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Universality and Short-Wavelength Approximations for Chaotic Wave Scattering

Martin Sieber

School of Mathematics, University of Bristol, Bristol BS8 1TW, UK
e-mail: m.sieber@bristol.ac.uk

***Abstract*—We give an overview of wave scattering in open cavities in which the ray dynamics is chaotic. In the limit of short wavelengths certain properties emerge that are universal and do not depend on the details of the cavity. These universal features are described by random matrix theory. We discuss in particular results that characterize the transmission probabilities and transmission times of waves through the cavity. Short-wavelength approximations that use statistical properties of long rays are able to explain this universality.**

I. Open chaotic cavities

Chaotic cavities are cavities in which the ray dynamics is chaotic. They find applications, for example, in microwave physics, in acoustics or in mesoscopics [1]–[5]. Here we give a brief overview of wave transport through chaotic cavities that are opened up by attaching semi-infinite leads. An example is shown in Fig. 1 in the form of a quarter stadium. These systems are motivated by the transport through quantum dots [5], [6]. We concentrate on the case of two leads, but the formalism can be generalised to an arbitrary number of leads.

Stationary waves inside the cavity and the leads satisfy the Helmholtz equation

$$(\nabla^2 + k^2)\psi(\boldsymbol{r}) = 0, \tag{1}$$

where k is the wavenumber. In addition one has to require appropriate boundary conditions, for example Dirichlet boundary conditions for which the wave function $\psi(\boldsymbol{r})$ vanishes at the boundary of cavity and leads.

One way to characterise the wave solutions for this system is to consider the corresponding scattering problem. In each lead there is a finite number of incoming and outgoing modes. These numbers are given by $M_i = \lfloor k w_i/\pi \rfloor$ where w_i is the width of the i-th lead.

The $M \times M$ scattering matrix S connects the M (flux normalised) incoming modes to the M outgoing modes, where $M = M_1 + M_2$. Due to flux conservation S is unitary, $S^\dagger S = 1$, and it has the block structure

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}. \tag{2}$$

Here r and t refer to reflection and transmission for incoming waves in lead 1, and r' and t' refer to reflection and transmission for incoming waves in lead 2. There are a number of quantities that can be obtained from the S-matrix and that can be used to characterise the transmission through the cavity. Some of them are discussed in the following section.

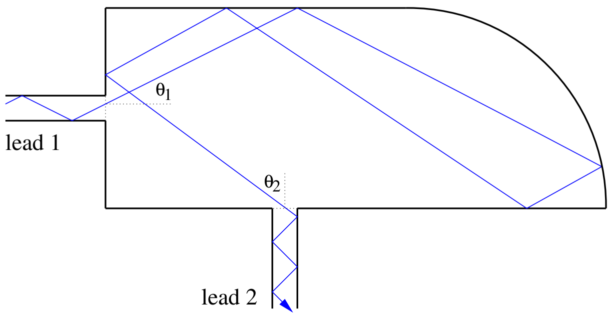

Fig. 1. An open chaotic cavity in the form of a quarter stadium with two attached semi-infinite leads. Also shown is a ray entering the cavity through lead 1 with angle θ_1, and leaving it through lead 2 with angle θ_2.

II. Transmission Eigenvalues and Time Delays

The total transmission coefficient T for incoming waves in lead 1 is

$$T = \sum_{a=1}^{M_1} \sum_{b=1}^{M_2} |t_{ba}|^2 = \mathrm{Tr}(t\,t^\dagger). \tag{3}$$

Similarly, the reflection coefficient is $R = \mathrm{Tr}(r\,r^\dagger)$, and unitarity of S requires that $R + T = M_1$.

The eigenvalues of $tt^\dagger$ are the transmission eigenvalues

$$T_1, \ldots, T_n\,, \qquad T_j \in [0,1]\,, \qquad n = \min(M_1, M_2)$$

Many important transmission properties can be obtained from the eigenvalues T_j. For example, the transmission coefficient $T = \sum_j T_j$, the variance of T, the shot noise $\langle \sum_j T_j(1-T_j) \rangle$ or the transmission moments $\frac{1}{n}\langle \sum_j T_j^k \rangle$. The expression of transmission properties in terms of the transmission eigenvalues is often referred to as Landauer-Büttiker formalism [7], [8].

A second set of quantities are related to the Wigner-Smith matrix Q [9], [10]

$$Q = -\mathrm{i}\, S^\dagger \frac{\partial S}{\partial E}. \tag{4}$$

The matrix Q is hermitian, $Q = Q^\dagger$, and its M eigenvalues are the proper time delays τ_j. They characterise temporal aspects of a wave scattering process. Similar as with the transmission eigenvalues T_j, the proper time delays τ_j can be used to express other quantities of interest. Among them are the Wigner time delay $\tau_W = \frac{1}{M}\mathrm{Tr} Q = \frac{1}{M}\sum_j \tau_j$, the variance of τ_W and the moments of the proper time delays $\frac{1}{M}\langle\sum_j \tau_j^k\rangle$. Interesting properties of the Wigner time delay include its relation to the total scattering phase shift $\tau_W(E) = -\frac{\mathrm{i}}{M}\frac{\mathrm{d}}{\mathrm{d}E}\ln\det S(E)$ and to the density of states $\tau_W(E) = \frac{2\pi}{M} d(E)$.

A remarkable property of chaotic cavities is that the statistical distributions of the transmission eigenvalues T_j and the proper time delays τ_j are expected to become universal in the limit of short wavelengths ($k \to \infty$) if the leads are sufficiently thin. These universal distributions are described by random matrix theory. This is discussed in the following.

III. Random Matrix Theory

There are two different approaches for applying random matrix theory to open chaotic cavities. In one approach the $M \times M$ scattering matrix is related to an $N \times N$ Hermitian matrix H that describes the eigenmodes of the corresponding closed cavity (without leads), and an $N \times M$ coupling matrix V that describes the coupling between inside and outside [11]. The relation is given by

$$S(E) = I - \mathrm{i}V^\dagger \frac{1}{E - \mathcal{H}_{\mathrm{eff}}} V\,, \qquad \mathcal{H}_{\mathrm{eff}} = H - \frac{\mathrm{i}}{2} V V^\dagger. \quad (5)$$

The matrix H can be chosen, for example, as a random Gaussian matrix and the matrix V can either be taken fixed or random [12].

In the second approach the scattering matrix is modelled directly by a random matrix. Derived from an information theoretic approach, the corresponding distribution of the S-matrix is uniquely parametrised by the average scattering matrix $\bar{S}$ and is described by the so-called Poisson kernel [13]. For perfect coupling ($\bar{S} = 0$) the relevant ensembles are the circular ensembles.

The two approaches for the scattering matrix can be shown to be equivalent in some cases [14].

For the transmission eigenvalues the results are the following. The joint probability density function is given by the Jacobi ensemble which has the form [15], [16]

$$P(T_1, T_2, \ldots, T_n) = \mathcal{N}_\beta \prod_{j=1}^{n} T_j^\alpha \prod_{1 \le j < k \le n} |T_j - T_k|^\beta. \quad (6)$$

Here $\alpha = \beta/2(|M_2 - M_1| + 1) - 1$ and $\mathcal{N}_\beta$ is a normalisation constant. The parameter β depends on time-reversal properties. Chaotic cavities are invariant under time reversal and then $\beta = 2$.

Using the distribution of the transmission eigenvalues (6), results for related quantities can be obtained. As an example, we state the result for the transmission coefficient ($\beta = 2$)

$$T = \frac{M_1 M_2}{M+1} = \frac{M_1 M_2}{M} - \frac{M_1 M_2}{M^2} + \frac{M_1 M_2}{M^3} - \ldots. \quad (7)$$

It deviates from the classical transmission that is given by the first term in this expansion, $M_1 M_2 / M$.

The corresponding results for the proper time delays where obtained in [17]. They are conveniently expressed in terms of the inverses of the proper time delays $\gamma_j = 1/\tau_j$. The joint probability density function of the γ_j is given by the Laguerre ensemble which has the form [17]

$$P(\gamma_1, \ldots, \gamma_n) = \mathcal{N}_\beta \prod_{j=1}^{n} \gamma_j^{\beta M/2}\, \mathrm{e}^{-\beta\gamma_j/2} \prod_{1 \le j < k \le n} |\gamma_j - \gamma_k|^\beta. \quad (8)$$

The applicability of random matrix theory to chaotic cavities was first established empirically, but it was later confirmed by asymptotic methods in the limit of short wavelengths. The following section discusses these asymptotic approaches.

IV. Approximations in the Limit of Short Wavelengths

The elements t_{ba} of the transmission matrix t can be approximated in the short-wavelength limit $k \to \infty$ in terms of rays that go from the incoming lead to the outgoing lead [18]

$$t_{ba} \approx \sum_{\gamma : a \to b} A_\gamma \exp(\mathrm{i}kL_\gamma). \quad (9)$$

An example of such a ray is shown in Fig. 1. The incoming and outgoing channels a and b determine the modulus of the angles with which the rays enter and leave the cavity (with respect to the normals at the openings)

$$\sin\theta_1 = \pm\frac{a\pi}{kw_1}\,, \qquad \sin\theta_2 = \pm\frac{b\pi}{kw_2}, \quad (10)$$

where $a \in \{1, \ldots, M_1\}$ and $b \in \{1, \ldots, M_2\}$. The sum over γ in (9) runs over the infinite number of such trajectories, L_γ is the length of the ray γ and A_γ an amplitude factor that depends on stability properties.

We discuss in the following how this approximation can be applied in order to reproduce the random matrix result (7) for the transmission coefficient T. The approximation for T is given by

$$\begin{aligned} T &= \left\langle \sum_{a,b} t_{ba}\, \overline{t_{ba}} \right\rangle_k \quad (11) \\ &\approx \left\langle \sum_{a,b} \sum_{\gamma,\gamma' : a \to b} A_\gamma\, \overline{A_{\gamma'}}\, \exp(\mathrm{i}k(L_\gamma - L_{\gamma'})) \right\rangle_k. \end{aligned}$$

This expression contains a double sum over rays γ and γ'. It involves a local average over the wavenumber k to smooth out fluctuations and to obtain the mean transmission. A central observation is that most terms in this double sum do not contribute because the average over the wavenumber k removes most of these highly oscillatory terms. The only important contributions come from pairs of rays that are correlated.

In a first approximation expression (11) was considered in the diagonal approximation where $\gamma = \gamma'$ [19]

$$T_{\mathrm{diag}} = \sum_{a,b} \sum_{\gamma : a \to b} |A_\gamma|^2\,. \quad (12)$$

This approximation can be evaluated by using a classical sum rule that is based on average properties of long rays in an open chaotic cavity. It can be obtained by considering the probability density that rays that enter the cavity with angle

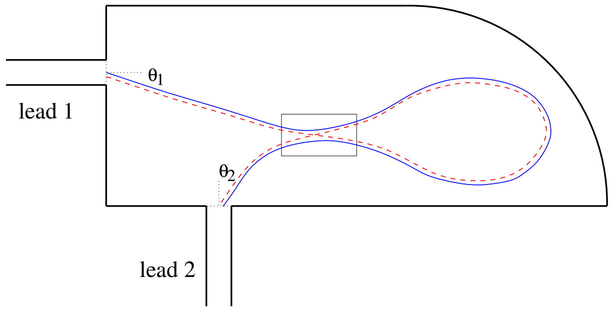

Fig. 2. A schematic view of a pair of trajectories that are correlated (blue full line and red dashed line). The two rays differ by traversing the loop in opposite directions. Both rays have a close self-encounter, indicated by the rectangular box, in which two stretches of a ray become almost (anti-)parallel.

θ_1 will leave it after time t with angle θ_2. One the one hand, this probability density can be related to the sum in (12). On the other hand it can be evaluated asymptotically by using the property of chaotic systems that long trajectories are equally likely to be anywhere in the cavity. One has to take furthermore into account that the trajectories have a probability to escape in time t that is characterised by the dwell time. The resulting sum rule for T_{diag} is

$$T_{\text{diag}} \sim \sum_{a,b} \frac{1}{M\tau_D} \int_0^\infty \mathrm{d}t \, e^{-t/\tau_D} \sim \frac{M_1 M_2}{M}, \tag{13}$$

where τ_D is the classical dwell time. The result is the classical transmission coefficient which is the leading order term in the random matrix result (7).

The corrections to the leading term involve correlated pairs of trajectories that are not identical. The first correction comes from rays that are almost identical, except that they contain a loop that is traversed by both rays in opposite directions. This is possible because the rays have a close self-encounter. A schematic view of such a correlated pair of rays is shown in Fig. 2. In reality the rays consist of straight line segments with specular reflections at the boundary as in Fig. 1, but the mechanism of correlation is the same. The two rays have a close self-encounter and they traverse the adjacent loop in opposite directions. The contributions of these pairs of rays were evaluated in [20], generalising methods for closed systems [21], and they give the next-to-leading order term in (7).

All higher order correlations were evaluated in [22], [23], also generalising methods for closed systems [24], [25]. They involve pairs of rays with arbitrary many self-encounters and arbitrary links between them, and they reproduce the complete expansion of the transmission coefficient T in (7) in agreement with random matrix theory. A central step in the evaluation in [22], [23] was the realisation that contributions of rays to T could be expressed in terms of diagrammatic rules. The diagrammatic rules reduced the calculations in the short-wavelength limit to a purely combinatorial problem of summing over various diagrams that expressed all possible correlations between rays. It opened the door for applying the ray approximation to the calculation of all moments of the transmission eigenvalues.

To illustrate the diagrammatic rules, let us apply them to the contributions of pairs of rays with one self-encounter as in Fig. 2. The rules are the following: there is a factor $1/M$ for every *link*. These are parts of the rays that connects encounter with encounter or encounter with lead. Furthermore, there is a factor $(-M)$ for every *encounter*, and a further factor of $M_1 M_2$ for the number of incoming and outgoing leads. The pair of rays in Fig. 2 have three links and one encounter and their contribution is

$$\left(\frac{1}{M}\right)^3 (-M)\, M_1 M_2 = -\frac{M_1 M_2}{M^2} \, . \tag{14}$$

This is in deed the second term in the expansion (7). Note that these diagrammatic rules were obtained from the properties of long rays in chaotic systems, similarly as the classical sum rule in (13).

In summary, the following results for the transmission moments $\mathcal{M}_k = \frac{1}{n}\langle \sum_j T_j^k \rangle$ were obtained: The first correction to the first moment $\mathcal{M}_1$ [20], the leading order for the second moment $\mathcal{M}_2$ [26], all orders for the first and second moment $\mathcal{M}_1$ and $\mathcal{M}_2$ [22], [23], the leading order of all moments $\mathcal{M}_k$ [27], the second order of all moments $\mathcal{M}_k$ [28], and finally all orders of all moments $\mathcal{M}_k$ [29]–[32]. All results are in agreement with random matrix theory.

For the proper time delays the results are not as complete, as is discussed in the following. One can base short-wavelength approximations for the moments of the proper time delays on the definition of the Wigner-Smith matrix in (4). This involves the same type of lead-connecting rays as for the transmission moments. However, one does not have similar diagrammatic rules in this case and this makes the calculations more complicated. Results that have been obtained for the moments of the proper time delays $m_k = \frac{1}{M}\langle \sum_j \tau_j^k \rangle$ in this way are: All orders for the first moment m_1 [33], the leading order for all moments m_k [34], and the first two orders for all moments m_k [28].

Recently, however, a different approximation in the short-wavelength limit has been derived for the time delays [35]. It is based on the so-called resonance approximation for the Wigner-Smith matrix [36]

$$Q = \hbar\, V^\dagger \frac{1}{(E - \mathcal{H}_{\text{eff}})^\dagger} \, \frac{1}{(E - \mathcal{H}_{\text{eff}})} V, \tag{15}$$

and it involves rays that start in a lead and end inside the cavity. With this new approximation it has now been possible to establish diagrammatic rules for the evaluation of ray correlations, and they have been applied to derive all orders for the second moment m_2 and the leading five orders for all moments m_k [35]. A remaining open problem is to obtain all orders for all moments m_k as was possible for the transmission eigenvalues.

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