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LOCAL CLUB CONDENSATION AND L-LIKENESS

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Abstract. We present a forcing to obtain a localized version of Local Club Condensation, a generalized Condensation principle introduced by Sy Friedman and the first author in [3] and [5]. This forcing will have properties nicer than the forcings to obtain this localized version that could be derived from the forcings presented in either [3] or [5]. We also strongly simplify the related proofs provided in [3] and [5]. Moreover our forcing will be capable of introducing this localized principle at κ while simultaneously performing collapses to make κ become the successor of any given smaller regular cardinal. This will be particularly useful when κ has large cardinal properties in the ground model. We will apply this to measure how much L-likeness is implied by Local Club Condensation and related principles. We show that Local Club Condensation at κ^+ is consistent with $\neg\square_\kappa$ whenever κ is regular and uncountable, generalizing and improving a result of the third author in [14], and that if $\kappa \geq \omega_2$ is regular, CC(κ^+) - Chang's Conjecture at κ^+ - is consistent with Local Club Condensation at κ^+, both under suitable large cardinal consistency assumptions.

1. Condensation and L-likeness

Besides the presentation of the forcing announced in the abstract, the central theme of this paper is the relationship between generalized Condensation principles (i.e. generalizations of consequences of Gödels Condensation Lemma) and other L-like principles; we investigate the question of how close to Gödels constructible universe the universe of sets has to be given that it satisfies certain generalized Condensation principles. For definitions of generalized Condensation principles that will be relevant to this paper see Section 2.

In [3], Sy Friedman and the first author showed that Local Club Condensation allows for the existence of very large large cardinals, far beyond those compatible with $\mathbf{V} = \mathbf{L}$ - namely they showed, by using the method of forcing, that Local Club Condensation is consistent with the existence of ω-superstrong cardinals. This was further improved in [4] by showing that Local Club Condensation and Acceptability are simultaneously consistent with the existence of ω-superstrong cardinals.

It is generally believed that the fine structural properties of $\mathbf{L}$ are necessary to prove that various square principles hold in $\mathbf{L}$. In [14], the third author showed that Strong Condensation for ω_2 is consistent with $\neg\square_{\omega_1}$ from a stationary limit of measurable cardinals, thus giving additional support to this belief. One of the main aims of this paper is to generalize his result to cardinals beyond ω_2, replacing Strong

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Condensation for ω_2 by Local Club Condensation at κ for $\kappa \geq \omega_2$[1] and reducing the consistency assumption to a 2-Mahlo cardinal.[2] We further investigate weaker square principles, Jónsson cardinals and Chang's Conjecture style principles, all in the context of generalized Condensation principles.

2. Condensation Principles

The definitions of Strong Condensation and Local Club Condensation apply to models $\mathbf{M}$ of set theory with a hierarchy of levels of the form $\langle M_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ with the properties that $\mathbf{M} = \bigcup_{\alpha\in\mathrm{Ord}} M_\alpha$, each M_α is transitive, $\mathrm{Ord}(M_\alpha) = \alpha$, if $\alpha < \beta$ then $M_\alpha \in M_\beta$ and if γ is a limit ordinal, $M_\gamma = \bigcup_{\alpha<\gamma} M_\alpha$. We will also let M_α denote the structure $(M_\alpha, \in, \langle M_\beta \mid \beta < \alpha\rangle)$,[3] where context will usually clarify the intended meaning. Moreover we denote $\langle M_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ or any of its restrictions by $\vec{M}$.

If $\mathcal{X}$ is a substructure of $(M_\alpha, \in, \vec{M})$ for some $\alpha \in \mathrm{Ord}$, we say that $\mathcal{X}$ *condenses* or *is a condensing substructure* of M_α if $\mathcal{X}$ is isomorphic to $(M_{\bar\alpha}, \in, \vec{M})$ for some $\bar\alpha \leq \alpha$. More generally, if $\mathcal{A}$ is a structure for a countable language of the form $\mathcal{A} = (M_\alpha, \in, \vec{M}, \ldots)$ for some $\alpha \in \mathrm{Ord}$ and $\mathcal{X}$ is a substructure of $\mathcal{A}$ with domain X, we say that $\mathcal{X}$ *condenses* or *is a condensing substructure* of $\mathcal{A}$ if $(X, \in, \vec{M}\restriction X)$ condenses.

Local Club Condensation is the statement that if α has uncountable cardinality κ and $\mathcal{A}_\alpha = (M_\alpha, \in, \vec{M}, \ldots)$ is a structure for a countable language, then there exists a continuous chain $\langle \mathcal{B}_\gamma \mid \omega \leq \gamma < \kappa\rangle$ of condensing substructures of $\mathcal{A}_\alpha$ whose domains have union M_α, where each $B_\gamma = \mathrm{dom}(\mathcal{B}_\gamma)$ is s.t. $|B_\gamma| = |\gamma|$ and $\gamma \subseteq B_\gamma$.

We will usually be in the situation that $\mathbf{M} = (\mathbf{L}[A], \in, A)$ for some $A \subseteq \mathrm{Ord}$ and $\vec{M} = \langle L_\alpha[A] \mid \alpha \in \mathrm{Ord}\rangle$. We say that $\mathbf{M}$ is of the form $\mathbf{L}[A]$ in that case.

If $\kappa = \lambda^+$ and λ is uncountable, *Local Club Condensation at* κ is the statement that $M_\kappa = H_\kappa$ and if $\alpha \in [\lambda, \kappa)$ and $\mathcal{A} = (M_\alpha, \in, \vec{M}, \ldots)$ is a structure for a countable language, then there exists a continuous chain $\langle \mathcal{B}_\gamma \mid \gamma < \lambda\rangle$ of condensing substructures of $\mathcal{A}$ whose domains have union M_α, where each $B_\gamma = \mathrm{dom}(\mathcal{B}_\gamma)$ is s.t. $|B_\gamma| < \lambda$ and $\gamma \subseteq B_\gamma$.[4]

Note: The reason why one need not include the case $\alpha = \kappa$ in the above is that if $\mathcal{A} = (M_\kappa, \in, \vec{M}, \ldots)$ is a structure for a countable language, then there is a continuous chain of transitive, elementary substructures of $\mathcal{A}$ of size λ that each have domain some $M_{\bar\alpha}$, $\bar\alpha < \kappa$ and have union M_κ. Now we may apply Local Club Condensation at κ to the least substructure of that chain to obtain a continuous chain $\langle \mathcal{B}_\gamma \mid \gamma < \kappa\rangle$ of condensing substructures of $\mathcal{A}$ whose domains have union M_κ, where each $B_\gamma = \mathrm{dom}(\mathcal{B}_\gamma)$ is s.t. $|B_\gamma| < \lambda$ for $\gamma < \lambda$, $|B_\gamma| = \lambda$ for $\gamma \in [\lambda, \kappa)$ and $\gamma \subseteq B_\gamma$ for every $\gamma < \kappa$.

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[1] Local Club Condensation at ω_2 implies Strong Condensation for ω_2, see Theorem 1 below.

[2] A cardinal κ is 2-Mahlo if the set of Mahlo cardinals below κ is a stationary subset of κ. In the first submitted version of this paper, our large cardinal assumption was a stationary limit of ω-Erdős cardinals. The key hint on how to further reduce this consistency assumption was given to the authors in personal communication by Boban Veličković.

[3] We assume some appropriate coding of the sequence $\langle M_\beta \mid \beta < \alpha\rangle$ here.

[4] If $\lambda = \theta^+$ is a successor cardinal, we may equivalently demand that $|B_\gamma| = \theta$.

In [3], the theorem below is shown assuming Local Club Condensation holds in $\mathbf{M}$, but all that is actually used is Local Club Condensation at κ, giving rise to the following (we will abbreviate the conclusion of the theorem by saying that *Transitive Condensation at* κ holds):

Theorem 1 (Friedman, Holy, Wu)**.** [4, Theorem 88] *If* $(\mathbf{M}, \in, \vec{M})$ *is a model of Local Club Condensation at* κ*, where* $\kappa = (\tau^+)^{\mathbf{M}}$*,* τ *is an* $\mathbf{M}$*-cardinal of uncountable cofinality,* $F = \langle f_\alpha \colon \alpha \in [\tau, \kappa)\rangle$ *where each* f_α *is a bijection from* τ *to* α *in* $\mathbf{M}$*,*

$$X \prec (M_\kappa, \in, \vec{M}, F, S)$$

where S *is a set of Skolem functions for the above structure and* X *is transitive below* τ*, then* X *condenses. In fact,* X *need not be an element of* $\mathbf{M}$ *for the above to hold.* □

Strong Condensation is the statement that for every ordinal α, there is a structure $\mathcal{A}_\alpha = (M_\alpha, \in, \vec{M}, \ldots)$ for a countable language such that each of its substructures condenses.[5] *Strong Condensation for* α ($\alpha \in \mathbf{Card}$) is the statement of Strong Condensation for a single cardinal α together with the assumption that $M_\alpha = H_\alpha$.

As was observed in [4], Local Club Condensation at ω_2 implies (by Theorem 1) Transitive Condensation at ω_2 which is easily seen to imply Strong Condensation for ω_2. In Section 4, we will observe that this is not the case for $\kappa \geq \omega_3$, i.e. that Local Club Condensation at κ does not imply Strong Condensation for κ.

We define one last version of Local Club Condensation that strengthens Local Club Condensation at κ (we will observe that this is a proper strengthening in Section 4).

If $\omega \leq \lambda < \lambda^+ < \kappa$, *Local Club Condensation in* $[\lambda, \kappa)$ is the statement that $M_\kappa = H_\kappa$ and if $\alpha \in [\lambda^+, \kappa)$ and $\mathcal{A}_\alpha = (M_\alpha, \in, \vec{M}, \ldots)$ is a structure for a countable language, then there exists a continuous chain $\langle \mathcal{B}_\gamma \mid \lambda \leq \gamma < \kappa\rangle$ of condensing substructures of $\mathcal{A}_\alpha$ whose domains have union M_α, where each $B_\gamma = \mathrm{dom}(\mathcal{B}_\gamma)$ is s.t. $|B_\gamma| = |\gamma|$ and $\gamma \subseteq B_\gamma$. If $\lambda = \omega$ in the above, we call the resulting principle *Local Club Condensation up to* κ.

Note: Whenever $\lambda_0 < \lambda_1 < {\lambda_1}^+ < \kappa$, Local Club Condensation in $[\lambda_0, \kappa)$ is stronger than Local Club Condensation in $[\lambda_1, \kappa)$ which in turn is stronger than Local Club Condensation at κ. If $\kappa = \lambda^{++}$, Local Club Condensation in $[\lambda, \kappa)$ is the same as Local Club Condensation at κ.

3. Easy observations regarding L-likeness

Local Club Condensation implies the GCH (see [3, Lemma 1]). If $\kappa = \lambda^+$ and λ is uncountable, Local Club Condensation at κ implies $2^\lambda = \kappa$ and $2^{<\lambda} = \lambda$, in fact it is easily seen to imply that $H_\lambda = M_\lambda$ has cardinality λ. We will see in Section 4 that it does not impose any further restrictions on the values of the continuum function. Strong Condensation for κ implies that the GCH holds below κ, in fact it implies that $M_\lambda = H_\lambda$ has cardinality λ for every uncountable $\lambda \leq \kappa$. We provide a proof of this last statement.

Lemma 2. *If* κ *is an uncountable cardinal, Strong Condensation for* κ *implies that for every uncountable cardinal* $\lambda \leq \kappa$*,* $M_\lambda = H_\lambda$*.*

[5] Strong Condensation was originally introduced by Hugh Woodin in [13].

Proof. This is part of the definition of Strong Condensation for κ if $\lambda = \kappa$. Thus we may assume that $\lambda < \kappa$.

Assume $x \in H_\lambda$. Let N be a condensing elementary substructure of M_κ that contains the transitive closure of x as a subset and has size less than λ. It follows that $x \in M_\lambda$, i.e. $H_\lambda \subseteq M_\lambda$.

Let $\mathcal{A}$ be a Skolemized structure on H_κ witnessing Strong Condensation for κ holds. For $\alpha < \lambda$, let N_α be the Skolem hull of α in $\mathcal{A}$. Each N_α condenses to some $M_{f(\alpha)}$, where $f(\alpha) < \lambda$ and $\{f(\alpha) \mid \alpha < \lambda\}$ is cofinal in λ. It follows that M_α has size less than λ whenever $\alpha < \lambda$. Since each M_α is transitive, it thus follows that $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha \subseteq H_\lambda$ has cardinality λ. □

We mention some further facts about Local Club Condensation, localizations of which may also be easily obtained.

Fact 3. [4, Lemma 95] *Local Club Condensation implies*

- $\diamondsuit_\kappa(E)$ *whenever* κ *is regular and* $E \subseteq \kappa$ *is stationary.*
- $\diamondsuit^+_\kappa$ *for all successor cardinals* κ*.*

Fact 4. [3, Theorem 37] *Local Club Condensation implies that whenever* κ *is regular, there is a* $\mathbf{\Delta}_1$*-definable wellorder of* H_{κ^+}*.*

Strong Condensation has some even more striking consequences:

Fact 5. [3, Theorem 4] *Strong Condensation implies that there is no* ω_1*-Erdős cardinal.*

Fact 6. [9, Corollary 1.13] *Strong Condensation for* ω_3 *implies that there is no precipitous ideal on* ω_1*.*

4. The Forcing Construction

In this section, we present our main forcing construction. This is a (strongly simplified and improved) variation of forcing constructions (and the corresponding proofs of their properties) from [3] and [5], that allows us to obtain Local Club Condensation at κ for a given regular cardinal κ while collapsing κ to become the successor of any given smaller regular uncountable cardinal λ.[6] When $\kappa = \lambda^+ = \omega_2$, the construction below is a significant simplification to obtain the main technical result of [14], namely a small forcing to obtain Strong Condensation for ω_2.

Assume κ is regular. We want to extend a given model $\mathbf{V}$ of set theory to a model of Local Club Condensation at κ while, for some regular cardinal $\lambda < \kappa$, collapsing all cardinals in (λ, κ) so that $\kappa = \lambda^+$ in the generic extension. We assume that $\mathbf{V}$ satisfies $2^{<\lambda} = \lambda$, $2^\lambda \leq \kappa$ and $\theta^{<\lambda} < \kappa$ for every $\theta < \kappa$, define a forcing iteration $P = P(\lambda, \kappa)$ and show that P-generic extensions of the universe satisfy Local Club Condensation at κ, model $\kappa = \lambda^+$ and $2^\lambda = \kappa$ and that forcing with P preserves λ and both the continuum function and all cardinals outside of the interval $[\lambda, \kappa)$. P will be $<\lambda$-directed closed and κ-cc. We define P inductively. $P_{<\lambda}$, the forcing up to λ, is just the product of length λ with $<\lambda$-sized support of the lottery of $\{0, 1\}$.[7]

If $\alpha \geq \lambda$, a *condition at* α is a pair (f_α, c_α) which is either trivial, i.e. $(f_\alpha, c_\alpha) = (\emptyset, \emptyset)$, or there is $\gamma_\alpha < \lambda$ such that

[6] This is supposed to include the case where $\kappa = \lambda^+$ initially, i.e. when no actual collapses are performed by the forcing.

[7] So $P_{<\lambda}$ is just the forcing to add a Cohen subset of λ, in disguise.

- $c_\alpha\colon \gamma_\alpha \to 2$ is such that $C_\alpha = \{\delta < \gamma_\alpha \mid c_\alpha(\delta) = 1\}$ is closed in λ, [8]
- $f_\alpha\colon \max(C_\alpha) \to \alpha$ is an injection and
- $f_\alpha[\max(C_\alpha)] \supseteq \max(C_\alpha)$.

If $p^0 = (f^0, c^0)$ and $p^1 = (f^1, c^1)$ are conditions at α, we let $p^1 \leq p^0$ iff p^0 is trivial or

(1) $f^1 \supseteq f^0$ and
(2) $c^1 \supseteq c^0$, i.e. $c^1 \restriction \mathrm{dom}(c^0) = c^0$.

Note that if we force with the poset consisting of conditions at α, ordered as above, this will generically add a bijection from λ to α.

While defining $P_{<\alpha}$ for $\alpha \in (\lambda, \kappa]$ we also define a function A with domain $[\lambda, \kappa)$ such that for every α, $A(\alpha)$ is a $P_{<\alpha}$-name for either 0 or 1. We fix a wellorder $\mathcal{W}$ of H_κ of order-type κ. Let $\beta \in [\lambda, \kappa)$ and assume $A \restriction \beta$ and $P_{<\beta}$ have been defined. Let $A(\beta)$ be the canonical $P_{<\beta}$-name for either 0 or 1 such that for any $P_{<\beta}$-generic $G_{<\beta}$, $A(\beta)^{G_{<\beta}} = 1$ iff $\beta = \prec\gamma, \prec\delta, \epsilon\succ\succ$,[9] $\dot{x}$ is the γ^{th} (in the sense of $\mathcal{W}$) $P_{<\delta}$-nice name for a subset of λ, $\epsilon < \lambda$ and $\epsilon \in \dot{x}^{G_{<\beta}}$.

Now assume $\alpha \in (\lambda, \kappa]$ and we defined $\langle P_{<\beta} \mid \beta < \alpha\rangle$ and $A \restriction \alpha$. Then $P_{<\alpha}$ is the set of all α-sequences p with $<\lambda$-support such that

- $p \restriction \beta \in P_{<\beta}$ for every $\beta < \alpha$ and if $\alpha = \beta + 1$, the following hold:
- $p(\beta) = (f_\beta, c_\beta)$ is a condition at β,
- if $C_\beta \neq \emptyset$, then $p \restriction \beta$ decides $A(\beta) = a_\beta \in \{0, 1\}$,
- $\forall \delta \in C_\beta\ p(\mathrm{ot}\, f_\beta[\delta]) = a_\beta$,

* We let C-supp$(p) = \{\beta \mid C_\beta \neq \emptyset\}$. The following need to hold for p as well:[10]

- $\gamma^p := \mathrm{supp}(p) \cap \lambda = \gamma_\beta = \mathrm{dom}(c_\beta)$ for any $\beta \in$ C-supp(p),
- $\exists \delta^p\, \forall \beta \in$ C-supp(p) $\max(C_\beta) = \delta^p$,
- $\forall \beta_0 < \beta_1$ both in C-supp(p),

$$f_{\beta_0}[\delta^p] \text{ is an initial segment of } f_{\beta_1}[\delta^p]$$

and

$$f_{\beta_1}[\delta^p] \setminus \beta_0 \neq \emptyset.$$

For p and q in $P_{<\alpha}$, we let $q \leq p$ iff $q \restriction \lambda \leq p \restriction \lambda$ and for every $\beta \in [\lambda, \alpha)$, $q(\beta) \leq p(\beta)$. We let $P = P_{<\kappa}$. Note that if $\beta < \alpha$ then $P_{<\beta} \subseteq P_{<\alpha}$.

Claim 7. *If $p \in P_{<\alpha}$, $\beta \in [\lambda, \alpha)$ and $\delta < \lambda$, then there is $q \leq p$ with $\beta \in$ C-supp(q) and $\delta^q > \delta$.*

Proof: We obtain an α-sequence $r = \langle (f^r_\zeta, c^r_\zeta) \mid \zeta < \alpha\rangle$ from p by extending $p \restriction \beta$ to $r \restriction \beta \in P_{<\beta}$ such that $r \restriction \beta$ decides $A(\beta) = a_\beta \in \{0, 1\}$ and setting $r \restriction [\beta, \alpha) = p \restriction [\beta, \alpha)$.[11] Choose $\xi > \delta$ such that $\gamma^{r\restriction\beta} + \xi = \xi$. We want to find a componentwise

[8] Thus if γ_α is a limit ordinal, C_α is bounded in γ_α. To avoid this case (which we don't), one could simply demand that γ_α is a successor ordinal.

[9] $\prec\cdot,\cdot\succ$ denotes the Gödel pairing function.

[10] The remaining clauses will help ensure both that our forcing is sufficiently closed and that the following proofs go through easily. None of these clauses (nor the last clause for a condition at α) were used in the forcing constructions presented in either [3] or [5]. While this made the presentation of the forcing constructions itself somewhat easier, it made the corresponding proofs much more difficult (and also the forcings provided there were not closed).

[11] As our forcing is no standard iteration, it is not necessarily the case here that $r \in P_{<\alpha}$, as for example possibly $\gamma^{r\restriction\beta} > \gamma^p$.

end-extension q of r with $q \in P_{<\alpha}$, $\delta^q = \xi$ and $\beta \in \text{C-supp}(q)$. It then follows that $q \leq p$, i.e. q is as desired.

Let $\text{C-supp}(r) = \{\zeta \mid C^r_\zeta \neq \emptyset\}$. For every $\zeta \in \text{C-supp}(r) \cup \{\beta\}$, we choose $c^q_\zeta \colon (\xi+1) \to 2$ such that $c^q_\zeta \restriction \text{dom}(c^r_\zeta) = c^r_\zeta$, $c^q_\zeta \restriction [\text{dom}(c^r_\zeta), \xi) = \vec{0}$ and $c^q_\zeta(\xi) = 1$. By our assumptions on ξ, we may extend f^r_ζ to f^q_ζ with domain ξ such that $f^q_\zeta[\xi] \supseteq \xi$ and such that whenever $\zeta_0 < \zeta_1$ are both in $\text{C-supp}(r) \cup \{\beta\}$, then $f^q_{\zeta_0}[\xi]$ is an initial segment of $f^q_{\zeta_1}[\xi]$ and $f^q_{\zeta_1}[\xi] \setminus \zeta_0 \neq \emptyset$, using that ξ was chosen sufficiently large. This now allows us to choose $q(\text{ot}\, f_\zeta[\xi]) = a_\zeta$ for $\zeta \in \text{C-supp}(r) \cup \{\beta\}$, where if $\zeta \neq \beta$, $a_\zeta \in \{0,1\}$ is such that either $p \restriction \zeta$ (if $\zeta > \beta$) or $r \restriction \zeta$ (if $\zeta < \beta$) decides $A(\zeta) = a_\zeta$. $\square$

The following useful fact can easily be extracted from the proof of Claim 7:

Fact 8. *If $p \in P_{<\alpha}$, $\beta \in [\lambda, \alpha)$, $q \in P_{<\beta}$ and $q \leq p \restriction \beta$, then there is $r \in P_{<\alpha}$ stronger than both p and q.* $\square$

Claim 9. *If $\beta < \alpha$, then $P_{<\beta}$ is a complete subforcing of $P_{<\alpha}$.*

Proof: Let X be a maximal antichain of $P_{<\beta}$ and let p be a condition in $P_{<\alpha}$. Then $p \restriction \beta \in P_{<\beta}$ is compatible with some element of X as witnessed by $q \in P_{<\beta}$ which is stronger than both. By Fact 8, p and q are compatible in $P_{<\alpha}$. $\square$

Notation: Given a decreasing sequence of conditions $\langle p^i \mid i < \delta\rangle$ in $P_{<\alpha}$, we say that $r = \langle r(\beta) \mid \beta < \alpha\rangle$ is the *componentwise union* of $\langle p^i \mid i < \delta\rangle$ if for every $\beta < \lambda$, $r(\beta) = \bigcup_{i<\delta} p^i(\beta)$, and for $\beta \geq \lambda$, $c^r_\beta = \bigcup_{i<\delta} c^{p^i}_\beta$ and $f^r_\beta = \bigcup_{i<\delta} f^{p^i}_\beta$. r is usually not a condition in $P_{<\alpha}$ as the c^r_β are not necessarily closed. We let $\text{C-supp}(r)$ denote $\{\beta \mid C^r_\beta \neq \emptyset\} = \bigcup_{i<\delta} \text{C-supp}(p^i)$.

Claim 10. *$P_{<\alpha}$ is $<\lambda$-closed.*

Proof: Let $\delta < \lambda$ be a limit ordinal and let $\langle p^i \mid i < \delta\rangle$ be a decreasing sequence of conditions in $P_{<\alpha}$. Let r be their componentwise union. Let $\gamma^r = \text{supp}(r) \cap \lambda = \bigcup_{i<\delta} \gamma^{p^i}$. Let $\delta^r = \bigcup_{i<\delta} \delta^{p^i}$. If $\delta^r < \gamma^r$, then r is a condition in $P_{<\alpha}$ and a lower bound for $\langle p^i \mid i < \delta\rangle$. Thus assume that $\delta^r = \gamma^r$ is a limit ordinal from now on. We want to form q out of r (by componentwise end-extension) such that $q \leq p^i$ for every $i < \delta$. We have to set $C^q_\beta = C^r_\beta \cup \{\delta^r\}$ for every $\beta \in \text{C-supp}(r)$.

As for every $i < \delta$, $f^{p^i}_\beta[\delta^{p^i}] \supseteq \delta^{p^i}$, we obtain that $f^r_\beta[\delta^r] = \bigcup_{i<\delta} f^{p^i}_\beta[\delta^{p^i}] \supseteq \delta^r$ for every $\beta \in \text{C-supp}(r)$. Similarly, if $\beta_0 < \beta_1$ are both in $\text{C-supp}(r)$, we obtain that $f^r_{\beta_0}[\delta^r]$ is an initial segment of $f^r_{\beta_1}[\delta^r]$ and β_0 and β_1 are both in $\text{C-supp}(p^i)$ for some $i < \delta$ and hence $f^{p^i}_{\beta_1}[\delta^{p^i}] \setminus \beta_0 \neq \emptyset$, implying that $f_{\beta_1}[\delta^r] \setminus \beta_0 \neq \emptyset$. This now allows us to choose $q(\text{ot}\, f_\beta[\delta^r]) = a_\beta$ for every $\beta \in \text{C-supp}(r)$, where a_β is such that some p^i decides $A(\beta) = a_\beta \in \{0,1\}$, and moreover set $c^q_\beta = c^r_\beta \cup \{(\delta_r, 1)\} \cup \{(\gamma, 0) \mid \gamma \in (\delta_r, \sup\{\text{ot}\, f_\beta[\delta^r] + 1 \mid \beta \in \text{C-supp}(r)\})\}$ for every $\beta \in \text{C-supp}(r)$, to obtain a condition q as desired. $\square$

If $\delta < \lambda$ is a limit ordinal, $\langle p^i \mid i < \delta\rangle$ is a decreasing sequence of conditions in $P_{<\alpha}$ and q is the lower bound of $\langle p^i \mid i < \delta\rangle$ as obtained in the proof of Claim 10, then we write $q = \bigcup_{i<\delta} p^i$.

Definition 11. *If Q is a poset, $D \subseteq Q$ is* directed *if any two elements of D have a lower bound in D. We say that Q is $<\lambda$-directed closed if for any directed $D \subseteq Q$ of size less than λ, there is a condition in Q below all elements of D.*

Corollary 12. *The proof of Claim 10 in fact shows that $P_{<\alpha}$ is $<\lambda$-directed closed.* □

Claim 13. *$P_{<\alpha}$ is κ-cc.*

Proof: Assume for a contradiction that X is an antichain of $P_{<\alpha}$ of size κ. By a Δ-System argument using $2^{<\lambda} = \lambda$, there is r of size $< \lambda$ and $Y \subseteq X$ of size κ such that for any p_0, p_1 in Y, C-supp$(p_0) \cap$ C-supp$(p_1) = r$. $2^{<\lambda} = \lambda$ now implies that κ-many conditions in Y are compatible in $P_{<\alpha}$, contradicting our assumption. □

Claim 14. *For every $\alpha < \kappa$, $P_{<\alpha}$ has size less than κ. $P \subseteq H_\kappa$ has size κ, forces that $\kappa = \lambda^+$ and $2^\lambda = \kappa$ and preserves λ and both cardinals and the continuum function below λ and at and above κ.*

Proof: The first and second statements follow since $\theta^{<\lambda} < \kappa$ for every $\theta < \kappa$. If G is P-generic over $\mathbf{V}$ and f_α^G denotes $\bigcup_{p\in G} f_\alpha^p$, then f_α^G is a bijection from λ to α for every $\alpha \in [\lambda, \kappa)$ by an easy density argument. That P forces $2^\lambda = \kappa$ follows from the first statement of the claim and Claim 13. The rest of the claim is immediate by Claim 10, Claim 13 and the fact that our assumptions imply that $\kappa^{<\kappa} = \kappa$. □

We will use the following easy fact, a proof of which may be found in [3].

Fact 15. *Assume β has regular cardinality ν and for every $\gamma \leq \beta$, f_γ is a bijection from* card γ *to γ. Then there is a club of $\delta < \nu$ such that*

$$f_\alpha[\delta] = f_\beta[\delta] \cap \alpha \text{ for all } \alpha \in f_\beta[\delta] \setminus \nu. \square$$

Claim 16. *P forces Local Club Condensation at $\kappa = \lambda^+$.*

Proof: Let G be P-generic. Let B be the generic predicate obtained by letting $B{\restriction}\lambda = A_{<\lambda}$, where $A_{<\lambda}$ denotes the generic Cohen subset of λ added by G restricted to $P_{<\lambda}$, and for $\alpha \in [\lambda, \kappa)$, $B(\alpha) = a_\alpha$, where $a_\alpha \in \{0, 1\}$ is s.t. some $p \in G$ decides $A(\alpha) = a_\alpha$. Note that $H_\kappa^{\mathbf{V}[G]} = L_\kappa[B]$, as Claim 14 and Claim 13 show that every subset of λ in $\mathbf{V}[G]$ has a $P_{<\alpha}$-nice name in H_κ for some $\alpha < \kappa$. We claim that $\langle M_\alpha \mid \alpha < \kappa\rangle$ witnesses Local Club Condensation at κ in $\mathbf{V}[G]$ with $M_\alpha = L_\alpha[B]$. Asume $\alpha \in [\lambda, \kappa)$ and let $\mathcal{A}_\alpha = \langle M_\alpha, \in, \langle M_\beta \mid \beta < \alpha\rangle, \ldots\rangle$ be a structure for a countable language. We may assume that $\mathcal{A}_\alpha$ is Skolemized. Note that for $\beta \in \alpha \setminus \lambda$ we have $B(\beta) = B(\text{ot } f_\beta[\delta])$ for all δ in the club $\bigcup_{p\in G} c_\beta^p \subseteq \lambda$. It follows that for a club C of $\delta < \lambda$, $B(\beta) = B(\text{ot } f_\beta[\delta])$ and moreover $f_\beta[\delta] = f_\alpha[\delta] \cap \beta$ for all $\beta \in f_\alpha[\delta] \setminus \lambda$; this is seen using Fact 15. For any $X \subseteq \alpha$ let $\mathcal{A}_\alpha(X)$ be the least substructure of $\mathcal{A}_\alpha$ containing X as a subset of its domain $A_\alpha(X)$. Consider the continuous chain $\langle \mathcal{A}_\alpha(f_\alpha[\delta]) \mid \delta \in D\rangle$, where D consists of all elements δ of C s.t. $\delta = f_\alpha[\delta] \cap \lambda$ and $f_\alpha[\delta] = A_\alpha(f_\alpha[\delta]) \cap \text{Ord}$. Then $\mathcal{A}_\alpha(f_\alpha[\delta])$ condenses for each $\delta \in D$. □

Note: If $\kappa = \lambda^+$ and λ is regular and uncountable, the above provides a cofinality-preserving forcing to obtain Local Club Condensation at κ, generalizing [14] and in the case of $\kappa = \omega_2$ providing a strongly simplified version of the proofs given in [14], [3] and [5]. Note moreover that if $\kappa \geq \omega_3$, we may perform the above forcing over a model of $2^{\aleph_0} = \aleph_2$, to obtain a model of Local Club Condensation at κ in which CH fails. This contrasts the situation with both Strong Condensation for κ and Local Club Condensation up to κ, as they both imply CH to hold.

5. Local Club Condensation and the negation of Square

In [14], the third author obtained Strong Condensation for ω_2 (which is implied by Local Club Condensation at ω_2) and $\neg\square_{\omega_1}$ starting from a stationary limit of measurable cardinals. In this section, building on the methods introduced in [14] and on the forcing construction of Section 4, we generalize his result to cardinals larger than ω_2 and also reduce the large cardinal hypothesis to a 2-Mahlo cardinal. For convenience, we assume GCH throughout.

Definition 17. *If κ is regular and greater than ω_1, $\square(\kappa)$ is the statement that there exists a sequence $\langle C_\alpha \mid \alpha < \kappa\rangle$ such that the following hold:*

(1) Whenever α is a limit ordinal, C_α is a closed unbounded subset of α.
(2) If β is a limit point of C_α then $C_\beta = C_\alpha \cap \beta$.
(3) There is no club $C \subseteq \kappa$ such that for every limit point α of C, $C_\alpha = C \cap \alpha$.

Definition 18. *If $\kappa = \lambda^+ > \omega_1$, $\square_\lambda$ is the statement that there exists a sequence $\langle C_\alpha \mid \alpha < \kappa\rangle$ such that (1) and (2) from above hold together with the following:*

(3) For every α, $\mathrm{ot}(C_\alpha) \leq \lambda$.*

Lemma 19. *If η is Mahlo, $\theta \geq \eta$ is regular, $\lambda < \eta$ is regular and $\mathcal{A}$ is a structure for a countable language with domain H_θ, then there is a pair of models M_0^* and M_1^* such that*

(1) M_0^* and M_1^* are both substructures of $\mathcal{A}$.
(2) M_0^* and M_1^* both have size λ.
(3) $\lambda \subseteq M_0^*, M_1^*$.
(4) Let $\bar\delta = \sup(\eta \cap M_0^* \cap M_1^*)$. Then $P_{\omega_1}(M_0^* \cap V_{\bar\delta}) \subseteq M_1^*$.
(5) $\min(M_0^* \setminus \bar\delta)$ has cofinality $\geq \lambda$
(6) $\bar\delta < \sup(M_0^* \cap \eta) = \sup(M_1^* \cap \eta)$ and the latter have cofinality ω.

Proof: Large parts of this proof are based on the proof of [12, Lemma 3.5] and much of the adaptation below was essentially suggested to us by Boban Veličković.

Let $A = H_\theta$ denote the domain of $\mathcal{A}$ and let $F\colon [A]^{<\omega} \to A$ be such that $X \prec \mathcal{A}$ whenever X is closed under F. Consider the following two player game with perfect information of length ω. I starts by playing $\rho \in [\lambda, \eta)$ with $\mathrm{cof}(\rho) > \omega$. Then II plays $\delta_0 \in (\rho, \eta)$ and $A_0 \in [V_\rho]^\lambda$. If II has played δ_i and A_i for some $i < \omega$, I responds by playing α_i and β_i such that $\delta_i < \alpha_i < \beta_i < \eta$. Moreover I has to choose α_0 such that $\mathrm{cof}(\alpha_0) \geq \lambda$. If I has played α_i and β_i for some $i < \omega$, II responds by playing δ_{i+1} and A_{i+1} such that $\beta_i < \delta_{i+1} < \eta$ and $A_i \in [V_\rho]^\lambda$.

I	ρ	α_0, β_0	α_1, β_1	$\ldots$
II		δ_0, A_0	δ_1, A_1	

Let X be the closure under F of $\lambda \cup \bigcup_{i<\omega} A_i \cup \{\alpha_i \mid i < \omega\}$. I wins the game if $\mathrm{Ord}(X) \subseteq \rho \cup \bigcup_{i<\omega}[\alpha_i, \beta_i)$.

If II wins, he knows so by a finite stage, i.e. the above is an open game for II. It follows by [6] that the game is determined, i.e. one of the players has a winning strategy. We will now show that if I has a winning strategy, then we can construct M_0^* and M_1^* as desired, and then show that II does not have a winning strategy.

Thus assume now that I has a winning strategy in the above game. We will play the game on two boards simultaneously, denoting moves on the first board as above and adding a * for moves on the second board in our notation. On the first and second board, let I start by playing $\rho = \rho^*$ according to her strategy. On the first board, let II respond with $\delta_0 = \rho + 1$ and $A_0 = \emptyset$. Assume δ_i and A_i are played on the first board. Let I respond with α_i and β_i according to her strategy. On the second board, let II respond with $\delta_i^* = \beta_i$ and $A_i^* = [V_\rho \cap \mathrm{cl}_F(\lambda \cup \{\alpha_j \mid j \leq i\}]^{\leq \omega}$ and let I respond with α_i^* and β_i^* according to her strategy. Now on the first board let II respond with $\delta_{i+1} = \beta_i^*$ and $A_{i+1} = \emptyset$.

After playing as above for ω-many stages, let $M_0^* = \mathrm{cl}_F(\lambda \cup \{\alpha_i \mid i < \omega\})$ and let $M_1^* = \mathrm{cl}_F(\lambda \cup \bigcup_{i<\omega} A_i^* \cup \{\alpha_i^* \mid i < \omega\})$. We claim that M_0^* and M_1^* are as desired. (1), (2) and (3) are obvious. If $\bar{\delta} = \sup(\eta \cap M_0^* \cap M_1^*)$, it follows that $\bar{\delta} \leq \rho$, thus $P_{\omega_1}(M_0^* \cap V_{\bar{\delta}}) \subseteq M_1^*$ by our choice of the A_i^*, i.e. (4) holds. (5) and (6) are again obvious from our construction.

We will thus finish the proof of Lemma 19 by the following:

Claim 20. *II does not have a winning strategy in the above game.*

Proof: Assume for a contradiction that II has a winning strategy in the above game. Let $\theta^* > \theta$ be sufficiently large and regular. Let $\mathcal{H}$ be a structure for a finite language on H_{θ^*} that contains everything relevant, in particular $\mathcal{A}$ and the winning strategy for II, as a constant, together with a wellorder of H_{θ^*}. Let $\langle M_i \mid i < \eta\rangle$ be a continuous increasing $\in$-chain of elementary substructures of $\mathcal{H}$ which are transitive below η, such that there is $\delta < \eta$ which is inaccessible and $M_\delta \cap \eta = \delta$, using that η is Mahlo. For $i < \eta$, let $\xi_i = M_i \cap \eta$ and note that $\xi_\delta = \delta$. Choose a singular strong limit cardinal $\rho < \delta$ such that if $N = H^{\mathcal{H}}(V_\rho \cup \{\xi_{\delta\cdot i} \mid 1 \leq i < \omega\})$, we get $N \cap \delta = \rho$. Since $|V_\rho| = \rho$, N is bounded in $\xi_{\delta\cdot i}$ for each $i \geq 1$. We claim that I wins by playing (independent of II's moves) ρ, $\alpha_i = \xi_{\delta\cdot(i+1)}$ and $\beta_i = \sup(N \cap \xi_{\delta\cdot(i+2)})$. This is because of elementarity of M_i for $i < \eta$, II is forced to play $\delta_i \in M_{\delta\cdot(i+1)}$ and thus $\delta_i < \alpha_i$. We obtain

$$\mathrm{cl}_F(\lambda \cup \bigcup_{i<\omega} A_i \cup \{\alpha_i \mid i < \omega\}) \subseteq H^{\mathcal{H}}(V_\rho \cup \{\alpha_i \mid i < \omega\}).$$

But now I has won the run of the game, for the intersection of the latter set with the ordinals was arranged to be contained in $\rho \cup \bigcup_{i<\omega}[\alpha_i, \beta_i)$. This gives the desired contradiction. □ □

Theorem 21. *Given κ 2-Mahlo and $\lambda < \kappa$ regular, $P(\lambda, \kappa)$ forces Local Club Condensation at $\kappa = \lambda^+$ and $\neg\square_\lambda$.*

Proof: Let $P = P(\lambda, \kappa)$. Assume for a contradiction that $q \in P$ forces that $\dot{C} = \langle \dot{C}_\eta \mid \eta < \kappa\rangle$ is a $\square_\lambda$-sequence in a P-generic extension. As q plays no role in the proof, we assume that $q = \mathbf{1}$.

If $\eta < \kappa$, $\dot{C}\restriction\eta$ is a name for an object that consists of $<\kappa$-many ordinals and thus using the chain condition of the forcing, there is $\xi < \kappa$ s.t. $\dot{C}\restriction\eta$ is in fact a $P_{<\xi}$-name. For a club of $\eta < \kappa$ we thus have a $P_{<\eta}$-name $\dot{C}\restriction\eta$. By the large cardinal properties of κ, we may choose such an $\eta < \kappa$ that is Mahlo. Assume some condition forces that $\dot{C}_\eta$ has a $P_{<\eta}$-name. Then $\dot{C}_\eta$ has order-type $\eta > \lambda$, as η is regular in any $P_{<\eta}$-generic extension, contradicting that $\dot{C}$ is a name for a $\square_\lambda$-sequence. Thus $\dot{C}_\eta$ doesn't have a $P_{<\eta}$-name.

By the above, there are $t_0 \perp t_1$ in P with $t_0 \restriction \eta = t_1 \restriction \eta$ and some $\xi < \eta$ such that t_0 and t_1 disagree about whether $\xi \in \dot{C}_\eta$. Let M_0^* and M_1^* be elementary substructures of $(H_\theta, \in, \eta, \lambda, \xi, t_0, t_1, \dot{C}_\eta, \ldots)$ for some large, regular θ as provided by Lemma 19, let δ denote $\sup(M_0^* \cap \eta) = \sup(M_1^* \cap \eta)$, let $\bar{\delta} = \sup(M_0^* \cap M_1^* \cap \eta)$. Let $M_0 \prec M_0^*$ be countable with $\sup(M_0 \cap \eta) = \delta$ and $t_0 \in M_0$, let $s_0 \leq t_0$ be (M_0, P)-complete, so that $s_0 = \bigcup_{i<\omega} p_i$ for some decreasing sequence of conditions $\langle p_i \mid i < \omega \rangle \subseteq M_0$ with $p_0 = t_0$ and such that whenever $D \in M_0$ is a dense subset of P there is $i < \omega$ such that $p_i \in D$.

Claim 22. *$s_0 \restriction \bar{\delta} \in M_1^*$.*

Proof: $s_0 \restriction \lambda \in H_\lambda$ and is thus an element of M_1^* by Clause 3 of Lemma 19. For every $i < \omega$, $p_i \restriction [\lambda, \bar{\delta}) = p_i \restriction [\lambda, \min(M_0^* \setminus \bar{\delta})) \in M_0^*$; the equation holds since $\mathrm{supp}(p_i) \subseteq M_0^*$. But by Clause 5 of Lemma 19, $p_i \restriction [\lambda, \bar{\delta}) \in V_{\bar{\delta}}$ holds as well, for M_0^* thinks that $p_i \restriction [\lambda, \bar{\delta}) \in V_{\min(M_0^* \setminus \bar{\delta})}$. By Clause 4 of Lemma 19, $\langle p_i \restriction [\lambda, \bar{\delta}) \mid i < \omega \rangle \in M_1^*$ and thus so is $s_0 \restriction [\lambda, \bar{\delta})$ for it is easily definable from that sequence. □

Let M_1 be a countable elementary submodel of M_1^* such that $s_0 \restriction \bar{\delta} \in M_1$ and $\sup(M_1 \cap \eta) = \delta$. Note that by Fact 8, $s_0 \restriction \bar{\delta}$ and t_1 are compatible. Let s_1 be stronger than both and (M_1, P)-complete. By the properties of M_0 and M_1, both s_0 and s_1 force that $\delta \in \mathrm{Lim}(\dot{C}_\eta)$. Thus both s_0 and s_1 force that $\xi \in \dot{C}_\eta \iff \xi \in \dot{C}_\delta$.

Claim 23. *$s_0 \restriction \eta$ and $s_1 \restriction \eta$ are compatible.*

Proof: $s_1 \restriction \bar{\delta} \leq s_0 \restriction \bar{\delta}$ and $\mathrm{supp}(s_0) \cap [\bar{\delta}, \eta) \subseteq M_0^*$ and $\mathrm{supp}(s_1) \cap [\bar{\delta}, \eta) \subseteq M_1^*$, and hence these supports are disjoint by the disjointness properties of M_0^* and M_1^*. □

We can now strengthen $s_0 \restriction \eta$ and $s_1 \restriction \eta$ to conditions agreeing about whether $\xi \in \dot{C}_\delta$, which clearly gives a contradiction to our choice of t_0 and t_1. □ □ Theorem 21

Note: One could replace *Local Club Condensation at λ^+* by *Local Club Condensation up to λ^+* (or, with a little more work, by *Local Club Condensation*) in the statement of Theorem 21. However this would require providing a σ-closed forcing construction to obtain this Condensation principle (and, in the case of Local Club Condensation, working around the κ-cc by standard reduction arguments). This is straightforward to do (but with a significant increase in complexity of notation) by combining the construction from Section 4 with some of the ideas from [5]. To keep things more easily readable and because the most obvious interaction between $\square_\lambda$ and Local Club Condensation (and thus the most interesting aspect of their independence) should naturally occur within the interval $[\lambda, \kappa]$, we decided not to present such a construction.

6. Variations of Square

We first improve Theorem 21 by showing that in fact a whole hierarchy of weaker square principles is forced to fail by $P(\lambda, \kappa)$. We again assume GCH throughout.

Definition 24. *$\square_\lambda^n$ is the statement that there exists a sequence $\langle C_\alpha : \alpha < \lambda^+ \rangle$ such that the following hold:*

1. *Each C_α has at most n-many elements.*
2. *Whenever α is a limit ordinal, elements of C_α are closed unbounded subsets of α of order-type at most λ.*
3. *If β is a limit point of $X \in C_\alpha$ then there is $Y \in C_\beta$ such that $Y = X \cap \beta$.*

Theorem 25. *Given κ 2-Mahlo, $\lambda < \kappa$ regular and $2 \leq n < \omega$, $P(\lambda, \kappa)$ forces Local Club Condensation at $\kappa = \lambda^+$ and $\neg\square^n_\lambda$.*

We need the following minor generalization of Lemma 19.

Lemma 26. *If η is Mahlo and $\theta \geq \eta$ is regular, $c \in H_\theta$ and $\lambda < \eta$ is regular, then there is a sequence of models $\langle M^*_i \mid i < \omega\rangle$ such that*

(1) $M^*_i \prec (H_\theta, \in, c, \eta, \lambda)$ for every $i < \omega$.
(2) Each M^*_i has size λ.
(3) $\lambda \subseteq M^*_i$ for every $i < \omega$.
(4) Let $\bar{\delta}_i = \sup(\eta \cap M^*_i \cap M^*_{i+1})$. Then $P_{\omega_1}(M^*_i \cap V_{\bar{\delta}_i}) \subseteq M^*_{i+1}$ for every $i < \omega$.
(5) $\langle \bar{\delta}_i \mid i < \omega\rangle$ is increasing.
(6) If $j > i$, $\sup(\eta \cap M^*_i \cap M^*_j) = \bar{\delta}_i$.
(7) $\min(M^*_i \setminus \bar{\delta}_i)$ has cofinality $\geq \lambda$.
(8) $\bar{\delta}_i < \sup(M^*_i \cap \eta) = \sup(M^*_j \cap \eta)$ and the latter have cofinality ω for any $i, j < \omega$.

Proof: We consider the game described in Lemma 19, for which player I has a winning strategy. Using this, we can simultaneously play on ω-many boards to produce the desired models, in a similar way as we produced two models in the proof of Lemma 19: Use a disjoint partition $\langle S_i \mid i < \omega\rangle$ of ω into infinite sets and at stage j play on board i if $j \in S_i$. We leave the (easy) details to the reader. □

Proof of Theorem 25: Fix $n < \omega$ and let $P = P(\lambda, \kappa)$. Assume for a contradiction that $q \in P$ forces that $\dot{C} = \langle \dot{C}_\eta \mid \eta < \kappa\rangle$ is a $\square^n_\lambda$-sequence in a P-generic extension. As q plays no role in the proof, we assume that $q = \mathbf{1}$.

If $\eta < \kappa$, $\dot{C}{\restriction}\eta$ is a name for an object that consists of $<\kappa$-many ordinals and thus using the chain condition of the forcing, there is $\xi < \kappa$ s.t. $\dot{C}{\restriction}\eta$ is in fact a $P_{<\xi}$-name. For a club of $\eta < \kappa$ we thus have a $P_{<\eta}$-name $\dot{C}{\restriction}\eta$. By the large cardinal properties of κ, we may choose such an $\eta < \kappa$ that is Mahlo. Choose $\dot{X}$ to be a name for an element of $\dot{C}_\eta$. Assume some condition forces that $\dot{X}$ has a $P_{<\eta}$-name. Then $\dot{X}$ has order-type $\eta > \lambda$, as η is regular in any $P_{<\eta}$-generic extension, contradicting that $\dot{C}$ is a name for a $\square^n_\lambda$-sequence. Thus $\dot{X}$ doesn't have a $P_{<\eta}$-name.

By the above, there are countably many incompatible conditions t_i in P with equal restrictions to η and some $\xi_i < \eta$ for $i < \omega$ such that for every $i < \omega$, t_i and t_{i+1} agree about whether $\xi_j \in \dot{X}$ for $j < i$ but disagree about whether $\xi_i \in \dot{X}$. Let $\langle M^*_i \mid i < \omega\rangle$ be an ω-sequence of elementary substructures of $(H_\theta, \in, \eta, \lambda, \langle \xi_i \mid i < \omega\rangle, \langle t_i \mid i < \omega\rangle, \dot{X}, \ldots)$ for some large, regular θ as provided by Lemma 26, let δ denote $\sup(M^*_0 \cap \eta) = \sup(M^*_i \cap \eta)$ for any $i < \omega$, let $\bar{\delta}_i = \sup(M^*_i \cap M^*_{i+1} \cap \eta)$.

Let $M_0 \prec M^*_0$ be countable with $\sup(M_0 \cap \eta) = \delta$ and $t_0 \in M_0$, let $s_0 \leq t_0$ be (M_0, P)-complete. Exactly as in the proof of Claim 22, we can now show that $s_0{\restriction}\bar{\delta}_0 \in M^*_1$. Now given s_i, let M_{i+1} be a countable elementary submodel of M^*_{i+1} such that $s_i{\restriction}\bar{\delta}_i \in M_{i+1}$ and $\sup(M_{i+1} \cap \eta) = \delta$. Note that $s_i{\restriction}\bar{\delta}_i$ and t_{i+1} are compatible and let s_{i+1} be stronger than both and (M_{i+1}, P)-complete. Using that $s_i{\restriction}\bar{\delta}_i \in M^*_{i+1}$ analogous to above, we may perform this construction for every $i < \omega$.

By the properties of the M_i, each s_i forces that $\delta \in \dot{X}$. Since $\dot{C}_\delta$ has size at most n, we can pick a P_η-name $\dot{Y}$ for an element of $\dot{C}_\delta$ and indices $i < j$ such that both s_i and s_j force that $\dot{X} \cap \delta = \dot{Y}$. Now both s_i and s_j force that $\xi_i \in \dot{X} \iff \xi_i \in \dot{Y}$.

Claim 27. *$s_i{\restriction}\eta$ and $s_j{\restriction}\eta$ are compatible.*

Proof: $s_j{\restriction}\bar{\delta}_i \leq s_i{\restriction}\bar{\delta}_i$ (this uses Clause 5 of Lemma 26). $\mathrm{supp}(s_i) \cap [\bar{\delta}_i, \eta) \subseteq M_i^*$ and $\mathrm{supp}(s_j) \cap [\bar{\delta}_i, \eta) \subseteq M_j^*$ - hence those supports are disjoint by the disjointness properties of M_i^* and M_j^* (Clause 6 of Lemma 26). $\square$

We can now strengthen $s_i{\restriction}\eta$ and $s_j{\restriction}\eta$ to conditions agreeing about whether $\xi_i \in \dot{Y}$ and hence about whether $\xi_i \in \dot{X}$, which clearly gives a contradiction to our choice of t_i and t_j. $\square\ \square_{\text{Theorem 25}}$

Now we consider $\square(\kappa)$. Note that our consistency strength assumption in Theorem 28 below is optimal, essentially because weakly compact cardinals are 2-Mahlo.

Theorem 28. *Given κ which is weakly compact and $\lambda < \kappa$ regular, $P(\lambda, \kappa)$ forces Local Club Condensation at λ^+ and $\neg\square(\lambda^+)$.*

Proof: Assume the GCH. Let $P = P(\lambda, \kappa)$. Assume for a contradiction that $q \in P$ forces that $\dot{C} = \langle \dot{C}_\eta \mid \eta < \kappa \rangle$ is a $\square(\kappa)$-sequence in a P-generic extension. As q plays no role in the proof, we assume that $q = \mathbf{1}$. Using the κ-cc of P, we may assume that $\dot{C} \subseteq V_\kappa$.

$$(V_\kappa, \in, P, \Vdash_P, \dot{C}) \models \Vdash_P \dot{C} \text{ is a } \square(\kappa)\text{-sequence.}$$

As κ being Mahlo is a $\mathbf{\Pi}^1_1$-property of V_κ, we may invoke $\mathbf{\Pi}^1_1$-indescribability of κ to find a Malo cardinal $\eta < \kappa$ such that

$$P_{<\eta} \Vdash \dot{C}{\restriction}\eta \text{ is a } \square(\eta)\text{-sequence.}$$

It follows that $\dot{C}_\eta$ cannot have a $P_{<\eta}$-name, as this would contradict that $\dot{C}{\restriction}\eta$ is forced to be a $\square(\eta)$-sequence in any $P_{<\eta}$-generic extension. But this now allows us to finish exactly as in the proof of Theorem 21. $\square$

Note: Similar to the remark at the end of the previous section, one could replace *Local Club Condensation at λ^+* by *Local Club Condensation up to λ^+* in the statement of Theorem 28. However we do not know whether it could also be replaced by *Local Club Condensation* in this case, for the proof of Theorem 28 heavily uses that $P \subseteq V_\kappa$. Moreover just like the proof for $\neg\square_\lambda$ was improved to $\neg\square^n_\lambda$ for $n < \omega$, one could improve the above from $\neg\square(\lambda^+)$ to $\neg\square(\lambda^+, n)$ for $n < \omega$, where the latter is defined correspondingly.

7. Condensation and Jónsson cardinals

In [3], it was shown that Strong Condensation refutes the existence of an ω_1-Erdős cardinal. We slightly improve this result by showing that it refutes the existence of an ω_1-Jónsson cardinal.

Definition 29. *If $\omega < \delta < \kappa$ are cardinals, κ is δ-Jónsson if for every first order structure $\mathcal{A}$ for a countable language with universe κ, there is $\mathcal{A}' \prec \mathcal{A}$ with universe A' such that $\mathrm{ot}(A') = \delta$.*

κ is κ-Jónsson or Jónsson if every structure $\mathcal{A}$ as above has a proper substructure $\mathcal{A}'$ with universe A' of size κ.

It is easy to see ([10]) that every δ-Erdős cardinal is δ-Jónsson. Whether the reverse implication holds is not known. The proof of the next result closely follows a proof by Keisler and Rowbottom (announced in [8]) which shows that if there is a Jónsson cardinal, then $\mathbf{V} \neq \mathbf{L}$. Their proof can be found in [7].

Theorem 30. *Assume κ is an uncountable cardinal. Strong Condensation for κ implies that κ is not Jónsson and that for no $\delta < \kappa$, κ is δ-Jónsson.*

Proof: First assume for a contradiction Strong Condensation for κ holds and κ is Jónsson. Let $\mathcal{A} = (M_\kappa, \in, \vec{M}, \ldots)$ be a structure for a countable language witnessing Strong Condensation for κ. As κ is Jónsson, $\mathcal{A}$ has a proper substructure $\mathcal{B}_0$ of cardinality κ. By our choice of $\mathcal{A}$, $(B_0, \in, \vec{M})$ is isomorphic to $(M_\kappa, \in, \vec{M})$; let π be the inverse of the collapsing isomorphism of $(B_0, \in)$, let $\delta > \omega$ be the critical point of π. Define U by

$$X \in U \iff X \subseteq \delta \wedge \delta \in \pi(X).$$

Since $M_\kappa = H_\kappa$, U is easily seen to be a δ-complete ultrafilter and hence δ is a measurable cardinal, contradicting the above-mentioned result of [3], for measurable cardinals are in particular ω_1-Erdős.

Now assume Strong Condensation for κ holds (note that this implies the GCH below κ), $\delta < \kappa$ and κ is δ-Jónsson. Assume that $\mathcal{A}$ is as above, but also includes δ as a constant. Let ρ be an isomorphism between κ and H_κ such that for every cardinal $\lambda < \kappa$, $\rho{\restriction}\lambda$ is an isomorphism between λ and H_λ (this uses the GCH below κ). This means that $\mathcal{A}$ is isomorphic (via ρ^{-1}) to a structure $\mathcal{B}$ on κ that has a substructure $\mathcal{B}'$ with underlying set B' of order-type δ. $\rho'' B'$ induces a substructure $\mathcal{A}'$ of $\mathcal{A}$ with underlying set A'. By Strong Condensation, A' condenses, say to $A'' \supseteq H_\delta$. Let π denote the elementary embedding from A'' to A. If $\mathrm{crit}(\pi) = \theta < \delta$, then $\mathcal{P}(\theta) \subseteq \mathrm{dom}\,\pi$ and θ is seen to be measurable, leading to a contradiction as above. If $\mathrm{crit}\,\pi \geq \delta$, this means that $H_\delta \cup \{\pi(\delta)\} \subseteq A'$ and hence $\delta \cup \{\rho^{-1}(\pi(\delta))\} \subseteq B'$ by our choice of ρ, contradicting that $\mathrm{ot}\, B' = \delta$. If $\pi = \mathrm{id}$, this means that $A'' = A' = H_\delta$ by our choice of ρ. But this contradicts elementarity of π, as $\delta \notin H_\delta$. □

In contrast to this, Local Club Condensation is clearly consistent with the existence of Jónsson cardinals, both inaccessible and accessible: Start with countably many measurable cardinals and force Local Club Condensation preserving those measurables by the techniques of [3] (preservation of measurables is a standard argument that is not carried out in that paper). In the extension, Local Club Condensation holds and the supremum of the measurable cardinals is accessible and Jónsson (see [11]). Preservation of a measurable while forcing Local Club Condensation clearly yields the consistency of Local Club Condensation with inaccessible Jónsson cardinals, as measurable cardinals are Jónsson.

8. Condensation and variants of Chang's Conjecture

Definition 31. *For infinite cardinals $\alpha, \beta, \gamma, \delta$ with $\alpha > \beta > \delta$ and $\alpha \geq \gamma > \delta$,*

$$(\alpha, \beta) \twoheadrightarrow (\gamma, \delta)$$

is the statement that for every countable language L with a unary predicate $A \in L$ and every L-structure $\mathcal{M} = (M, A^{\mathcal{M}}, \ldots)$ with $\mathrm{card}\, M = \alpha$ and $\mathrm{card}\, A^{\mathcal{M}} = \beta$, there exists $\mathcal{N} = (N, A^{\mathcal{N}}, \ldots)$ such that

(1) $\mathcal{N}$ is a substructure of $\mathcal{M}$ and
(2) $\mathrm{card}\, N = \gamma$ and $\mathrm{card}(A^{\mathcal{N}}) = \delta$.

Theorem 32. *Strong Condensation for α refutes $(\alpha, \beta) \twoheadrightarrow (\gamma, \delta)$.*

Proof: Assume for a contradiction that Strong Condensation for α holds and $(\alpha,\beta)\twoheadrightarrow(\gamma,\delta)$. Thus $(M_\alpha,\in,\vec{M},\beta)$ has an elementary substructure N with $\operatorname{card} N=\gamma$ and $\operatorname{card}(N\cap\beta)=\delta$, which condenses to some $M_{\bar\alpha}$; let π denote the collapsing map. By taking the Skolem Hull of $N\cup(\delta+1)$ in that structure w.r.t. some wellordering of M_α, we may as well assume that $\delta+1\subseteq N$. Hence $\pi(\delta)=\delta$. As $\operatorname{card}(N\cap\beta)=\delta$, $\pi(\beta)<\delta^+$. Since, using Lemma 2, $H_{\delta^+}=M_{\delta^+}\subseteq M_{\bar\alpha}$, it follows that $M_{\bar\alpha}\models\pi(\beta)$ is not a cardinal, contradicting elementarity of N. □

Theorem 33. *Local Club Condensation at κ^{++} refutes $(\kappa^{++},\kappa^+)\twoheadrightarrow(\kappa^+,\kappa)$ for any infinite cardinal κ.*

Proof: Let F be as in Theorem 1. Assume for a contradiction Local Club Condensation at κ^{++} holds and $(\kappa^{++},\kappa^+)\twoheadrightarrow(\kappa^+,\kappa)$. Thus $(M_{\kappa^{++}},\in,\vec{M},\kappa^+,F)$ has an elementary substructure N with $\operatorname{card} N=\kappa^+$ and $\operatorname{card} N\cap\kappa^+=\kappa$. We may assume that $(\kappa+1)\subseteq N$ (and hence N is transitive below κ^+) as in the proof of Theorem 32 and thus by Theorem 1, N condenses to some $M_\alpha\supseteq M_{\kappa^+}=H_{\kappa^+}$ (the final equality is mentioned in Section 3). Let $\overline{\kappa^+}$ denote the M_α-version of κ^+. $\overline{\kappa^+}\in(\kappa,\kappa^+)$, contradicting $M_\alpha\supseteq H_{\kappa^+}$. □

Are any nontrivial instances of Chang's Conjecture consistent with Local Club Condensation? Under sufficient large cardinal assumptions, the following fact answers this positively.

Fact 34. *Assume $\delta^+<\kappa$ and κ is δ^+-Jónsson. Then $(\kappa,\delta^+)\twoheadrightarrow(\delta^+,\delta)$.*

Proof: Let $\mathcal{A}$ be a Skolemized structure on κ using δ^+ as a predicate. Then by δ^+-Jónssonness of κ, there is an $M\prec\mathcal{A}$ of order-type δ^+. $\delta^{++}\in M$ by elementarity and thus M is bounded in δ^+ (as otherwise $ot(M)>\delta^+$). But this means that $M\cap\delta^+$ has cardinality at most δ. We may enlarge M to contain δ as a subset (by taking the Skolem Hull of $M\cup\delta$ in $\mathcal{A}$), so that $M\cap\delta^+$ has cardinality δ. Since any structure of size κ with a unary predicate T of size δ^+ has an extension which is isomorphic to some extension of $\mathcal{A}$ and identifies T and δ^+, this shows that $(\kappa,\delta^+)\twoheadrightarrow(\delta^+,\delta)$. □

We next present a positive result for small cardinals, strongly based on a proof of James Baumgartner in [1]. Let $\mathrm{CC}(\kappa)$ denote the statement that for every $\lambda<\kappa$, $(\kappa,\lambda)\twoheadrightarrow(\omega_1,\omega)$. We will show that, assuming the existence of an ω_1-Erdős cardinal κ, we may collapse κ to become the successor of any regular cardinal $\lambda\geq\omega_2$ and obtain $\mathrm{CC}(\kappa)$ together with Local Club Condensation at κ. By Theorem 33, this cannot work for $\lambda=\omega_1$.

Definition 35. *A cardinal κ is α-Erdős if for any Skolemized structure $\mathcal{A}$ for a countable language with universe κ and for any closed unbounded $C\subseteq\kappa$, there is $I\subseteq C$ of order-type α such that I is a set of indiscernibles for $\mathcal{A}$ that is* remarkable*, i.e. whenever $\alpha_0,\ldots,\alpha_i,\ldots,\alpha_n$ and $\beta_i,\ldots,\beta_n$ are increasing sequences from I with $\alpha_{i-1}<\beta_i$, τ is a term in the language of $\mathcal{A}$ and $\tau^{\mathcal{A}}(\alpha_0,\ldots,\alpha_n)<\alpha_i$, then*

$$\tau^{\mathcal{A}}(\alpha_0,\ldots,\alpha_n)=\tau^{\mathcal{A}}(\alpha_0,\ldots,\alpha_{i-1},\beta_i,\ldots,\beta_n).$$

Definition 36. [1] *Suppose $\mathcal{A}$ is a Skolemized structure for a countable language with universe κ. For each $k<\omega$ let f_k be given so that $f_k\colon\kappa^n\to\kappa$ for some $n<\omega$ that may depend on k. We say $I\subseteq\kappa$ is* remarkable *for $\mathcal{A}$ and the f_k if for any $\alpha\in H^{\mathcal{A}}(I)$ there is $k<\omega$ and increasing $\alpha_0,\ldots,\alpha_{n-1}$ from I so that*

$\alpha = f_k(\alpha_0, \dots, \alpha_{n-1})$ and if $\alpha_{i-1} \leq \alpha < \alpha_i$ then for any increasing $\beta_i, \dots, \beta_{n-1}$ from I with $\alpha_{i-1} < \beta_i$ we have $\alpha = f_k(\alpha_0, \dots, \alpha_{i-1}, \beta_i, \dots, \beta_{n-1})$ and $\alpha < \beta_i$. We say that a cardinal κ is α-remarkable *if for any $\mathcal{A}$ as above there exist f_k as above so that for any closed unbounded set $C \subseteq \kappa$ there is $I \subseteq C$ of order-type α s.t. I contains none of its limit points and is remarkable for $\mathcal{A}$ and the f_k.*

Lemma 37. [1, Proposition 5.1] *If κ is ω_1-remarkable, then* CC(κ) *holds.* □

Definition 38. *We say that $\langle P_{<\alpha} \mid \alpha \leq \kappa \rangle$ is an* Easton bounded iteration of length κ *if whenever $\alpha_0 < \alpha_1 < \kappa$, $P_{<\alpha_0}$ is a complete subforcing of $P_{<\alpha_1}$, and $P_{<\alpha}$ is the direct limit of $\langle P_{<\beta} \mid \beta < \alpha \rangle$ if α is inaccessible.*

The proof of Theorem 39 below is almost an exact copy of one of the two cases of [1, Theorem 5.2], which however refers to what are called *Easton-like* partial orderings (those are particular kinds of products with Easton support). All that we do below is to essentially observe that the same proof can be carried out for Easton bounded iterations of length κ. For the benefit of the reader, we provide the basic framework of the proof (and omit the proofs of several auxiliary lemmas, which can all be found in [1]).

Theorem 39. *Let κ be ω_1-Erdős and let $\langle P_{<\alpha} \mid \alpha \leq \kappa \rangle$ be an Easton bounded iteration of length κ s.t. $P_{<\alpha}$ has size $< \kappa$ for $\alpha < \kappa$, and $P = P_{<\kappa}$ is $<\omega_2$-directed closed. Then $\Vdash_P \kappa$ is ω_1-remarkable and hence* CC(κ) *holds by Lemma 37.*

Proof: We may assume that $P_{<\alpha} \in V_\kappa$ for $\alpha < \kappa$, and so $P \subseteq V_\kappa$. Let $\dot{\mathcal{A}}$ be a term for a Skolemized structure with universe κ. Let

$$\mathcal{B} = (V_\kappa, \in, <, P, \Vdash_\phi),$$

where $<$ is a well-ordering of V_κ, ϕ ranges over all formulas of the language of $\dot{\mathcal{A}}$ and $\Vdash_\phi$ is the relation

$$\{(p, \alpha_0, \dots, \alpha_{n-1}) \mid p \Vdash \dot{\mathcal{A}} \models \phi(\alpha_0, \dots, \alpha_{n-1})\}.$$

Let $\langle f_k \mid k < \omega \rangle$ enumerate all functions of the form $f \colon \kappa^n \to \kappa$ definable over $\mathcal{B}$. By the κ-cc of P, every club subset of κ in $\mathbf{V}^P$ contains a club subset of κ in $\mathbf{V}$. Let $C \subseteq \kappa$ be club. Let $I \subseteq C$ be a cofinal remarkable set of indiscernibles for $\mathcal{B}$ with order-type ω_1 (we say that I is cofinal if whenever $\alpha_0, \dots, \alpha_n$ is an increasing sequence of elements of I and τ is a term in the language of $\mathcal{B}$, then $\tau^{\mathcal{B}}(\alpha_0, \dots, \alpha_{n-1}) < \alpha_n$; this property can be ensured by shrinking C before applying the large cardinal properties of κ to obtain I). By standard arguments we may assume that I consists only of Mahlo cardinals and $\forall \alpha < \beta \ P_\alpha \in V_\beta$ for every $\beta \in I$. In particular, I contains none of its limit points. We will show that in $\mathbf{V}^P$, I contains a set of order-type ω_1 remarkable for $\dot{\mathcal{A}}$ and the f_k.

For $X \subseteq I$, we say G is P-generic over $H^{\mathcal{B}}(X)$ if $G \subseteq P \cap H^{\mathcal{B}}(X)$ is a filter meeting every dense subset of P lying in $H^{\mathcal{B}}(X)$, the Skolem Hull of X in $\mathcal{B}$.

The following lemmas can be proven exactly as in [1] by using our requirements on P, so we will just provide their statements and refer to the corresponding lemmas in [1].

Lemma 40. [1, Lemma 5.3] *Let G be P-generic over $\mathbf{V}$. In $\mathbf{V}[G]$ suppose $J \subseteq I$ is uncountable and $G \cap H^{\mathcal{B}}(J)$ is P-generic over $H^{\mathcal{B}}(J)$. Then J is remarkable for $\dot{\mathcal{A}}$ and the f_k.* □

We thus want to find $J \subseteq I$ as in the hypothesis of the above lemma. We work in $\mathbf{V}$. Let $F = \{X \subseteq I \mid X \text{ is countable and has no last element}\}$. For $X, Y \in F$ put $X < Y$ if X is a proper initial segment of Y.

Lemma 41. [1, Lemma 5.4] *Suppose $X, Y \in F$, $X < Y$ and $\beta = \min(Y \setminus X)$. If G is P-generic over $H^{\mathcal{B}}(X)$ then G is P_β-generic over $H^{\mathcal{B}}(Y)$.* □

Lemma 42. [1, Lemma 5.5] *Suppose $X, Y \in F$, $X < Y$ and G is P-generic over $H^{\mathcal{B}}(X)$. Then there is $H \supseteq G$ such that H is P-generic over $H^{\mathcal{B}}(Y)$.* □

Lemma 43. [1, Lemma 5.6] *Suppose that for $n \in \omega$, $X_n \in F$, G_n is P-generic over $H^{\mathcal{B}}(X_n)$, $X_n < X_{n+1}$ and $G_n \subseteq G_{n+1}$. Then $\bigcup_{n\in\omega} G_n$ is P-generic over $H^{\mathcal{B}}(\bigcup_{n\in\omega} X_n)$.* □

Using the above lemmas, it is straightforward to find $G \subseteq P \cap H^{\mathcal{B}}(I)$ such that G is P-generic over $H^{\mathcal{B}}(I)$. Since by Corollary 12 P is $<\omega_2$-directed closed, there is $p \in P$ stronger than any condition in G. Hence if $\dot{G}$ is the canonical name for the P-generic,

$$p \Vdash \dot{G} \cap H^{\mathcal{B}}(I) \text{ is } P\text{-generic over } H^{\mathcal{B}}(I)$$

and we are thus finished proving Theorem 39. □

Corollary 44. *Given the consistency of an ω_1-Erdős cardinal and $2 < n < \omega$, Local Club Condensation at ω_n is consistent with* $\mathrm{CC}(\omega_n)$*. More generally, assuming the* GCH*, if κ is ω_1-Erdős and $\omega_2 \leq \lambda < \kappa$ is regular, we may obtain a forcing extension in which κ becomes λ^+, all cardinals up to λ and $\geq\kappa$ and the* GCH *are preserved, and* $\mathrm{CC}(\kappa)$ *and Local Club Condensation at κ hold. In fact, a similar statement can be obtained using only weaker assumptions on the continuum function, namely those made in Section 4.*

Proof: If κ and λ are as in the second statement above, force with $P = P(\lambda, \kappa)$. □

Note: Similar to the remark at the end of Section 5, one could replace *Local Club Condensation at κ* by *Local Club Condensation in $[\omega_1, \kappa)$* in the statement of Corollary 44. This would require a forcing construction to obtain the latter principle by $<\omega_2$-directed closed forcing which (as explained in Section 5) we omitted for the sake of simplicity. It is worth noting however that forcing Local Club Condensation up to κ seems to require us to add new subsets of ω_1 in general and hence any forcing to obtain this principle should not be $<\omega_2$-directed closed. As the proof of Theorem 39 seems to crucially require this closure property, we do not know whether Local Club Condensation up to κ is consistent with $\mathrm{CC}(\kappa)$ for any $\kappa > \omega_2$ (they are inconsistent for $\kappa = \omega_2$ by Theorem 33 above).

We close this section by considering weak Chang's Conjecture.

Definition 45. [2] *Assume κ is a successor cardinal. The* weak Chang Conjecture *for κ,* $\mathrm{wCC}(\kappa)$*, is the following assertion: Whenever $\mathcal{A}$ is a first order structure for a countable language, with universe A and $\kappa^+ \subseteq A$, then there is $\alpha < \kappa$ such that for all $\beta < \kappa$ there is $X \prec \mathcal{A}$ with $X \cap \kappa \subseteq \alpha$ and $\mathrm{ot}(X \cap \kappa^+) > \beta$.*[12]

We present a well-known auxiliary lemma.

[12]The same definition would make perfect sense if κ were inaccessible. The resulting principle though is easily seen to be inconsistent with ZFC.

Lemma 46 (folklore)**.** *Assume $N \prec \mathcal{A}$ for some Skolemized structure $\mathcal{A}$ on H_κ. Assume $\lambda_0 < \lambda_1$ are regular cardinals below κ and $\lambda_0, \lambda_1 \in N$. There is $M \prec \mathcal{A}$ such that $\sup(M \cap \lambda_1) = \sup(N \cap \lambda_1)$, M is transitive below λ_0 and $M \cap \lambda_0 = \sup(N \cap \lambda_0)$.*

Proof: Let $M = H^{\mathcal{A}}(\sup(N \cap \lambda_0) \cup N)$. Assume that $\xi \in M \cap [\sup(N \cap \lambda_1), \lambda_1)$. Then for some $\delta_0, \ldots, \delta_m \in N$, $\gamma_0, \ldots, \gamma_n < \sup(N \cap \lambda_0)$ and some formula φ in the language of $\mathcal{A}$, $M \models \xi$ is the unique x with $\varphi(x, \gamma_0, \ldots, \gamma_n, \delta_0, \ldots, \delta_m)$. Let ν be such that M thinks that ν is the supremum of all $x < \lambda_1$ which are uniquely defined by some formula φ in the language of $\mathcal{A}$ of the form $\varphi(x, \zeta_0, \ldots, \zeta_n, \delta_0, \ldots, \delta_m)$ for some $\zeta_0, \ldots, \zeta_n < \lambda_0$. By regularity of λ_1, $\nu < \lambda_1$. By elementarity, $\nu \in N$. As $\xi < \nu$, this implies that $\sup(M \cap \lambda_1) = \sup(N \cap \lambda_1)$. The final statement of our claim follows similarly, noting that if $\sup(N \cap \lambda_0)$ is not a cardinal, we have $M = H^{\mathcal{A}}(\mathrm{card}(\sup(N \cap \lambda_0)) \cup N)$. □

Theorem 47. *Local Club Condensation at κ^+ refutes* wCC(κ) *whenever $\kappa = \lambda^+$ is a successor cardinal.*

Proof: Assume that both Local Club Condensation at κ^+ and wCC(κ) hold with κ regular. Assume $\mathcal{A} = (M_{\kappa^+}, \in, \vec{M}, F, \ldots)$ is a Skolemized structure for a countable language with F as in Theorem 1 and assume α is for $\mathcal{A}$ as in Definition 45. For each $\beta < \kappa$, let X_β be a witnessing structure, i.e. $X_\beta \prec \mathcal{A}$ with $X_\beta \cap \kappa \subseteq \alpha$ and $\mathrm{ot}(X_\beta \cap \kappa^+) > \beta$. We may assume that each X_β is transitive below κ by Lemma 46 and hence condenses by Theorem 1. Let π_β denote the collapsing map of X_β, $\bar{X}_\beta$ its transitive collapse. Then $\lambda + 1 \subseteq X_\beta$ and therefore $\pi_\beta(\kappa) \in (\lambda, \alpha]$. But $\bar{X}_\beta \supseteq M_\beta$, hence for sufficiently large $\beta < \kappa$, $\bar{X}_\beta \models \alpha \cong \lambda$, contradicting elementarity of X_β. □

9. Open Questions

Question 48. *Does Lemma 19 hold true if η is assumed only to be inaccessible?*

If the above has a positive answer, we would obtain the result of Theorem 21 from the optimal consistency assumption, namely a Mahlo cardinal, i.e. a positive answer to the following question.

Question 49. *Does Theorem 21 hold true if κ is assumed only to be Mahlo?*

Question 50. *Assume λ is regular and* GCH *holds. Under sufficient large cardinal hypothesis, can one force to obtain a model of Local Club Condensation at λ^+ and $\neg\square^\omega_\lambda$ while preserving all cofinalities $\leq \lambda$?* [13]

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[13] $\square^\omega_\lambda$ is the same as $\square^n_\lambda$ for $n < \omega$ except that one allows for the C_α to be at most countable.

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