously one obtains

$$|q^{(j+1)} - q^{(j)}| \leq R_0^3 d_j^2/4. \tag{35}$$

As for η, write $\mathcal{L}_{\chi^{(j)}}^s \eta^{(j+1)} = -\mathcal{L}_{\chi^{(j)}}^{s-1}\partial_t\chi^{(j)}$ then, similarly, by the second of (18)

$$|\eta^{(j+1)} - \eta^{(j)}| \leq \frac{4\varepsilon_j}{\omega d_j^2}\sum_{s\geq 1}\frac{(s-1)!}{s!}\left(\frac{4e^2\varepsilon_j}{\omega d_j^4}\right)^{s-1} \leq \frac{d_j^2}{e^2}\sum_{s\geq 1}\left(\frac{4e^2\varepsilon_j}{\omega d_j^4}\right)^s \overset{(26)}{\leq} \frac{R_0^2}{4e^2}d_j^4. \tag{36}$$

Our aim is now to determine the final value of R_0, by proceeding as follows. As F is supposed to be analytic on $\mathcal{D}_R$, suppose $R_0 \leq R^4 \leq 1/16$. We have $|f_{\underline{\alpha}}|_+ \leq M_F R^{-|\underline{\alpha}|} \leq M_F R_0^{-|\underline{\alpha}|/4}$, hence (use (15))

$$\|F(p,q,t)\|_{R_0} \leq M_F \sum_{|\underline{\alpha}|\geq 3} R_0^{|\underline{\alpha}|/4} \leq 4M_F R_0^{9/4} =: \varepsilon_0.$$

[5] use inequality $\log(1-x) \geq -2x\log 2$ (valid for all $x\in[0,1/2]$).

By substituting the latter into (32) one gets

$$R_0 \leq \min\{(3^6 2^{-25}\pi^{-12}e^{-2}\omega M_F^{-1})^4, R^4\}. \tag{37}$$

Finally recall that $\tilde{J}^{(0)} = \omega x$ that is $\tilde{g}^{(0)} = \omega$. Hence, in order to guarantee that the choice of $\tilde{m}_0$ and of $\tilde{M}_0$ of Sec. 5.2 is well defined, we need to show that $\left\|\partial_x \Delta^{(0)}\right\|_{R_0} \leq \omega/4$. Recall that $\Delta^{(0)} = \sum_{k\geq 2} f_{\underline{k}} x^k$, hence we get (use again the analyticity of F on $\mathcal{D}_R$), $\left\|\partial_x \Delta^{(0)}\right\|_{R_0} \leq M_F \sum_{k\geq 2} k R_0^k \leq 8 M_F R_0^2$. It is immediate to realize that the latter is smaller than $\omega/4$, for all $\omega \in (0,1]$, under the condition (37). This completes the choice of $\underline{u}^{(0)}$.
In conclusion, by using (31) in (34), (35) and (36) we get,

$$\max\left\{\sum_{j\geq 0}|p^{(j+1)} - p^{(j)}|, \sum_{j\geq 0}|q^{(j+1)} - q^{(j)}|, \sum_{j\geq 0}|\eta^{(j+1)} - \eta^{(j)}|\right\} \leq R_0/4$$

(we used $R_0 < 1 < e^2/\omega$, trivially from (37)). Hence, by the Weierstraß theorem, the limit (7) converges to a transformation, $\mathcal{M} : \mathcal{D}_{R_*} \to \mathcal{D}_{R_0}$, which is analytic for all $t \in \mathbb{R}^+$. Hence $(p^{(\infty)}, q^{(\infty)}, \eta^{(\infty)})$, denote the canonical variables on $\mathcal{D}_{R_*}$ (and $(p^{(0)}, q^{(0)}, \eta^{(0)}) := (p, q, \eta)$ those on $\mathcal{D}_{R_0}$) and the Hamiltonian $H^{(\infty)}$, formally defined after (7), is an analytic function on $\mathcal{D}_{R_*}$ as well, and is in the desired Moser normal form.

6. An outline of the proof of Theorem 2.3

In this section we describe the necessary modifications in the proof of Thm. 2.1 in order to get its "strong" version. However, we stress that the crucial point is the following: if we suppose the existence of the integral $\int_{\mathbb{R}^+} f_{\underline{\alpha}}^{(j)}(t)dt$ (guaranteed by the exponential decay of $F^{(j)}$), then (14) exists on $\mathbb{R}^+$ also for $\lambda = 0$ i.e. the r.h.s. of the homological equation can contain also terms with $\alpha_1 = \alpha_2$.

Formal scheme. The definition of $\tilde{J}^{(j)}$ and of $\tilde{F}^{(j)}$ is not necessary, we suppose that $H^{(j)}$ is directly of the form

$$H^{(j)} = \omega pq + \eta + F^{(j)}(p, q, t). \tag{38}$$

The initial Hamiltonian is exactly of the form above, so we can set $H^{(0)} := H$. Suppose that $\chi^{(j)}$ is chosen in a way to satisfy the homological equation

$$\mathcal{L}_{\chi^{(j)}}(\omega pq + \eta) + F^{(j)} = 0, \tag{39}$$

it is sufficient to define

$$F^{(j+1)} := \sum_{s\geq 1} \frac{s}{(s+1)!}\mathcal{L}_{\chi^{(j)}} F^{(j)}, \tag{40}$$

in order to have $H^{(j+1)}$ of the form (38). By expanding $\chi^{(j)} = \sum_{\underline{\alpha}} c_{\underline{\alpha}}(t) p^{\alpha_1} q^{\alpha_2}$ and $F^{(j)}$ as well[6], we get this time, for all $\underline{\alpha}$

$$\dot{c}_{\underline{\alpha}}^{(j)}(t) + \hat{\lambda} c_{\underline{\alpha}}^{(j)}(t) = f_{\underline{\alpha}}^{(j)}(t), \tag{41}$$

with $\hat{\lambda} := \omega(\alpha_1 - \alpha_2)$ *purely real*.

[6]Note that in this case the Taylor expansion of $F^{(j)}$ will contain also terms with $\alpha_1 = \alpha_2$.

Bounds on the homological equation. The easy structure of eq. (41) simplifies remarkably the proof of the equivalent of Prop. 4.1, which states, in this case, as follows

Proposition 6.1. *Suppose that there exists $M^{(j)} > 0$ such that $\left\| F^{(j)}(p,q,t) \right\|_{R_j,\mathbb{R}^+} \le M^{(j)} \exp(-at)$. Then for all $\delta \in (0,1)$ the solution of (41) satisfies*

$$\left\| \chi^{(j)}(p,q,t) \right\|_{(1-\delta)R_j;\mathbb{R}^+}, \left\| \partial_t \chi^{(j)}(p,q,t) \right\|_{(1-\delta)R_j;\mathbb{R}^+} \le 4M^{(j)} a^{-1} \delta^{-3}. \tag{42}$$

Proof. (Sketch) If $\underline{\alpha}$ is such that $\hat{\lambda} > 0$ then choose $c_{\underline{\alpha}}^{(j)}(0) = 0$. In this way[7]

$$|c_{\underline{\alpha}}^{(j)}(t)| \le M^{(j)} R_j^{-|\underline{\alpha}|} \int_0^t e^{\hat{\lambda}(s-t)} e^{-as} ds \le M^{(j)} a^{-1} R_j^{-|\underline{\alpha}|}. \tag{43}$$

If $\hat{\lambda} \le 0$ set $\hat{\lambda} \to -\hat{\lambda}$ with $\hat{\lambda} \ge 0$ and choose $c_{\underline{\alpha}}^{(j)}(0) := -\int_0^{+\infty} \exp(-\hat{\lambda}s) f_{\underline{\alpha}}^{(j)}(s) ds$. A similar procedure yields the same estimate as (43) and then $\left\| \chi^{(j)}(p,q,t) \right\|_{(1-\delta)R_j;\mathbb{R}^+} \le Ma^{-1}\delta^{-2}$. By using the obtained estimates and (41), one gets the second of (42). □

Quantitative part. We define now $\underline{\hat{u}}^{(j)} := (d_j, \varepsilon_j, R_j)$, with $\underline{\hat{u}}_* := (0, 0, \hat{R}_*)$. Statement of Lemma 5.1 modifies as follows

Lemma 6.2. *Suppose that for some $j \in \mathbb{N}$, there exists $\underline{\hat{u}}^{(j)}$ with $\hat{u}_l^{(j)} > (\hat{u}_*)_l$ for $l = 1,2,3$, satisfying*

$$\left\| \tilde{F}^{(j)}(p,q,t) \right\|_{R_j;\mathbb{R}^+} \le \varepsilon_j e^{-at}, \tag{44}$$

for all $(p,q,t) \in \mathcal{Q}_{R_j} \times \mathbb{R}^+$. Then, under the condition

$$\frac{4e^2 \varepsilon_j}{\omega a \hat{R}_*^2 d_j^6} \le \frac{1}{2}, \tag{45}$$

it is possible to determine $\hat{u}_l^{(j+1)} \in [(\hat{u}_*)_l, \hat{u}_l^{(j)})$ for $l = 1,2,3$ such that (44) is satisfied by $F^{(j+1)}$ as defined in (40).

The proof of this Lemma and of the rest of the Theorem is straightforward *mutatis mutandis*. We only mention that condition (32) is replaced by $\varepsilon_0 \le 3^6 8^{-7} e^{-2} \pi^{-12} a\omega \hat{R}_*^2$, implying

$$\hat{R}_0 \le \min\{(3^6 2^{-25} \pi^{-12} e^{-2} \omega a M_F^{-1})^4, \hat{R}^4\}$$

i.e. $\hat{R}_0 \sim a^4$ as $a \to 0$ as announced after the statement of Thm. 2.3.

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[7] We are using $\exp(\hat{\lambda}(s-t)) \le 1$ (and a similar bound in the case $\hat{\lambda} \le 0$). These bounds, used for simplicity and sufficient for our purposes, hide the property $|c_{\underline{\alpha}}^{(j)}(t)| \to 0$ as $t \to +\infty$ i.e. the (reasonable) phenomenon for which $\chi^{(j)}$ is asymptotically vanishing for all j, that is, the canonical transformation $\mathcal{M}_{\mathcal{S}}$ reduces to the identity.

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