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Nonlinear modal interaction analysis for a three degree-of-freedom system with cubic nonlinearities

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Abstract

The majority of work in the literature on modal interaction is based on two degree-of-freedom nonlinear systems with cubic nonlinearities. In this paper we consider a three degree-of-freedom system with nonlinear springs containing cubic nonlinear terms. First the undamped, unforced case is considered. Specifically the modal interaction case that occurs when all the underlying linear modal frequencies are close is considered (i.e. $\omega_{n1} : \omega_{n2} : \omega_{n3} \simeq 1 : 1 : 1$). In the case considered, due to the symmetric of the system, the first mode is linear and not coupled with the other two modes. The analysis is carried out by using a normal form transformation to obtain the nonlinear backbone curves of the undamped, unforced response. In addition, the frequency response function (FRF) of the corresponding lightly damped and harmonically forced system obtained by the continuation software AUTO-07p is compared with the backbones curve to show its validity for predicting the nonlinear resonant frequency and amplitude. A comparison of the results gives an insight into how modal interactions in the forced-damped response can be predicted using just the backbone curves, and how this might be applied to predict resonant responses of multi-modal nonlinear systems more generally.

Key words: Backbone curve, 3-DoF nonlinear oscillator, Nonlinear modal interaction, Cubic nonlinearity, Second-order normal form method

1 Introduction

In this paper, we consider the nonlinear modal behaviour of a three-degree-of-freedom (3-DOF) lumped mass system. In particular we consider the potential modal interactions that can occur by analysing the backbone curves, i.e the response of the equivalent undamped, unforced system. This is because, in common with the majority of vibration examples that lend themselves to modal analysis, the lightly damped dynamic behaviour is largely determined by the properties of the underlying Hamiltonian dynamic system.

The motivation for this study is that when multi-degree-of-freedom systems have weak nonlinearities, then internal resonance effects become significant. In fact these types of resonance effects have extensively been studied because they are often related to unwanted vibration effects in structures. Most of the literature is for undamped, unforced systems, and includes beams, cables, membranes, plates and shells — see for example [1, 8, 14]. Several different analytical approaches have been used to approach this type of problem, such as perturbation methods, [9] nonlinear normal modes (NNMs) [6, 11, 13] or normal form analysis [2, 7, 10]. Similar 3-DOF systems have been analysed using NNMs in the context of nonlinear vibration suppression [5].

In this paper we demonstrate the resonance effects by considering two configurations of an in-line three-degree-freedom(3-DOF) nonlinear oscillators with small forcing and light damping. In Section 2 we describe the two different configurations of in-line oscillator. Then, in Section 3, we apply a normal form transformation method to the 3-DOF systems. Having found the normal form, in Section 4 the backbone curves are computed. These curves are then used to infer the dynamic behaviour of the system, which in turn can be used or interpret the forced, damped behaviour. Conclusions are drawn in Section 5.

2 In-line nonlinear 3-DOF oscillators

The first 3-DOF system considered here is shown schematically in Figure 1. Three lumped massed, all of mass m, have displacements x_1, x_2 and x_3 and are forced sinusoidally at amplitudes P_1, P_2 and P_3 respectively at frequency Ω. The two outside masses are connected to the ground via linear viscous dampers with damping constant c_n, and via nonlinear springs, with linear spring stiffness k_n and cubic stiffness κ_n. The middle mass is also connected to the two side masses via a linear viscous damper and nonlinear springs.

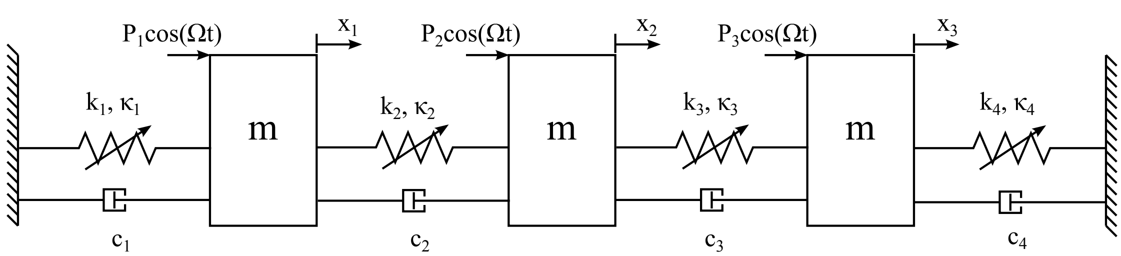

Fig. 1: In-line, 3-DOF nonlinear oscillator. The middle mass is respectively connecting to the two masses at the end and the two end masses are connected to the ground via a linear spring, a nonlinear cubic spring and a linear viscous damper. All three masses are excited by the single-frequency-sinusoidal forces.

The second 3-DOF system configuration is shown schematically in Figure 2. All three masses are connect to the ground via a linear viscous damper,c_n, and a linear springs, k_n, and the middle mass is connected to the two end masses via a linear viscous dampers and the cubic nonlinear springs.

Generally, the governing motion equations of the multi-degree-of-freedom (MDOF) nonlinear vibration system under single-frequency-sinusoidal excitations can be expressed as,

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{N_x}(\mathbf{x}) = \mathbf{P}\cos(\Omega t), \tag{1}$$

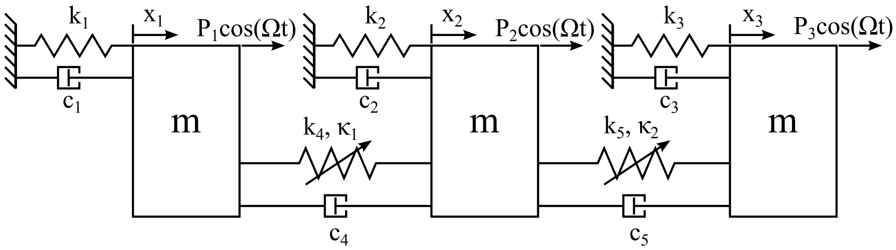

Fig. 2: In-line, 3-DOF nonlinear oscillator.All three masses are connected to the ground with a linear spring and a viscous damper and the middle mass links the masses at two ends via a linear spring, a nonlinear cubic spring and a linear viscous damper. All three masses are excited by the single-frequency-sinusoidal forces.

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices $(n \times n)$, respectively; $\mathbf{x}$, $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$ are the displacement, velocity and acceleration vectors $(n \times 1)$, respectively; $\mathbf{N_x}(\mathbf{x})$ is the nonlinear force vector $(n \times 1)$ and $\mathbf{P}$ is the external force amplitude vector $(n \times 1)$.

For both of our nonlinear oscillator configurations, the linear damping matrix $\mathbf{C}$ and external sinusoidal force amplitude vector can be written as,

$$\mathbf{C} = \begin{bmatrix} C_1 & -C_4 & 0 \\ -C_4 & C_2 & -C_5 \\ 0 & -C_5 & C_3 \end{bmatrix}, \qquad \mathbf{P} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}. \tag{2}$$

But their stiffness matrices and nonlinear terms vectors are different. For the nonlinear oscillator in Figure 1, its specific stiffness matrix $\mathbf{K}$ and the cubic nonlinear force vector $\mathbf{N_x}(\mathbf{x})$ are given by,

$$\mathbf{K}^I = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix}, \qquad \mathbf{N}^I_{\mathbf{x}}(\mathbf{x}) = \begin{pmatrix} \kappa_1 x_1^3 + \kappa_2 (x_1 - x_2)^3 \\ \kappa_2 (x_2 - x_1)^3 + \kappa_3 (x_2 - x_3)^3 \\ \kappa_3 (x_3 - x_2)^3 + \kappa_4 x_3^3 \end{pmatrix}, \tag{3}$$

while for another oscillator which is shown in Figure 2, its stiffness matrix $\mathbf{K}$ and nonlinear force vector $\mathbf{N_x}(\mathbf{x})$ (superscripts are used to indicate the two oscillators),

$$\mathbf{K}^{II} = \begin{bmatrix} k_1 + k_4 & -k_4 & 0 \\ -k_4 & k_2 + k_4 + k_5 & -k_5 \\ 0 & -k_5 & k_3 + k_5 \end{bmatrix}, \qquad \mathbf{N}^{II}_{\mathbf{x}}(\mathbf{x}) = \begin{pmatrix} \kappa_1 (x_1 - x_2)^3 \\ \kappa_1 (x_2 - x_1)^3 + \kappa_2 (x_2 - x_3)^3 \\ \kappa_2 (x_3 - x_2)^3 \end{pmatrix}. \tag{4}$$

3 Application of the second-order normal form method

Now, we apply a normal form technique which is the second-order normal form method [10] to investigate the modal interaction within the nonlinear 3-DOF oscillator. Here, we firstly consider the case with no damping and no forcing. In this case the equation of motion for both systems is given by

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{N_x}(\mathbf{x}) = \mathbf{0}. \tag{5}$$

First by applying the linear modal transformation to Equation 5 to decouple the linear terms, we get the linear modal decomposition equation in terms of the new modal coordinate $\mathbf{q}$ as

$$\ddot{\mathbf{q}} + \mathbf{\Lambda}\mathbf{q} + \mathbf{N_q}(\mathbf{q}) = \mathbf{0}, \tag{6}$$

where $\mathbf{q}$ is the modal coordinates and $\mathbf{\Lambda}$ is a diagonal matrix of the squares of the corresponding linearised natural frequencies ω_{n1}, ω_{n2} & ω_{n3}, and the $\mathbf{N_q}(\mathbf{q})$ is the nonlinear modal force vector,

$$\mathbf{N_q}(\mathbf{q}) = \mathbf{n_q}\mathbf{q}^*, \tag{7}$$

where coefficient matrix $\mathbf{n_q}$ and nonlinear element vector $\mathbf{q}^*$ are, in general

$$\mathbf{n_q^T} = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \\ \alpha_{14} & \alpha_{24} & \alpha_{34} \\ \alpha_{15} & \alpha_{25} & \alpha_{35} \\ \alpha_{16} & \alpha_{26} & \alpha_{36} \\ \alpha_{17} & \alpha_{27} & \alpha_{37} \\ \alpha_{18} & \alpha_{28} & \alpha_{38} \\ \alpha_{19} & \alpha_{29} & \alpha_{39} \\ \alpha_{10} & \alpha_{20} & \alpha_{30} \end{bmatrix}, \qquad \mathbf{q}^* = \begin{bmatrix} q_1^3 \\ q_1^2 q_2 \\ q_1 q_2^2 \\ q_2^3 \\ q_2^2 q_3 \\ q_2 q_3^2 \\ q_3^3 \\ q_1 q_3^2 \\ q_1^2 q_3 \\ q_1 q_2 q_3 \end{bmatrix}. \tag{8}$$

Substituting $q_n \to u_n = u_{np} + u_{nm}$ gives the functional form of $\mathbf{u}^*$ vector and coefficient vector $\mathbf{n_u}$ as below, and simultaneously, values for $\beta_{k,l}$ can be computed using

$$\beta_{k,l} = \left[\sum_{n=1}^{N} \{(s_{npl} - s_{nml})\omega_{rn}\} \right]^2 - \omega_{rk}^2, \tag{9}$$

with the assumption that $\omega_{r2} = r\omega_{r1}$ and $\omega_{r3} = r'\omega_{r1}$,

$$
\mathbf{u}^* = \begin{bmatrix}
u_{1p}^3 \\
u_{1p}^2 u_{2p} \\
u_{1p} u_{2p}^2 \\
u_{2p}^3 \\
u_{1p}^2 u_{2p} \\
u_{1p} u_{1m} u_{2p} \\
u_{1m}^2 u_{2p} \\
u_{1p}^2 u_{2m} \\
u_{1p} u_{1m} u_{2m} \\
u_{1m}^2 u_{2m} \\
u_{1p} u_{2p}^2 \\
u_{1p} u_{2p} u_{2m} \\
u_{1p} u_{2m}^2 \\
u_{1m} u_{2p}^2 \\
u_{1m} u_{2p} u_{2m} \\
u_{1m} u_{2m}^2 \\
u_{2p}^3 \\
u_{2p}^2 u_{2m} \\
u_{2p} u_{2m}^2 \\
u_{2m}^3 \\
u_{2p}^2 u_{3p} \\
u_{2p} u_{2m} u_{3p} \\
u_{2m}^2 u_{3p} \\
u_{2p}^2 u_{3m} \\
u_{2p} u_{2m} u_{3m} \\
u_{2m}^2 u_{3m} \\
u_{2p} u_{3p}^2 \\
u_{2p} u_{3p} u_{3m} \\
u_{2p} u_{3m}^2 \\
u_{2m} u_{3p}^2 \\
u_{2m} u_{3p} u_{3m} \\
u_{2m} u_{3m}^2 \\
u_{3p}^3 \\
u_{3p}^2 u_{3m} \\
u_{3p} u_{3m}^2 \\
u_{3m}^3 \\
u_{1p} u_{3p}^2 \\
u_{1p} u_{3p} u_{3m} \\
u_{1p} u_{3m}^2 \\
u_{1m} u_{3p}^2 \\
u_{1m} u_{3p} u_{3m} \\
u_{1m} u_{3m}^2 \\
u_{1p}^2 u_{3p} \\
u_{1p} u_{1m} u_{3p} \\
u_{1m}^2 u_{3p} \\
u_{1p}^2 u_{3m} \\
u_{1p} u_{1m} u_{3m} \\
u_{1m}^2 u_{3m} \\
u_{1p} u_{2p} u_{3p} \\
u_{1p} u_{2p} u_{3m} \\
u_{1p} u_{2m} u_{3p} \\
u_{1p} u_{2m} u_{3m} \\
u_{1m} u_{2p} u_{3p} \\
u_{1m} u_{2p} u_{3m} \\
u_{1m} u_{2m} u_{3p} \\
u_{1m} u_{2m} u_{3m}
\end{bmatrix}
\quad
\mathbf{n_u^T} = \begin{bmatrix}
\alpha_{11} & \alpha_{21} & \alpha_{31} \\
3\alpha_{11} & 3\alpha_{21} & 3\alpha_{31} \\
3\alpha_{11} & 3\alpha_{21} & 3\alpha_{31} \\
\alpha_{11} & \alpha_{21} & \alpha_{31} \\
\alpha_{12} & \alpha_{22} & \alpha_{32} \\
2\alpha_{12} & 2\alpha_{22} & 2\alpha_{32} \\
\alpha_{12} & \alpha_{22} & \alpha_{32} \\
\alpha_{12} & \alpha_{22} & \alpha_{32} \\
2\alpha_{12} & 2\alpha_{22} & 2\alpha_{32} \\
\alpha_{12} & \alpha_{22} & \alpha_{32} \\
\alpha_{13} & \alpha_{23} & \alpha_{33} \\
2\alpha_{13} & 2\alpha_{23} & 2\alpha_{33} \\
\alpha_{13} & \alpha_{23} & \alpha_{33} \\
\alpha_{13} & \alpha_{23} & \alpha_{33} \\
2\alpha_{13} & 2\alpha_{23} & 2\alpha_{33} \\
\alpha_{13} & \alpha_{23} & \alpha_{33} \\
\alpha_{14} & \alpha_{24} & \alpha_{34} \\
3\alpha_{14} & 3\alpha_{24} & 3\alpha_{34} \\
3\alpha_{14} & 3\alpha_{24} & 3\alpha_{34} \\
\alpha_{14} & \alpha_{24} & \alpha_{34} \\
\alpha_{15} & \alpha_{25} & \alpha_{35} \\
2\alpha_{15} & 2\alpha_{25} & 2\alpha_{35} \\
\alpha_{15} & \alpha_{25} & \alpha_{35} \\
\alpha_{15} & \alpha_{25} & \alpha_{35} \\
2\alpha_{15} & 2\alpha_{25} & 2\alpha_{35} \\
\alpha_{15} & \alpha_{25} & \alpha_{35} \\
\alpha_{16} & \alpha_{26} & \alpha_{36} \\
2\alpha_{16} & 2\alpha_{26} & 2\alpha_{36} \\
\alpha_{16} & \alpha_{26} & \alpha_{36} \\
\alpha_{16} & \alpha_{26} & \alpha_{36} \\
2\alpha_{16} & 2\alpha_{26} & 2\alpha_{36} \\
\alpha_{16} & \alpha_{26} & \alpha_{36} \\
\alpha_{17} & \alpha_{27} & \alpha_{37} \\
3\alpha_{17} & 3\alpha_{27} & 3\alpha_{37} \\
3\alpha_{17} & 3\alpha_{27} & 3\alpha_{37} \\
\alpha_{17} & \alpha_{27} & \alpha_{37} \\
\alpha_{18} & \alpha_{28} & \alpha_{38} \\
2\alpha_{18} & 2\alpha_{28} & 2\alpha_{38} \\
\alpha_{18} & \alpha_{28} & \alpha_{38} \\
\alpha_{18} & \alpha_{28} & \alpha_{38} \\
2\alpha_{18} & 2\alpha_{28} & 2\alpha_{38} \\
\alpha_{18} & \alpha_{28} & \alpha_{38} \\
\alpha_{19} & \alpha_{29} & \alpha_{39} \\
2\alpha_{19} & 2\alpha_{29} & 2\alpha_{39} \\
\alpha_{19} & \alpha_{29} & \alpha_{39} \\
\alpha_{19} & \alpha_{29} & \alpha_{39} \\
2\alpha_{19} & 2\alpha_{29} & 2\alpha_{39} \\
\alpha_{19} & \alpha_{29} & \alpha_{39} \\
\alpha_{10} & \alpha_{20} & \alpha_{30} \\
\alpha_{10} & \alpha_{20} & \alpha_{30} \\
\alpha_{10} & \alpha_{20} & \alpha_{30} \\
\alpha_{10} & \alpha_{20} & \alpha_{30} \\
\alpha_{10} & \alpha_{20} & \alpha_{30} \\
\alpha_{10} & \alpha_{20} & \alpha_{30} \\
\alpha_{10} & \alpha_{20} & \alpha_{30} \\
\alpha_{10} & \alpha_{20} & \alpha_{30}
\end{bmatrix}
\quad
\beta^T = \omega_{r1}^2 \begin{bmatrix}
8 & 9 - r^2 & 9 - r'^2 \\
0 & 1 - r^2 & 1 - r'^2 \\
0 & 1 - r^2 & 1 - r'^2 \\
8 & 9 - r^2 & 9 - r'^2 \\
r^2 + 4r + 3 & 4r + 4 & r^2 + 4r + 4 - r'^2 \\
r^2 - 1 & 0 & r^2 - r'^2 \\
r^2 - 4r + 3 & -4r + 4 & r^2 - 4r + 4 - r'^2 \\
r^2 - 4r + 3 & -4r + 4 & r^2 - 4r + 4 - r'^2 \\
r^2 - 1 & 0 & r^2 - r'^2 \\
r^2 + 4r + 3 & 4r + 4 & r^2 + 4r + 4 - r'^2 \\
4r^2 + 4r & 3r^2 + 4r + 1 & 4r^2 + 4r + 1 - r'^2 \\
0 & 1 - r^2 & 1 - r'^2 \\
4r^2 - 4r & 3r^2 - 4r + 1 & 4r^2 - 4r + 1 - r'^2 \\
4r^2 - 4r & 3r^2 - 4r + 1 & 4r^2 - 4r + 1 - r'^2 \\
0 & 1 - r2 & 1 - r'2 \\
4r^2 + 4r & 3r^2 + 4r + 1 & 4r^2 + 4r + 1 - r'^2 \\
9r^2 - 1 & 8r^2 & 9r^2 - r'^2 \\
r^2 - 1 & 0 & r^2 - r'^2 \\
r^2 - 1 & 0 & r^2 - r'^2 \\
9r^2 - 1 & 8r^2 & 9r^2 - r'^2 \\
4r^2 + 4rr' + r'^2 - 1 & 3r^2 + 4rr' + r'^2 & 4r^2 + 4rr' \\
r'^2 - 1 & r'^2 - r^2 & 0 \\
4r^2 - 4rr' + r'^2 - 1 & 3r^2 - 4rr' + r'^2 & 4r^2 - 4rr' \\
4r^2 - 4rr' + r'^2 - 1 & 3r^2 - 4rr' + r'^2 & 4r^2 - 4rr' \\
r'^2 - 1 & r'^2 - r^2 & 0 \\
4r^2 + 4rr' + r'^2 - 1 & 3r^2 + 4rr' + r'^2 & 4r^2 + 4rr' \\
r^2 + 4rr' + 4r'^2 - 1 & 4rr' + 4r'^2 & r^2 + 4rr' + 3r'^2 \\
r^2 - 1 & 0 & r^2 - r'^2 \\
r^2 - 4rr' + 4r'^2 - 1 & -4rr' + 4r'^2 & r^2 - 4rr' + 3r'^2 \\
r^2 - 4rr' + 4r'^2 - 1 & -4rr' + 4r'^2 & r^2 - 4rr' + 3r'^2 \\
r^2 - 1 & 0 & r^2 - r'^2 \\
r^2 + 4rr' + 4r'^2 - 1 & 4rr' + 4r'^2 & r^2 + 4rr' + 3r'^2 \\
9r'^2 - 1 & 9r'^2 - r^2 & 8r'^2 \\
r'^2 - 1 & r'^2 - r^2 & 0 \\
r'^2 - 1 & r'^2 - r^2 & 0 \\
9r'^2 - 1 & 9r'^2 - r^2 & 8r'^2 \\
4r' + 4r'^2 & 1 + 4r' + 4r'^2 - r^2 & 1 + 4r' + 3r'^2 \\
0 & 1 - r^2 & 1 - r'^2 \\
-4r' + 4r'^2 & 1 - 4r' + 4r'^2 - r^2 & 1 - 4r' + 3r'^2 \\
-4r' + 4r'^2 & 1 - 4r' + 4r'^2 - r^2 & 1 - 4r' + 3r'^2 \\
0 & 1 - r^2 & 1 - r'^2 \\
4r' + 4r'^2 & 1 + 4r' + 4r'^2 - r^2 & 1 + 4r' + 3r'^2 \\
3 + 4r' + r'^2 & 4 + 4r' + r'^2 - r^2 & 4 + 4r' \\
r'^2 - 1 & r'^2 - r^2 & 0 \\
3 - 4r' + r'^2 & 4 - 4r' + r'^2 - r^2 & 4 - 4r' \\
3 - 4r' + r'^2 & 4 - 4r' + r'^2 - r^2 & 4 - 4r' \\
r'^2 - 1 & r'^2 - r^2 & 0 \\
3 + 4r' + r'^2 & 4 + 4r' + r'^2 - r^2 & 4 + 4r' \\
(1 + r + r')^2 - 1 & (1 + r + r')^2 - r^2 & (1 + r + r')^2 - r'^2 \\
(1 + r - r')^2 - 1 & (1 + r - r')^2 - r^2 & (1 + r - r')^2 - r'^2 \\
(1 - r + r')^2 - 1 & (1 - r + r')^2 - r^2 & (1 - r + r')^2 - r'^2 \\
(1 - r - r')^2 - 1 & (1 - r - r')^2 - r^2 & (1 - r - r')^2 - r'^2 \\
(1 - r - r')^2 - 1 & (1 - r - r')^2 - r^2 & (1 - r - r')^2 - r'^2 \\
(1 - r + r')^2 - 1 & (1 - r + r')^2 - r^2 & (1 - r + r')^2 - r'^2 \\
(1 + r - r')^2 - 1 & (1 + r - r')^2 - r^2 & (1 + r - r')^2 - r'^2 \\
(1 + r + r')^2 - 1 & (1 + r + r')^2 - r^2 & (1 + r + r')^2 - r'^2
\end{bmatrix}
\tag{10}
$$

By viewing the β matrix, the zero terms represent the unconditionally-resonant terms which should be kept in the dynamic equation for u. Furthermore, there are also additional conditionally-resonant terms depending on the value of r and r'. For example, $r, r' = 1, 3, \frac{1}{3}$ will lead further zero terms in the β matrix. For both unconditionally- and conditionally-resonant terms, the corresponding terms in $\mathbf{n_u}$ as equal to those in $\mathbf{n}$. Here, as an instead examples, consider the case that r and $r' \neq 1, 3, \frac{1}{3}$, such that there is no conditionally resonance terms and the resulting dynamic equations are, (Note that this special case is inconsistent with the $\omega_{n1} : \omega_{n2} : \omega_{n3} \simeq 1 : 1 : 1$ case considered elsewhere in this paper)

$$\begin{aligned}
&\ddot{u}_1 + \omega_{n1}^2 u_1 + 3\alpha_{11}\left[u_{1p}^2 u_{1m} + u_{1p} u_{1m}^2\right] + 2\alpha_{13}\left[u_{1p} u_{2p} u_{2m} + u_{1m} u_{2p} u_{2m}\right] + 2\alpha_{18}\left[u_{1p} u_{3p} u_{3m} + u_{1m} u_{3p} u_{3m}\right] = 0,\\
&\ddot{u}_2 + \omega_{n2}^2 u_2 + 3\alpha_{24}\left[u_{2p}^2 u_{2m} + u_{2p} u_{2m}^2\right] + 2\alpha_{22}\left[u_{1p} u_{1m} u_{2p} + u_{1p} u_{1m} u_{2m}\right] + 2\alpha_{26}\left[u_{2p} u_{3p} u_{3m} + u_{2m} u_{2p} u_{3m}\right] = 0,\\
&\ddot{u}_3 + \omega_{n3}^2 u_3 + 3\alpha_{37}\left[u_{3p}^2 u_{3m} + u_{3p} u_{3m}^2\right] + 2\alpha_{35}\left[u_{2p} u_{2m} u_{3p} + u_{2p} u_{2m} u_{3m}\right] + 2\alpha_{39}\left[u_{1p} u_{1m} u_{3p} + u_{1p} u_{1m} u_{3m}\right] = 0.
\end{aligned} \tag{11}$$

Substituting $u_{ip} = (U_i/2)\, e^{j(\omega_{ri}t - \phi_i)}$ and $u_{im} = (U_i/2)\, e^{-j(\omega_{ri}t - \phi_i)}$ into Equation 11 and balancing the coefficients of $e^{j\omega_{ri}t}$ and $e^{-j\omega_{ri}t}$, we can get the time-independent equations

$$\begin{aligned}
\left[-\omega_{r1}^2 + \omega_{n1}^2 + 3\alpha_{11}U_1^2 + 2\alpha_{13}U_2^2 + 2\alpha_{18}U_3^2\right] U_1 = 0,\\
\left[-\omega_{r2}^2 + \omega_{n2}^2 + 3\alpha_{24}U_2^2 + 2\alpha_{22}U_1^2 + 2\alpha_{26}U_3^2\right] U_2 = 0,\\
\left[-\omega_{r3}^2 + \omega_{n3}^2 + 3\alpha_{37}U_3^2 + 2\alpha_{35}U_2^2 + 2\alpha_{39}U_1^2\right] U_3 = 0.
\end{aligned} \tag{12}$$

Three solutions to Equation 12 can be found by successively setting U_1 , U_2 and U_3 to zero and the results are,

$$S1: \qquad U_1 \neq 0, U_2 = U_3 = 0, \qquad \omega_{r1}^2 = \omega_{n1}^2 + \frac{3}{4}\alpha_{11}U_1^2, \tag{13}$$

$$S2: \qquad U_2 \neq 0, U_1 = U_3 = 0, \qquad \omega_{r2}^2 = \omega_{n2}^2 + \frac{3}{4}\alpha_{24}U_2^2, \tag{14}$$

$$S3: \qquad U_3 \neq 0, U_1 = U_2 = 0, \qquad \omega_{r3}^2 = \omega_{n3}^2 + \frac{3}{4}\alpha_{37}U_3^2. \tag{15}$$

Here $S1$, $S2$ and $S3$ are the expressions of the backbone curve branches of the modal coordinates for the non-modal-interaction case.

4 Backbone curve and FRF results

In order to show how the backbone curve obtained by the second-order normal form method can present the information of the modal interaction within a nonlinear oscillator, we choose the second nonlinear system (shown in Figure 2) with specific parameters as a example for illustration. Here, to simplify the mathematical presentation, we choose the oscillator to be symmetric, which means the linear stiffness $k_1 = k_2 = k_3 = k$ and $k_4 = k_5 = k'$ and the cubic nonlinear stiffness $\kappa_1 = \kappa_2 = \kappa$.

Therefore, through linear modal transformation, the linear modal natural frequencies can be calculated to be $\omega_{n1} = \sqrt{k}$, $\omega_{n2} = \sqrt{k + k'}$ and $\omega_{n3} = \sqrt{k + 3k'}$. To make these modal frequencies be close, k' is supposed to be small and there will exist the possibility for the nonlinear modes to interact. Hence, for a $1 : 1 : 1$ resonance we set $r = r' = 1$. Submitting the frequencies ratios values into the β matrix derived in Section 3, picking out the resonant terms into $\mathbf{n}$ and following the same process in last section, we obtain the time-independent equations like Equation 12 to be,

$$\begin{aligned}
&\left[-\omega_{r1}^2 + \omega_{n1}^2\right] U_1 = 0, \\
&\left[-\omega_{r2}^2 + \omega_{n2}^2 + \frac{3}{4}\mu\left\{U_2^2 + (18+9p)U_3^2\right\}\right] U_2 = 0, \\
&\left[-\omega_{r3}^2 + \omega_{n3}^2 + \frac{3}{4}\mu\left\{27U_3^2 + (6+3p)U_2^2\right\}\right] U_3 = 0.
\end{aligned} \tag{16}$$

where $\mu = \kappa/m$ and $p = e^{j2(|\phi_2-\phi_3|)}$. The $|\phi_2 - \phi_3|$ represents the phase difference between mode 2 and 3. Using the same procedure as that in Section 3, there exist three independent backbone branches labelled $S1$, $S2$ and $S3$,

$$S1: \qquad U_1 \neq 0, U_2 = U_3 = 0, \qquad \omega_{r1}^2 = \omega_{n1}^2, \tag{17}$$

$$S2: \qquad U_2 \neq 0, U_1 = U_3 = 0, \qquad \omega_{r2}^2 = \omega_{n2}^2 + \frac{3}{4}\mu U_2^2, \tag{18}$$

$$S3: \qquad U_3 \neq 0, U_1 = U_2 = 0, \qquad \omega_{r3}^2 = \omega_{n3}^2 + \frac{81}{4}\mu U_3^2. \tag{19}$$

It can be noticed that the branch for u_1 is linear and it is actually not coupled with the other two modes, where mode 2 and 3 would potentially interact with each other which will be affected by p. If u_2 and u_3 are both active, Equation 16 can be written as

$$\Omega^2 = \omega_{n2}^2 + \frac{3}{4}\mu\left\{U_2^2 + (18+9p)U_3^2\right\} = \omega_{n3}^2 + \frac{3}{4}\mu\left\{27U_3^2 + (6+3p)U_2^2\right\}. \tag{20}$$

For Equation 20 to be real, the phase difference terms should be, $p = \pm 1$. $p = 1$ and $p = -1$ represent the in-unison and out-of-unison resonances respectively. The full discussion of the value chosen of p and its corresponding physical meaning can be found in [3, 16]. So, here by setting $p = +1$ yields two extra backbone curves, labelled $S4^+$ and $S4^-$, with the phase difference

$$S4^+ : |\phi_2 - \phi_3| = 0, \qquad S4^- : |\phi_2 - \phi_3| = \pi, \tag{21}$$

and their corresponding backbone branch expressions,

$$\begin{aligned}
S4^\pm : \qquad & U_2^2 = \frac{\omega_{n2}^2 - \omega_{n3}^2}{6\mu}, \\
S4^\pm : \qquad & \Omega^2 = \frac{9\omega_{n2}^2 - \omega_{n3}^2}{8} + \frac{81}{4}\mu U_3^2.
\end{aligned} \tag{22}$$

The case where $p = -1$ yields a further two backone curves, denoted $S5^+$ and $S5^-$. They are characterised by the phase differences,

$$S5^+ : |\phi_2 - \phi_3| = +\pi/2, \qquad S5^- : |\phi_2 - \phi_3| = -\pi/2. \tag{23}$$

Substituting $p =_1$ into Equation 20 gives the amplitude and response frequency relationships

$$\begin{aligned}
S5^\pm : \qquad & U_2^2 = \frac{2(\omega_{n2}^2 - \omega_{n3}^2)}{3\mu} - 9U_3^2, \\
S5^\pm : \qquad & \Omega^2 = \frac{3\omega_{n2}^2 - \omega_{n3}^2}{2}.
\end{aligned} \tag{24}$$

It should be noted that if μ is positive Equation 22 and Equation 24 will be complex (assumed $\omega_{n3} > \omega_{n2}$) and lose its physical meaning. Hence, the $S4^{\pm}$ and $S5^{\pm}$ branches exist and the corresponding nonlinear modal interaction only happens when the cubic spring stiffness κ is negative, i.e. the system is a softening nonlinear case. Also it can be seen from Equation 24 that as the frequency of $S5^{\pm}$ will not vary with the mode amplitude and the mode amplitude cannot infinitely increase $S5^{\pm}$ backbone branch will be a vertical line with a limited length.

Figure 3 shows the backbone curves for the cases where $\omega_{n1} = 1$, $\omega_{n2} = 1.005$, $\omega_{n3} = 1.015$ and $\kappa = -0.05$. All panels show the backbone curves in the projection of response frequency against a displacement. The first column shows the amplitude of displacement of the fundamental response of u_1, u_2 and u_3 and the second shows that of the lumped masses x_1, x_2 and x_3. The labelled $S1$, $S2$ and $S3$ branches are the independent resonant backbone curve. The $S4^+$, $S4^-$, $S5^+$ and $S5^-$ branches are the interacting ones in which case mode 2 and mode 3 are both activated and the mode phase differences are 0, π, $+\pi/2$ and $-\pi/2$ respectively. Note that as $S5^{\pm}$ will coincide with $S1$, $S5^{\pm}$ backbone curves are indicated by by short cross lines in Figure 3.

To further demonstrate the ability of the backbone curve in determining the response of the system to an external forcing, we show a brief example of the relationship between the forced response and the backbone curves. The same fundamental system as above is used and it is forced in the second mode, $[P_{m1}, P_{m2}, P_{m3}] = [0, 15, 0] \times 10^{-4}$ (corresponding physical mass force is $[P_1, P_2, P_3] = [15, 0, -15] \times 10^{-4}$) with a damping ratio of $\zeta = 0.001$. The forced response has been computed from an initial steady state solution, found with numerical integration in MATLAB, which is then continued in forcing frequency using the software AUTO-07p.

Figure 4 shows the forced response of the first and third masses, X_1 and X_3 (due to the system symmetry) for the system whose backbone curves are presented in the first panel of the second column in Figure 3. In Figure 4, it can be observed that the response of the typical softening Duffing oscillator is following the $S2$ backbone where the response is confined to just the second mode. On the right stable and left unstable Duffing oscillator response branches, two branch bifurcation points, marked by quadrangles, respectively leading to two branches which follow $S4^+$ and $S4^-$ backbone curves respectively. Still one couple of response branches is stable and the other is unstable.l These branches are the responses composed of both second and third modes with the phase difference 0 and π respectively. In the center part of Figure 4, there are four stable branches which also bifurcate from the Duffing oscillator response and they appear to be attracted to $S5^{\pm}$. There four curves seems to be the response composed of second mode and third mode but with the phase difference $+\pi/2$ and $-\pi/2$. Besides, it can be seen that all the peaks amplitude points are close to our backbone curve branches. Therefore, the result shows that the backbone curve obtained by the second-order normal form method provides a good way for understanding the modal interaction within the nonlinear 3-DOF oscillator system.

5 Conclusions

In this paper, we have considered the nonlinear modal behaviour of a three-degree-freedom (3-DOF) lumped mass system. In particular we considered the potential modal interactions that can occur by analysing the backbone curves of the undamped, unforced system. This

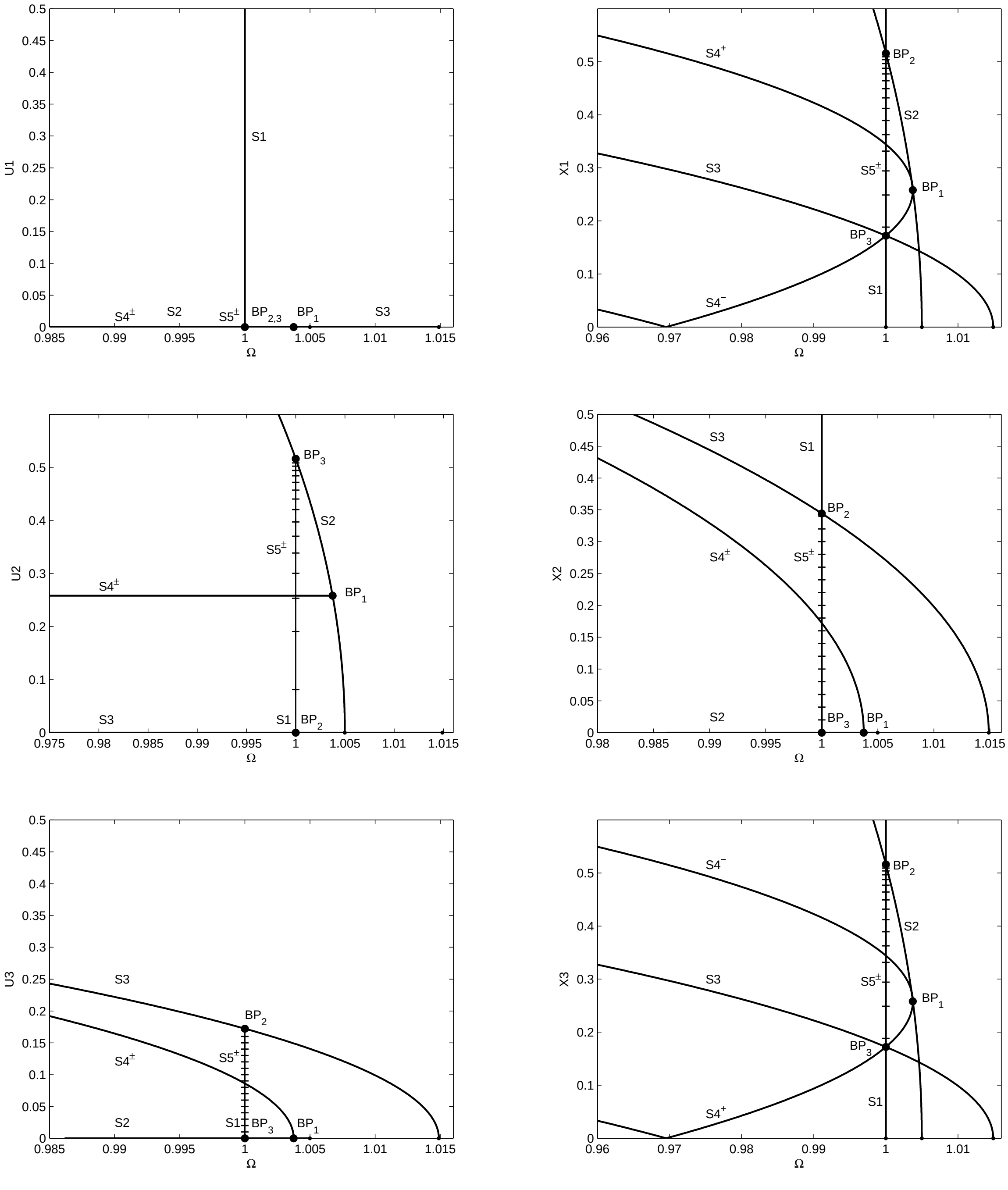

Fig. 3: Backbone curves for the oscillator with the physical parameters $k_1 = k_2 = k_3 = 1$, $k_3 = k_4 = 0.01$ and $\kappa_1 = \kappa_2 = -0.05$, so the modal natural frequencies are $\omega_{n1} = 1$, $\omega_{n2} = 1.005$ and $\omega_{n3} = 1.015$. The panels in the first and second column show the modal and physical results respectively. Stable solutions are shown with solid lines, whereas unstable solutions are represented by dashed line. Bifurcation points are noted by *BP*. Note that as $S5^{\pm}$ would overlap $S1$, $S5^{\pm}$ backbone curves are indicated by short cross lines for distinction.

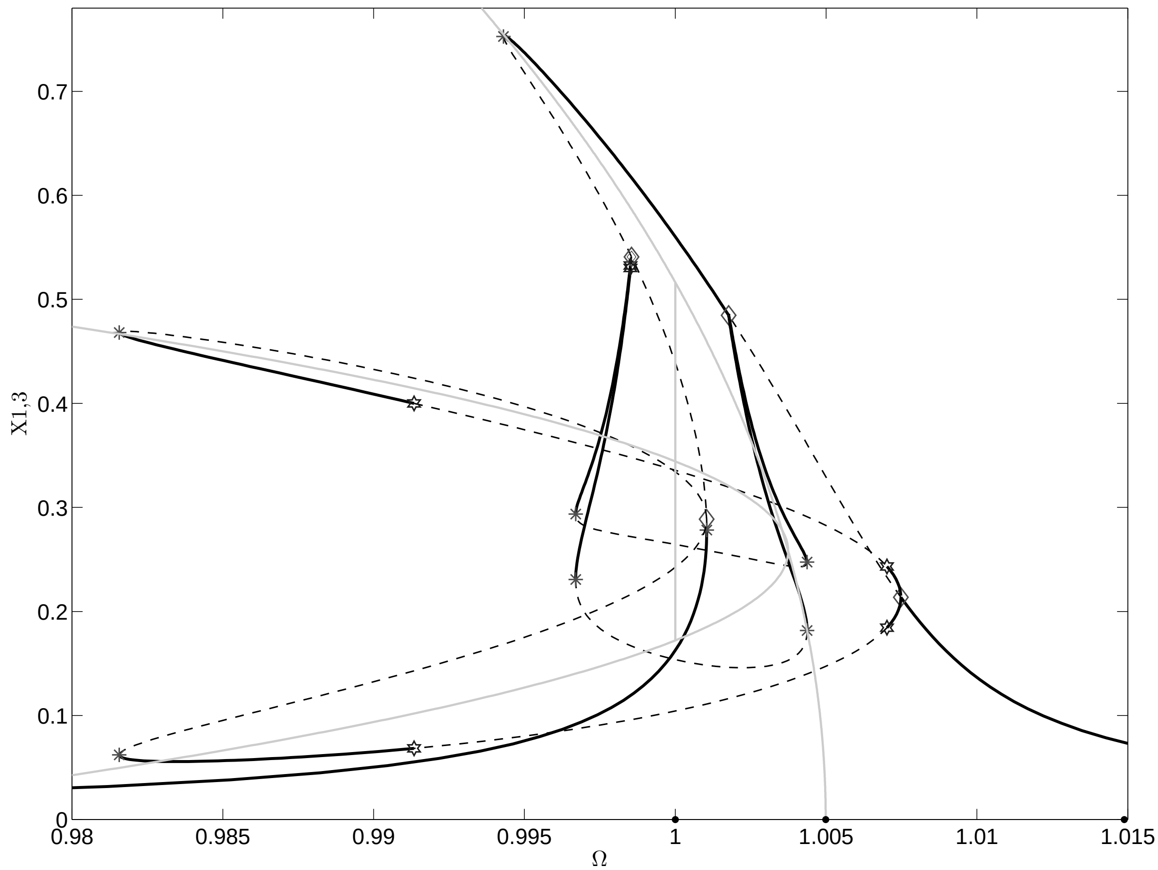

Fig. 4: Backbone curves and the amplitude response of the first and third masses when the system is forced in only the second mode (i.e. $[P_{m1}, P_{m2}, P_{m3}] = [0, 0.0015, 0]$) with parameters $m = 1$, $\omega_{n1} = 1$, $\omega_{n2} = 1.005$, $\omega_{n3} = 1.015$, $\kappa_1 = \kappa_2 = -0.05$, $c_1 = c_2 = c_3 = 0.002$, $c_4 = c_5 = 0$ and $[P_1, P_2, P_3] = [0.0015, 0, -0.0015]$. The diamond, start and asterisk indicate branch, torus bifurcation and fold points. The black solid lines and thin dash lines represent the stable and unstable amplitude response respectively. The grey lines represent the backbone curves. Note that as forcing is in the second mode only, the backbone branches $S1$(containing mode 1 only) and $S3$(containing mode 3) have not been plotted here.

is an important topic because the majority of vibration examples that relate to modal analysis are lightly damped and therefore the dynamic behaviour is largely determined by the properties of the underlying undamped dynamic system.

First we considered the undamped, unforced case was considered. In particular the modal interaction case that occurs when all the underlying linear modal frequencies are close was considered (i.e. $\omega_{n1} : \omega_{n2} : \omega_{n3} \simeq 1 : 1 : 1$). In this case the first mode is linear because of the symmetry of the system and the rest two mode will potentially interact with each other when the special parameters are chosen. We showed how this system can be analysed using a normal form transformation to obtain the nonlinear backbone curves of the undamped, unforced response. Following this the frequency response function (FRF) of the corresponding lightly damped and harmonically forced system was obtained using the continuation software AUTO-07p. This result was compared with the backbones curve to show its validity for predicting the nonlinear resonant frequency and amplitude.

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