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Large Cardinals, Inner Models and Determinacy: an introductory overview

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Abstract. The interaction between large cardinals, determinacy of two person perfect information games, and inner model theory has been a singularly powerful driving force in modern set theory during the last three decades. For the outsider the intellectual excitement is often tempered by the somewhat daunting technicalities, and the seeming length of study needed to understand the flow of ideas. The purpose of this article is to try and give a short, albeit rather rough, guide to the broad lines of development.

1 Introduction

The following article grew out of a series of three tutorials at the conference on the development of large cardinal hypothesis and their connections to proofs of the determinacy of two person perfect information Gale-Stewart games. The intention was to give a perspective, not to specialists in set theory, who in any case either would have no need for such, or would know where to go find it, but to those interested in the foundations of set theory already with some technical knowledge of Zermelo-Fraenkel axiomatics, and the construction of Gödel's universe L, but who may not have seen a systematic production of the hierarchy of increasing strong axioms of infinity arising from embeddings of the universe V and in particular the connections of these axioms with the descriptive set theory of the real continuum. It was hoped that the lectures would provide a fast ascent without oxygen to some of the peaks of the last twenty years. In this we believed that a lot could be learnt, not in detail, nor even attempting to provide a working knowledge (for which a longer term devotion would be required, and again for which there are thorough texts) but that would bring about at least some familiarity with the technical tools, and some insight as to why they had been developed, and how they had been used.

We should also like to urge the reader to study Jensen's masterly, and non-technical, overview [9] if he or she has not already done so.

1.1 Overview

The roots of modern set theory lie surprisingly deep in the history of the subject, although it is admittedly rather young compared to other areas of mathematics. It is well known that it concerns itself still with the question that vexed its founding father: Cantor's Continuum Problem (or Hypothesis, hence CH) that any set of real numbers is either equinumerous with all of $\mathbb{R}$ or is countable. However some of the early analysts such as the those from the French school, Lebesgue, Borel, Baire, and then the Russians, Luzin and more particularly Suslin, introduced both further questions and new methods of representation that are still key today. Today *descriptive set theory* is that branch of set theory which seeks to investigate, not the whole real

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continuum, but *definable* parts or fragments of it. In doing this two things are achieved: (i) one may concentrate on perhaps simpler parts of the continuum for which the questions are more tractable and (ii) the definable parts are rather concretely given, and as such the results proven may give more definite information to the analyst who is uninterested in foundational questions. At the same time this gives clues to the most general question concerning the whole continuum. The roots of descriptive set theory can also be traced back to Cantor himself: he first tried (and succeeded) to solve the CH for closed sets. His ambition was to proceed through increasingly complicated sets, until, somehow he had conquered the whole continuum. He, and Bernstein, showed that a closed set is the union of a perfect set with a countable set of isolated points. Thus closed sets enjoyed the "perfect subset property" (PSP). CH for closed sets follows: if a closed set is uncountable the perfect part is non-empty and can easily be shown to be equinumerous with $\mathbb{R}$. Borel defined a hierarchy of more complicated sets: one starts with the class of open sets at the base level, and then builds up levels by the process of taking complementation and countable unions of the sets created so far. This *Borel Hierarchy* is built in precisely ω_1-many stages, and was much studied. It was found that PSP holds for Borel sets.

Suslin then showed the same for the analytic, or $\mathbf{\Sigma}_1^1$ sets (more detailed definitions are below, but an analytic subset of $\mathbb{R}$ (depicted on the x-axis), is one that is the projection of a Borel set in the x, y-plane down to that x-axis). This can be seen as a further step in Cantor's original plan: now any analytic set satisfies CH. However the program stopped there. Later Gödel was to show that in L, his *constructible hierarchy*, although CH was true, *co-analytic* or $\mathbf{\Pi}_1^1$ sets failed to have the PSP. Hence it was consistent with the axioms of ZFC that PSP failed at this level. Hence the analysts had gone as far as they could have done using analytical techniques. However the PSP is only an example of one of several properties. The property of being *Lebesgue Measurable* (LM) was also shown to hold for first Borel, then analytic (and since complements of measurable sets are measurable too, also co-analytic) sets. Similarly for the Baire property (BP, that is, a set A with the Baire property is almost an open set, in that there is an open set O so that the symmetric difference of the two sets A and O is 'thin', in the sense of being the union of countably many nowhere dense sets), and moreover for the Uniformisation Property: that co-analytic regions of the plane could be uniformized by co-analytic functions. Together this clutch of properties became (more recently) collectively known as the *regularity properties*. Again the BP and LM could both consistently fail for the same reason as for the PSP: in the Gödel universe $\mathbf{\Delta}_2^1$ sets failed to have them.

Proceeding in another direction Banach had considered Lebesgue's problem of defining a translation-invariant countably additive measure on $\mathbb{R}$. He replaced the translation-invariance part of the requirement with that of asking the measure of a single point be non-zero. Was such a measure possible? Again this was ruled out by an Axiom of Choice construction, as Vitali had shown would happen for any measure satisfying Lebesgue's requirements. However Banach's requirements allowed one to generalise away from $\mathcal{P}(\mathbb{R})$ to consider measures on $\mathcal{P}(S)$ for any set S (Banach 1930). The additivity of such a measure, were it to exist, would be that of the cardinality of S. If such existed for an uncountable set S, with $|S| = \kappa$ then κ was called *real-valued measurable*. Ulam (1930) showed that a real-valued measurable cardinal was weakly inaccessible in Hausdorff's sense. If the measure was atomless then $2^{\aleph_0} > \kappa$. In this way *large cardinals* and *measures* enter the scene. Ulam also defined the ultimately more consequential notion of a 2-valued measure κ-additive measure. With Tarski the cardinals κ admitting such measures on their power sets were shown to be strongly inaccessible, but not until the work of Hanf (1964) was it shown that the least measurable cardinal could not be the least inaccessible.

Shortly thereafter Scott used the technique of ultrapowers from model theory to show that the existence of a measurable cardinal implied $V \neq L$ (Theorem 2.5 (1961) below). This technique has become ubiquitous in modern set theory, with the realisation that a measurable cardinal allows an ultrapower of the whole universe V into some proper *inner model* M, with a first order truth-preserving *elementary embedding*: $j : V \longrightarrow_e M$. Embeddings of the universe V with stronger properties arose from the work of Solovay and of Reinhardt, and these were defined using ultrafilters on different base sets. The cardinals that were the first point moved, or the *critical point* of such embeddings were called *supercompact*, and the embeddings they produced had markedly different aspects to those from simple measures alone. Together with large cardinals defined from other differing directions, these would eventually all be seen to lie in a well ordered spectrum of increasing strength giving rise to stronger and stronger embedding properties, which to a first approximation can be said to be about how much of V is carried over into the target model M. (We give definitions of the more prominent of these again below).

"Inner Models" in this account are transitive class models of ZFC (this can be given a first order formulation). Gödel's L was the first inner model to be discovered (besides V itself of course). *Relativised constructibility* is easy to define: one enlarges the language of set theory to include (one or more) additional predicates into the first order Definability operator. Thus models $\langle L[\mu], \in, \mu \rangle$ where μ is a two valued κ-additive measure on some κ may be defined. Solovay defined such and their properties were investigated by Silver and Kunen. $L[\mu]$ was seen to be a generalisation of L: GCH held and there was a global well order of its universe. The definability of that well order on its real continuum was however Δ^1_3 rather than Δ^1_2 which had been the case for L.

Other generalised notions of definability were possible: instead of definability relative to a predicate one may start with a set R and define:

$$L_0(R) = R; \quad L_{\alpha+1}(R) = \mathrm{Def}(\langle L_\alpha(R), \in \rangle); \quad Lim(\lambda) \longrightarrow L_\lambda(R) = \bigcup_{\alpha < \lambda} L_\alpha(R).$$

The universe $L(R) = \bigcup_{\alpha \in On} L_\alpha(R)$ is then an inner model of ZF (not necessarily AC as there is not necessarily a well order of R definable over any of the levels $L_\alpha(R)$). For $R = \mathbb{R}$ this has been a model of tremendous importance as we shall see. One can regard it as a laboratory for very generalised notions of definability over analysis: it contains the projective sets of analysis right down at the bottom, and allows for definitions over analysis but containing more and more ordinals.

There was a further striking result concerning measurability.

Theorem 1.1 (Solovay (1965)) (ZFC) *If there is a measurable cardinal, then all Σ^1_2 sets of reals have the regularity properties of* LM, BP, PSP.

This was probably the first indication that large cardinals could really influence how the continuum would look to an analyst: by assuming some measure on some remote cardinal the nature of $V_{\omega+1}$ is fundamentally different.

However researchers were already deriving the regularity properties from a quite unexpected quarter. This was the notion of a *two person perfect information game* (see Def. 3.1 below.) Curiously Zermelo had first published on the existence of winning strategies for finite games, but Polish mathematicians such as Steinhaus, Banach and Mazur in the 1920's took up the question of infinite length games. As an example such a game G_A is based on a subset $A \subseteq \omega^\omega$ (the 'payoff set') and then two players alternate choosing integers $n_0, m_0, n_1, m_1, \ldots$ (with perfect knowledge of the other's moves) and after ω-many moves, the mutually constructed infinite

sequence $z \in \omega^\omega$ may or may not lie in A. If it does player I is deemed to have own. Otherwise player II wins.

It was then asked for which payoff sets A, could a player have a winning strategy (in the obvious sense)? In their formulations $\mathbb{R}$ was taken as ω^ω with moves taken from ω. In descriptive set theory it is usual to consider the reals as ω^ω rather than the Euclidean $\mathbb{R}$: ω^ω is homeomorphic to the irrationals, and only the countable set of rationals is 'omitted'. Moreover the topology determined by finite sequences (see below) is very conducive to descriptive set theoretical arguments: the resulting Baire space is an example of a well-studied space, being a Polish space, *i.e.* a separable complete metric space; moreover it is homeomorphic to the direct product of any finite number of copies of itself. Questions of dimension are thus made irrelevant: proving a result about the descriptive set theory of the one-dimensional space ω^ω, in general one has the result for the k-dimensional $(\omega^\omega)^k$.

With this topology one sees that if $A \subseteq \omega^\omega$ is an open set, and Player I has a winning strategy, then any run of the game using that strategy has essentially been won by I by a finite stage. Mazur had already conjectured the connection between the class of A for which the games G_A are *determined* games (meaning one of the players has a winning strategy) and the Baire property for that class. This was proven by Banach in 1935.

We now know that determinacy implies the regularity properties. Let Γ be a class of subsets of ω^ω. If G_A is determined for all $A \in \Gamma$ we write '$\mathrm{Det}(\Gamma)$'. If Γ has some minimal closure properties (this is 'adequacy' in Moschovakis's terminology [23]) which we shall not define here, but are enjoyed by all the definable projective classes Σ_n^1, Π_n^1 etc, then as the combined work of Banach, Mazur, Oxtoby (BP), Mycielski-Swierczkovski (LM), Davis (PSP) (all by 1964) showed:

Theorem 1.2 (ZFC) *Assume* $\mathrm{Det}(\Gamma)$. *Then all sets in* Γ *have the regularity properties* BP, LM, PSP.

Actually some more can be extracted from these proofs: assuming $\mathrm{Det}(\Gamma)$ the point class $\exists^\mathbb{R}\Gamma$ also enjoys these regularity properties. Here $\exists^\mathbb{R}\Gamma$ is the point class of *projections* of sets $A \in \Gamma$. (Here it is particularly useful to be able to prove results simultaneously in any finite dimension.) That is $B \in \exists^\mathbb{R}\Gamma$ iff $\exists A \in \Gamma$ $B = \{x \mid \exists y (x,y) \in A\}$.

By Martin's result on Borel Determinacy, in other words, that ZFC $\vdash \mathrm{Det}(\mathbf{\Delta}_1^1)$ (Theorem 3.8 below), we thus have the regularity properties for $\exists^\mathbb{R}\mathbf{\Delta}_1^1 = \mathbf{\Sigma}_1^1$. (See below for the definition of these classes within the *projective hierarchy*; in brief the *projective sets* are those definable in analysis, that is definable over the standard model of second order number theory.) The conclusion of the regularity properties for $\mathbf{\Sigma}_1^1$ is of course nothing more than the work of the earlier analysts. However now we have that $\mathrm{Det}(\mathbf{\Pi}_1^1)$ yields them for $\mathbf{\Sigma}_2^1$.

In short, determinacy is, unexpectedly perhaps, yielding the regularity properties on Baire space (and this transfers to other complete Polish spaces, such as Cantor space 2^ω, as well as back to the Euclidean $\mathbb{R}$).

Solovay had obtained in 1965 (as mentioned above) the very same $\mathbf{\Sigma}_2^1$ regularity results from a measurable cardinal. It turned out that determinacy for co-analytic sets follows too:

Theorem 1.3 (Martin 1970) (ZFC) *The existence of a measurable cardinal implies* $\mathrm{Det}(\mathbf{\Pi}_1^1)$.

It turned out that $\mathrm{Det}(\Pi_1^1)$ was equivalent to a non-trivial elementary embedding of L to L (the Martin-Harrington Theorem 3.9 below).

Martin later (1980) went on to show that $\mathrm{Det}(\mathbf{\Pi}_2^1)$ followed from the existence of a non-trivial elementary embedding $j : V_\lambda \longrightarrow_e V_\lambda$. This is one of the strongest hypotheses known for a large cardinal property, and it led to the feeling that PD, the determinacy of all point classes

in the projective hierarchy was stronger than all large cardinal hypotheses then known. This was amplified by Woodin's result that PD was indeed provable from a hypothesis even stronger than the one Martin used. Later results, coming from quite a different direction, of Foreman, Magidor and Shelah showed that a supercompact cardinal (defined below) implied all sets of reals in $L(\mathbb{R})$ were Lebesgue measurable. This hypothesis was much weaker than the Martin-Woodin hypotheses, and opened up the possibility that perhaps PD might not be so strong after all, since LM, being one of its consequences, was not.

So, it was after these results that it was discovered that embeddings of proper inner models (not of the whole of V into an inner model) are equivalent, broadly speaking, to that of the determinacy of pointclasses within the projective hierarchy (although such had been established for some point classes strictly within $\mathbf{\Delta^1_2}$). In the case of a Π^1_1 set A the point was to use a representation of A as a tree on (finite sequences from) $\omega \times \omega^{<\omega}$ in an auxiliary open game. The latter game would be in a different space of moves but importantly would be an open game there, and thus would (in ZFC) have a winning strategy σ^*. Indiscernibility (from Silver's indiscernibles from the embedding of L to L) was used to 'integrate out' a strategy σ for the original integer game G_A from σ^*. This rough scheme applies for the higher pointclasses Π^1_n: *homogeneous* tree representations of the set A were now on $\omega \times \omega^{<\lambda}$ for larger ordinals λ and the homegeneity guarantees that there are sufficiently coherent towers of measures on the tree to run the argument. A subtle kind of cardinal was defined, the Woodin cardinal, which allowed the correct properties of these trees to be propagated through the increasing complexity of the projective hierarchy.

When the dust settled:

Theorem 1.4 (Martin-Steel 1985) (ZFC)) *The existence of n Woodin cardinals with a measurable above them all, implies* $\mathrm{Det}(\mathbf{\Pi^1_{n+1}})$.

Theorem 1.5 (Martin-Steel, Woodin 1985) (ZFC) *The existence of infinitely many Woodin cardinals with a measurable cardinal above them all, implies* $\mathrm{AD}^{L(\mathbb{R})}$.

The notion of a Woodin cardinal is much weaker than a supercompact, so the original thought concerning the consistency strength of PD turned out to be quite incorrect. In fact an exact equiconsistency obtains (see Theorem 3.19 below). Now we have that large cardinals are sufficient to prove definable determinacy, and that conversely definable determinacy implies the existence of inner models of those large cardinals. The question arises as to whether we should wish to adopt definable determinacy as a new 'axiom'. For a full account of this discussion the reader is referred to the excellent account of Koellner's [**12**], but to summarise:

There is now a point of view that there is a case for assuming *projective determinacy*, or PD, as a natural 'completion' for the theory of the hereditarily countable sets: if one takes the theory $\mathrm{ZFC}^- + \mathrm{PD} + $ "Every set is countable", then this theory proves that all projective sets of reals have the regularity properties. Moreover the only sentences (as yet) known to be independent of the theory are Gödel-like sentences (*cf.* [**33**], [**32**]). If we move to a more global perspective and consider $\mathrm{AD}^{L(\mathbb{R})}$, then we have a very robust phenomenon concerning it: it appears (empirically) that any theory extending ZFC of sufficient strength implies it outright (not just its consistency we emphasise). This is the case of course of theories that are also themselves mutually inconsistent when put together; but they all have $\mathrm{AD}^{L(\mathbb{R})}$ as part of their common core consequences. If we were ever to choose one such theory as constituting a new 'axiom' we should be sure to get $\mathrm{AD}^{L(\mathbb{R})}$. Woodin has also shown that $\mathrm{AD}^{L(\mathbb{R})}$ is remarkable in that it is also implied by its consequences:

Theorem 1.6 (Woodin) (ZFC) *If all sets of reals enjoy the* LM, PSP *properties, and* $\mathrm{Unif}(\Sigma^2_1)$ *also holds then* $\mathrm{AD}^{L(\mathbb{R})}$.

(The latter Uniformisation property also holds in $L(\mathbb{R})$ assuming AD there). We thus cannot find a different theory or axiom implying the regularity consequences in $L(\mathbb{R})$ without *automatically* getting AD there. Koellner in [**12**] makes a comparison with an empirical science: we are remarkably obtaining back from our 'empirical consequences' of the theory, that is the observations of the regularity properties, the theory itself. This of course does not happen in physical theories, but the very remarkability of this phenomenon is one reason why many set theorists, especially those on or near the realist wing, feel that determinacy hypotheses such as $\mathrm{AD}^{L(\mathbb{R})}$ are compelling assertions about the set-theoretical universe.

Woodin has also shown that large cardinals in inner models are obtainable from determinacy hypotheses.

Theorem 1.7 (Woodin) (ZFC) *The theories* $\mathrm{ZF} + \mathrm{AD}$ *and* $\mathrm{ZFC} +$ "There exist infinitely many Woodin cardinals" *are equiconsistent.*

There are some equivalents over ZFC for levels of projective determinacy which we shall not detail here (*cf.* [**13**]).

Woodin has also demonstrated a very impressive array of results concerning attempts to develop a logic, the Ω-logic, that would factor out the baleful effects of the set-forcing independence results. One is thus interested in sentences, theories ... which survive attempts to alter their truth value by forcing arguments. Such sentences *etc.* are deemed "generically absolute".

Theorem 1.8 (Woodin) (ZFC) *Assume there is a proper class of Woodin cardinals. Then the (first order) theory of* $L(\mathbb{R})$ *is generically absolute. It thus cannot be altered by any set forcing.*

See Theorem 3.20 below. In particular as it only requires infinitely many Woodin cardinals to establish that $\mathrm{AD}^{L(\mathbb{R})}$, and as any set forcing will still leave infinitely many Woodin cardinals intact, every attempt to build a model using (set) forcing techniques will still result in a generic extension in which AD holds in the $L(\mathbb{R})$ of that generic extension. Now compare that with the following:

Theorem 1.9 (Woodin) (ZFC) *Suppose there is a proper class of inaccessible cardinals and that the theory of* $L(\mathbb{R})$ *is generically absolute. Then* $\mathrm{AD}^{L(\mathbb{R})}$ *holds.*

As with Theorem 1.6, the last theorem can also be viewed as another *recovery theorem*: $\mathrm{AD}^{L(\mathbb{R})}$ is being recovered from its consequences. Here we are witnessing that it must hold if the kind of generic absoluteness that flows from the existence of large cardinals holds true.

1.2 *Preliminaries:*

Our notation and formalisms are quite standard and can be found in many text books, but in particular we mention [**8**] and [**10**]. We let $\mathcal{L}_{\dot{\in}}$ denote the first order language of set theory; by ZF we mean a formulation of first order Zermelo-Fraenkel. We occasionally mention the theory ZF^{-} which is ZF with the power set axiom dropped. For ZF^{-} it is usual to include the Collection Scheme, rather than the Replacement Scheme, as the latter without Power Set is too weak to prove some basic facts.

By $\mathcal{L}_{\dot{\in},\dot{A}}$ we mean the standard language of set theory with an (optional) predicate $\dot{A}$. In both these languages we freely make use of terms abstracts $t = \{z \mid \varphi(\vec{y}, z)\}$ as if they were part of the languages. ZF_A is then a formulation of ZF with instances of the predicate $\dot{A}$ allowed in the axioms. A set is *transitive*, $\mathrm{Trans}(x)$, if every element of x is at the same time a subset of x.

The class of all ordinals is denoted On; Sing is the class of singular ordinals; Card is the class of all cardinals (we assume AC throughout and that cardinals are initial ordinals).

SingCard, Reg are the classes of singular cardinals and regular cardinals respectively. Inacc is the class of (strongly) inaccessible cardinals. For a limit ordinal τ, by the *"cofinality of τ"*, $\mathrm{cf}(\tau)$, is meant the least δ so that there is a function $f : \delta \longrightarrow \tau$ with $\mathrm{ran}(f)$ unbounded in τ ($\tau \in Reg$ then, if $cf(\tau) = \tau$). The rank function,"$\rho(x) = \alpha$" is Δ_1^{ZF} and the relation of y and α: "$y = V_\alpha$" is Π_1^{ZF}. We let $^x y$ denote the set of all functions from the set x to y. It is often customary to deviate from the practice when $x = \omega$ and we are considering, *e.g.*, $^\omega \omega$ or $^\omega X$ as spaces and then we shall write ω^ω or X^ω.

It is useful to fix a recursive enumeration $\langle \tau_i \mid i < \omega \rangle$ of Seq $=_{\mathrm{df}}$ $^{<\omega}\omega$ so that:

$$(i)\tau_0 = (), \tau_1 = (0); (ii)\, i \geq |\tau_i|\, (=_{\mathrm{df}} \mathrm{lh}(\tau_i)); (iii)\ \tau_i \subset \tau_j \longrightarrow i < j.$$

(We may similarly define recursive enumerating sequences $\langle \tau_i^k \mid i < \omega \rangle$ for $^k(^{<\omega}\omega)$.) The *Kleene-Brouwer ordering* is defined as follows: for $s, t \in\ ^{<\omega}\mathrm{On}$ put

$$s <_{\mathrm{KB}} t \leftrightarrow s \supset t \vee (s(i) < t(i) \text{ where } i \in \omega \text{ is least with } s(i) \neq t(i)).$$

Finally recall that a set $B \subseteq \mathbb{R}$ is Π_n^1 if it can be defined in Z_2 with

$$x \in B \leftrightarrow \exists f_1 \forall f_2 \cdots Q f_n \forall k R(k, \overrightarrow{f_i \restriction k}, \overline{x} \restriction k)\ \text{ with } R \in \Sigma_1^0$$

with the $f_i \in \mathbb{N}^{\mathbb{N}}$ and $k \in \mathbb{N}$. (Z_2 is a formalisation of second order number theory, or analysis, see [**28**].) A set is $\Pi_n^1(a)$ for some $a \in \mathbb{R}$ if the parameter a is allowed into the above defining Σ_1^0 relation R. The class $\mathbf{\Pi_n^1}$ is the union over a of the classes $\Pi_n^1(a)$. As the typeface indicates, a boldface point class indicates that, in general, real parameters are present in the definition of a member if that class. A *lightface* point class such as Σ_1^1 (also called lightfaced analytic), or Π_n^1, is one which involves no such parameters. As is usual in set theory, we make little distinction between $\mathbb{R}$ and ω^ω, or even $\mathcal{P}(\omega)$ on occasion.

2 Inner Models, Elementary Embeddings, and Covering Lemmas

Gödel constructed an *inner model* of V (here the hierarchy of *wellfounded* sets $\mathrm{WF} = \bigcup_{\alpha \in \mathrm{On}} V_\alpha$) by taking a "definable power set" operation $x \longrightarrow \mathrm{Def}(x)$ where x is shorthand for the first order structure $\langle x, \in \rangle = \langle x, \in \restriction x \times x \rangle$. By the parsimonious nature of the Def function Gödel showed that the sets so constructed were very regular, that they appeared in a definable order (thus a global axiom of choice was verified to hold in L) and that few subsets of any cardinal were admitted; the last justifies the GCH, the generalised continuum hypothesis as holding in L.

The Def function can be augmented as a *relativised definability operation* $x \longrightarrow \mathrm{Def}_A(x)$ using definability over $\langle x, A \restriction x, \in \restriction x \times x \rangle$ in the wider $\mathcal{L}_{\dot\in, \dot A}$ language; this gives us the ability to construct both from the ordinals as well as using information from the predicate A. These two definability functions are given by $\Delta_1^{\mathrm{ZF}^-}$ or $\Delta_1^{\mathrm{ZF}_A^-}$ terms and hence are absolute. Given a model W (thus given by some class term) we define the *relativisation* of a formula φ (or a theory T) denoted by φ^W (or T^W respectively), to be the formula φ with all quantifiers $\forall v_i, \exists v_j$ replaced by $\forall v_i \in W, \exists v_j \in W$ (respectively the same for all formulae in T). We think of this as expressing that the formula φ (or theory T) holds in W.

We let "$V = L$" abbreviate the statement that every set is constructible, or more formally: $\forall x \exists \alpha (x \in L_\alpha)$. That $(V = L)^L$ was shown by Gödel ([**6**]), as well as $(\mathrm{AC})^L$, $(\mathrm{GCH})^L$ thereby establishing: $\mathrm{Con}(\mathrm{ZF}) \implies \mathrm{Con}(\mathrm{ZFC} + \mathrm{GCH})$. Why should $V = L$? As this is a hierarchy of sets constructed by syntactico-semantical means, and not responding to any particular intuitions about set existence, most set-theorists do not believe $V = L$ (but see the discussion in [**9**]).

We shall see good reasons for asking whether there are other inner models, some associated with the notions of elementary embeddings. We first define our terms.

Definition 2.1 (Inner Model of ZF) $IM(M) \leftrightarrow \mathrm{Trans}(M) \wedge \mathrm{On} \subseteq M \wedge (\mathrm{ZF})^M$.

In the above we are assuming that M is given by some term of the language. The notion of being an inner model of ZF actually has a first order formalisation: it is well known (see Jech [**8**]) that

$$\mathrm{ZF} \vdash IM(M) \leftrightarrow \forall u \subseteq M \exists v \supseteq u(\mathrm{Trans}(v) \wedge \mathrm{Def}(\langle v, \in \rangle) \subseteq M).$$

The notion of an inner model of ZFC is simply the above with AC added at the relevant points.

We single out the following beautiful theorem as the motivation for this account:

Theorem 2.2 (ZF) *The following are equivalent:*
   a) $\exists j : L \longrightarrow_e L$.
   b) $\exists \gamma(\omega_2 \leq \gamma \in \mathrm{SingCard} \wedge (\gamma \notin \mathrm{SingCard})^L)$;
   c) $\mathrm{Det}(\Pi_1^1)$

Here a) expresses that there is a *truth-preserving* map between L and itself (see the Definition 2.3 below) which is non-trivial: that is, it is not the identity map. This is actually a culminative theorem established over several years: a)$\leftrightarrow$b) Jensen [**3**]; a) $\rightarrow$ c) Martin [**17**]; c)$\rightarrow$ a) Martin-Harrington [**7**]. Part a) of the above theorem, is sometimes stated as: "$0^\#$ *exists*". For the c) $\rightarrow$ a) direction, Martin actually showed that the determinacy of games at the third level of the difference hierarchy of analytic sets, proved the existence of $0^\#$. This was then reduced by Harrington to a single analytic in the work cited. For an elegant proof of the Harrington result see [**24**].

One should consider the overall form of this theorem, even if it is anachronistically stated, it is a deep result: it takes concepts that *a priori* have no connection with each other, but by deep analysis are shown to be equivalent. The first, a), involves an elementary embedding of L to itself.

Much large cardinal theory is about which ultrafilters can or do exist on (large) sets; in particular when those large sets are the power set of some cardinal of an inner model, then there is usually an equivalent formulation in terms of elementary embeddings, as defined below, of that inner model such as stated at a).

It is often stated that "$0^\#$ *exists*" is the first large cardinal axiom, in that it transcends Gödel's L in a drastic fashion: such embeddings cannot be added by Cohen style forcing arguments, and so can never be established by using 'small large' cardinals where the latter are those consistent with with universe being L itself (such as inaccessible, weakly compact, or ineffable cardinals *etc.*). Indeed from properties of $0^\#$, one can show that the L becomes a very thin inner model of V, and in some way becomes very similar to the class of ordinals alone.

The second clause b), is the negation of a form of the so-called *Covering Lemma* due to Jensen. What is being negated is the assertion that *all* singular cardinals are seen by L to be singular: whilst L may have many 'cardinals' that it does not see, or have, functions for mapping them to smaller ordinals, functions that may exist in V, the assertion is that, nevertheless, L can recognise the singularity of any singular cardinal of V. It must then possess functions cofinal in these cardinals. If this failed for just one singular cardinal, then $0^\#$ would exist. The Covering Lemma (which has stronger forms with more ramifications - see below) then asserts that the cardinality and cofinality structure of V is reflected inwardly to some extent into that of L. If a) holds however, this picture entirely collapses. The final clause c) is then at first glance quite extraordinary: we are asked to believe that the determinacy of coanalytic games, thus something right down at the bottom of the ramified hierarchy of sets, has a global effect: that it engenders an elementary embedding $j : L \longrightarrow_e L$. It seems to have nothing to do with the other two clauses.

It is one of the triumphs of the California school of set theory in the 1980's that this picture relating large cardinals and determinacy for the projective pointclasses, and on to sets in $L(\mathbb{R})$ came fully into focus, and was conclusively analysed. The equivalences in the theorem thus also show that determinacy is not some arcane curiosity in the set theorists' Wunderkammer, but holds a central role in our current understanding of what the subject is about.

We explore the background to Jensen's a)↔b). Again we have to define some terms:

Definition 2.3 *Let M, N be inner models of* ZF*, $j : M \longrightarrow N$ is an* elementary embedding *if the function j takes elements $x \in M$ to elements $j(x) \in N$ in a 'truth preserving way': for any formula $\varphi(v_0, \ldots, v_{n-1})$ and any $\vec{x} = x_0, \ldots, x_{n-1} \in M$, then*

$$\varphi(\vec{x})^M \leftrightarrow \varphi(\overrightarrow{j(x)})^N.$$

In this case we write: $j : M \longrightarrow_e N$. (We shall always assume that $j \neq$ id, and shall write cp(j) for the critical point: the least ordinal α so that $j(\alpha) > \alpha$, if it exists.)

(ii) If the above holds, but with the formulae restricted to a certain class, eg. the Σ_k formulae, then we write $j : M \longrightarrow_{\Sigma_k} N$.

In the above scheme, we have assumed that the models M, N satisfy $IM(M)$, $IM(N)$ above and are given by terms of our basic set theoretical language, and the same holds true for j. Our embeddings in this paper will all have critical points in the ordinals. It is an easy consequence of the ZF axioms (using the definition of the rank function, the V_α hierarchy, and Replacement) that if $j : M \longrightarrow_{\Sigma_1} N$, then by a (meta-theoretic) induction on k we may prove $j : M \longrightarrow_{\Sigma_k} N$ for any $k \in \omega$.

If $\exists j : L \longrightarrow_e L$ then we may define a *derived measure* $U = U_j$ on $\kappa = \text{cp}(j)$ as follows: we set

$$X \in U_j \leftrightarrow X \in \mathcal{P}(\kappa)^L \wedge \kappa \in j(X).$$

Then U_j is a *normal measure* ([**10**],p 52) on $\mathcal{P}(\kappa)^L$. Suppose we have $j : M \longrightarrow_e N$; $\text{cp}(j) = \kappa$, define $U = U_j$. Then we construct an ultrapower defining:

$$|\operatorname{Ult}(M, U)| = \{[f]_\sim : f \in {}^\kappa M \cap M\}$$

where

$$f \sim g \leftrightarrow \{\alpha \mid f(\alpha) = g(\alpha)\} \in U$$

and on which we can define a pseudo-$\in$ relation E:

$$fEg \leftrightarrow \{\alpha \mid f(\alpha) \in g(\alpha)\} \in U.$$

Because of where U_j comes from, we are guaranteed E is wellfounded on $\operatorname{Ult}(M, U)$; we may define by recursion along E the Mostowski-Shepherdson transitivising collapse isomorphism $\pi : (\operatorname{Ult}(M, U_j), E) \longrightarrow (\overline{N}, \in)$ with $\overline{N}$ transitive, and we have, with $k(\pi([f])) = j(f)(\kappa)$:

$$j : (M, \in) \longrightarrow (N, \in)$$

$$k$$

$$\pi : ((Ult(M, U_j), E)) \cong (\overline{N}, \in)$$

We can, and often do, have $(M, \in) = (V, \in)$. Also *starting* from *any* κ-complete U on $\mathcal{P}(\kappa)^V$, (for an uncountable regular κ), then we can define $\mathrm{Ult}(V, U)$ and we may prove outright that the E relation on $\mathrm{Ult}(V, U)$ will be wellfounded. In general for wellfounded cases we may then define the map π as above.

Taking these facts together then:

Theorem 2.4 (ZFC) *Let $\kappa > \omega$. The following are equivalent:*
(a) There is a κ-complete non-principle ultrafilter on $\mathcal{P}(\kappa)$.
(b) $\exists j : V \longrightarrow_e M$ with $\mathrm{cp}(j) = \kappa$.

Scott took the construction of an ultrapower from model theory and used it to prove the first theorem demonstrating that a 'large cardinal' was incompatible with $V = L$.

Theorem 2.5 (Scott [**27**]) (ZF) *$\exists\kappa(\kappa$ a measurable cardinal) $\Longrightarrow V \neq L$.*

Proof: If $V = L$, let κ be the least such measurable cardinal (MC), form the ultrapower and so the embedding above. Then from $j : V \longrightarrow_e N$, and elementarity we have: $(V = L)^V \longrightarrow (V = L)^N$; so as $\mathrm{Trans}(N)$, $N = L$. However "κ is the least MC" implies that ("$j(\kappa)$ is the least MC")^N. However $N = V \wedge j(\kappa) > \kappa$!   Q.E.D.

The assumption of this theorem implies $\exists j : V \longrightarrow_e M$, but again by Gödel's results on the absoluteness of the L-construction, $L^M = L$, so $j \restriction L : L \longrightarrow_e L$. Note that no first order formula $\varphi(v_0)$ can differentiate between κ and $j(\kappa)$: $\varphi(\kappa)^L \leftrightarrow \varphi(j(\kappa))^L$. Moreover both ordinals are inaccessible cardinals in the sense of L.

So we investigate the consequences of this embedding from L to L and shall discover that such *indiscernibility* of these critical points in fact characterises such embeddings. Kunen showed ([**14**]) that if $\exists \bar{j} : L \longrightarrow_e L$, then a number of consequences follow:

(i) Then there is such a j: $L \longrightarrow_e L$ with $\mathrm{cp}(j) < \omega_1$. Moreover defining U_0 from such a j with critical point κ_0 least, we are guaranteed wellfoundedness of *iterated ultrapowers*: that is we may define $j_{01} : L \longrightarrow_e L$ by taking the ultrapower of L by U_0; define U_1 on $\mathcal{P}(\kappa_1)^L$ where $\kappa_1 =_{\mathrm{df}} j_{01}(\kappa_0)$, and then $\mathrm{Ult}(L, U_1)$ will also be wellfounded. We then may transitivise it, and obtain L again. We thus have an embedding map $j_{12} : L \longrightarrow_e L$ with critical point κ_1; we then define $\kappa_2 = j_{12}(\kappa_1)$ and U_1 on $\mathcal{P}(\kappa_2)^L$. The process may be iterated without breaking down, forming a directed system $\langle \langle N_\alpha \rangle, j_{\alpha\beta}, \kappa_a, U_\alpha \rangle_{\alpha \leq \beta \in \mathrm{On}}$ with (in this case) all $N_\alpha = L$ and elementary maps into direct limits at limit stages λ, and the κ_α forming a class C of L-inaccessibles, which is closed and unbounded below any uncountable cardinal.

(ii) The iteration points of such ultrapowers enjoy full-blooded indiscernibility properties in L if $\varphi(v_0, \ldots, v_n)$ is any formula of $\mathcal{L}$ and $\vec{\gamma}, \vec{\delta}$ any two ascending sequences from $[C]^{n+1}$ then $(\varphi(\vec{\gamma}) \leftrightarrow \varphi(\vec{\delta}))^L$.

Definition 2.6 (The $0^{\#}$-mouse) *Let $j_{\alpha\beta}$ etc. be as above. Let*

$$M_0 = \langle L_{\kappa_0^{+L}}, \in, U_0 \rangle.$$

This structure is called the "$0^{\#}$-mouse" which itself has iterated ultrapowers using maps that are the restrictions of the

$$j_{\alpha\beta} \restriction M_\alpha : M_\alpha \longrightarrow_e M_\beta \text{ where } M_\alpha = \langle L_{\kappa_\alpha^{+L}}, \in, U_\alpha \rangle \text{ etc.}$$

Remark 2.7 *The viewpoint is shifted to that of the* mouse (M_0) *generating the model (in this case L). All of this is a paradigm for generalised constructible inner models K - the core models.*

By these means we argue for

Theorem 2.8 (ZF) *If $\exists j : L \longrightarrow_e L$ then $\forall \gamma > \omega((\gamma \in \mathrm{Card}) \longrightarrow (\gamma\,\mathrm{Inacc})^L)$.*

Proof: The above implies that $C \cap \gamma$ is unbounded below γ. However C is closed, so $\gamma \in C$. Each $\gamma \in C$ is inaccessible in L. Q.E.D.

Theorem 2.9 (Jensen)(ZF) *Suppose $\gamma \in \mathrm{SingCard}$, $\gamma \geq \omega_2$ but $(\gamma \in \mathrm{Reg})^L$. Then $\exists j : L \longrightarrow_e L$, with $j \neq \mathrm{id}$; that is "$0^\#$ exists."*

Proof: Suppose $\neg\exists j : L \longrightarrow_e L$, but γ is chosen least with $\gamma \in \mathrm{SingCard}$ but $(\gamma \in \mathrm{Reg})^L$. Without loss of interest, we shall assume that (i) $\mathrm{cf}(\gamma) > \omega$ (ii) $\delta < \gamma \longrightarrow \delta^\omega < \gamma$. Let $\tau = \mathrm{cf}(\gamma)$. By assumption then $\tau < \gamma$ and so we may choose $X_0 \subseteq \gamma$ with $|X_0| = \tau$ but X_0 unbounded in γ. By (ii) we'll assume also that for some $X \supset X_0$ we have (a) $\gamma \in X \prec L_{\gamma^+ L}$ (b) $^\omega X \subseteq X$ (c) $|X| = \tau^\omega < \gamma$.

Let $\pi : \langle X, \in \rangle \longrightarrow \langle M, \in \rangle = \langle L_{\bar\delta}, \in \rangle$ be the collapsing isomorphism with $\pi(\gamma) = \delta$ say. Then we have:

(1) $\mathrm{cf}(\delta) = \tau$ also, with $\overline{|\delta|} = |M| = |X| < \gamma$.

Suppose we had $\mathcal{P}(\delta)^M = \mathcal{P}(\delta)^L$. Then we could define a measure derived from π^{-1} in the usual manner: let $\alpha = \mathrm{crit}(\pi^{-1})$ and define U by

$$Z \in U \iff Z \in \mathcal{P}(\delta)^M \wedge \alpha \in \pi^{-1}(X).$$

Then U would be a *countably complete* ultrafilter on $\mathcal{P}(\delta)^L$ (that is why we chose $^\omega X \subseteq X$ as this implies $^\omega M \subseteq M$), and this in turn implies that $\mathrm{Ult}(L, U)$ is wellfounded. However that implies $\exists j : L \longrightarrow_e L$.

$$
\begin{array}{ccccc}
(X, \in) & \cong & (L_{\bar\delta}, \in) & \subseteq & (L, \in) \\
\mathrm{id} \downarrow & \pi^{-1} \nearrow & j \supset \pi^{-1} \nearrow & & \vdots \\
(L_{\gamma^+}, \in) & \subseteq & (L, \in) & \cong & (Ult(L, U), E)
\end{array}
$$

Hence we must have: $\mathcal{P}(\delta)^M \subsetneq \mathcal{P}(\delta)^L$. So:

(2) $\exists \beta \geq \bar\delta (\mathrm{Def}(L_\beta) \cap \mathcal{P}(\delta)) \nsubseteq L_{\bar\delta}$.

Choose β least so that (2) holds. By so-called fine structural methods Jensen showed how there is a superstructure L_η for some $\eta > \gamma^{+L}$ and a sufficiently elementary map $\tilde\pi \supset \pi^{-1}$, $\tilde\pi : L_\beta \longrightarrow L_\eta$, and because there is a 'new' subset of δ definable over L_β there must also be a 'new' subset of $\gamma = \pi^{-1}(\delta)$ that is not in L_η. However this is absurd as by L's construction $(\mathcal{P}(\gamma) \subset L_{\gamma^+})^L$. Q.E.D.

Remark 2.10 The assumptions (i) and (ii) can be dropped, but not without some difficulty, in particular when $cf(\gamma) = \omega$; however the format of the argument remains roughly the same.

Theorem 2.11 ((Jensen - the full L-Covering Lemma) (ZF+$\neg 0^\#$))
For any $X \subseteq \mathrm{On}$, if $|X| > \omega$ then there exists $Y \in L$ with (a) $|Y| = |X|$ and (b) $Y \supseteq X$.

The above result Theorem 2.9 is then a corollary of this. As are:

Corollary 2.12 (ZF+$\neg 0^{\#}$ exists) *(a) Let $(\tau \in \mathrm{Reg})^{L}$ with $\tau \geq \omega_{2}$, then $\mathrm{cf}(\tau) = |\tau|$.*
(b) Let $\tau \in \mathrm{SingCard}$. Then $\tau^{+} = \tau^{+L}$.

Remark 2.13 Part (b) of the last Corollary above is sometimes called WCL the Weak Covering Lemma. The reason is that for other inner models M we may have $\mathrm{WCL}(M)$ provable (obtained by replacing L by M in the Corollary's statements) whilst the full $\mathrm{CL}(M)$ is not. There is a result of Mitchell that shows that if there are sufficiently many measurable cardinals in an inner model, there may be many so-called Prikry-sequences (generic for Prikry forcings associated with the measurables) in V that must mess up the full Covering Lemma.

Generalizations

If $0^{\#} = M_{0}$ exists as above, perhaps there is no non-trivial $j : L[0^{\#}] \longrightarrow_{e} L[0^{\#}]$ and then we have a $\mathrm{CL}(L[0^{\#}])$? This is indeed the case; however if this new assumption fails then we have "$(0^{\#})^{\#}$". We then get a theorem along the lines of a new Covering Lemma in the form of $\mathrm{CL}(L[0^{\#}])$ which holds iff $\neg j : L[0^{\#}] \longrightarrow_{e} L[0^{\#}]$. $(0^{\#})^{\#}$ is again a countable object and we can repeat this process. After we have done this uncountably often our #-like mouse objects are no longer countable and we have to resort to uncountable mice M.

Instead of toiling inductively through seas of such objects, Dodd & Jensen first proved the following:

Theorem 2.14 (Dodd-Jensen [**4**]) (ZF) *There is an inner model K^{DJ}, so that if there is no inner model with a measurable cardinal, then (a) there is no non-trivial embedding $j; K^{\mathrm{DJ}} \longrightarrow_{e} K^{\mathrm{DJ}}$ and (b) $\mathrm{CL}(K^{\mathrm{DJ}})$.*

This was the first core model to go beyond L (if one discounts the models $L[0^{\#}]$ etc).

Theorem 2.15 (Steel [**30**]) (ZFC) *If there is no inner model for a Woodin cardinal, then there is a model K^{Steel}, which is again rigid, and over which $\mathrm{WCL}(K^{\mathrm{Steel}})$ holds.*

Perhaps there is some ultimate model which, although not L, is an inner model $L[E]$ say, which is the "core" of V, with E coding up all possible 'measures' or means of generating embeddings of models, and hopefully one has say $\mathrm{WCL}(L[E])$? Maybe, but to date the following 'inner model program' has worked on the following inductive template: assume you have built core models $K^{\Gamma'}$ for any strictly weaker large cardinal assumption Γ' than "There is an inner model with a Large Cardinal Γ". Now assume there is no inner model for such a cardinal as Γ and build K^{Γ}, which, you should show, still under this assumption, is rigid and over which $\mathrm{WCL}(K^{\Gamma})$ holds. Finally, if the latter fails you have (by contraposition) an inner model with a Γ-cardinal. Continue. This programmatic approach to filling out V with wider and thicker inner models depending on the strength of the large cardinals existing in inner models of V, has become known as the "*Inner Model program*". Currently, however, the programme is stuck at the level roughly of Γ a Woodin limit of Woodin cardinals.

Further remarks:

Theorem 2.16 (Magidor, [**16**]) (ZF) *Assume $\neg 0^{\#}$. Then if $X \subseteq \mathrm{On}$ is uncountable and closed under the primitive recursive set functions, then $X = \bigcup_{n<\omega} Y_{n}$ with each $Y_{n} \in L$.*

This is thus a decomposition rather than a covering theorem, but its proof is essentially (but perhaps surprisingly) a variant on Jensen's argument. The primitive recursive set functions mentioned are a mild collection of absolute functions on sets generalizing those on numbers, *cf* [**2**].

For many constructions, ordinal combinatorial properties up to a successor cardinal γ^+ are needed. The significance of the WCL over a model M say is that $\gamma^+ = \gamma^{+M}$ for certain cardinals of V (including the singular cardinals). The requisite ordinal combinatorial property may be established in the inner model M. Providing that the property is then sufficiently simple that it is absolute to V, we then have that that the combinatorial property is valid in V as well and is available for us for that original construction. The Weak Covering Lemma (under the rigidity assumption as always) has thus delivered for us a piece of absolute knowledge about V: one such combinatorial principle is called:

$\square_\gamma$: *There is* $\langle C_\alpha \mid \alpha \in \mathrm{Sing} \cap \gamma^+ \rangle$ *with*
(i) $C_\alpha \subseteq \alpha$ *is closed; it is unbounded in* α*, if* $\mathrm{cf}(\alpha) > \omega$*;*
(ii) $\mathrm{otp}(C_\alpha) < \alpha$*;*
(iii) $\beta \in C_\alpha \longrightarrow C_\beta = \beta \cap C_\alpha$*.*

Such a sequence can be thought of as a uniformly presented sequence of witnesses to the singularity of ordinals below γ^+, which cohere or glue together nicely as expressed in (iii). This is useful for inductive constrictions up to γ^+, and in particular for singular γ. Since the properties (i)-(iii) are absolute between M and V, if such can be constructed in M and $\gamma^+ = \gamma^{+M}$ we have such a good sequence in V.

Remark 2.17 Jensen first established $\square_\gamma$ for all $\gamma \in \mathrm{Card}$, in L. This was then established in the core models over the years as they were developed (in K^{DJ} by the author, in K^{Steel} by Schimmerling-Zeman [26], and with other results by other set theorists for intermediate models which we have not defined, see also [25] for an overview.) As intimated, principles such as $\square_\gamma$ can be seen as identifying very uniform singularising functions for singular ordinals. Certain large cardinals have an effect of implying the non-existence of certain such sequences in V. Other *forcing axioms* outright contradict the existence of any $\square_\gamma$ sequences at all. The study of such principles and their various weakenings has told us a lot about the interrelationship between large cardinals and inner models. The reference [25] gives an outline of which inner models can admit such sequences: it was a fundamental question which these authors solved as to which canonical inner models could in the end support such sequences: since it was known that large cardinals contradicted $\square$-principles, it was germane to find precisely where these principles broke down.

Remark 2.18 It is possible to establish the "correct successor cardinal" computation for certain classes of regular limit cardinals γ other than singulars alone: for γ weakly compact in V, (Jensen originally for L); γ Ramsey (Mitchell [21] originally for K^{DJ}, later in [22] for K^{Steel}); γ Jonsson (Welch [31] for K^{Steel}) again thereby showing that for cardinals in these classes, their successors in V also enjoy certain combinatorial properties. However for inaccessibles in general this will fail.

3 Determinacy: The Martin-Harrington Theorem

We have mentioned extremely briefly the history of determinacy; a complete source for this material is Moschovakis [23]. We have stated that the regularity properties hold for the projective sets, if all such sets are determined. In this section we define the basic terms relating to determinacy, and outline the route leading from large cardinals to projective determinacy. In particular we demonstrate the key concepts of *weakly homogeneous tree* and the *(weakly) homogeneously Suslin* property for sets of reals (Defs. 3.11 and 3.14 below). It is the descriptive set-theoretical analysis and representation of sets *via* trees, that goes back implicitly to Suslin for Σ^1_1-sets, and

for trees with special measures on them, that enable these theorems. The large cardinals enter the picture, as they are needed to provide the special measurability properties on the trees to make the analysis work. However it was then seen later, that the determinacy of sets which became thus provable, in turned proved 'reversals', in the form of demonstrations of the existence of *inner models* of the large cardinals which had been needed to prove the determinacy in the first place. (Notice that determinacy statements alone, being as they are, statements about the existence of strategies, which are essentially themselves reals, and so statements about elements of $V_{\omega+1}$, cannot prove that large cardinals actually exist in V - as the latter statements are about, say, some large $V_{\kappa+1}$, but it can, and does, prove the existence of inner models of those large cardinals.) The relationship between the determinacy of co-analytic sets and embeddings of L was thus the first evidence of a very rich picture that subsequently emerged. What was at first dark and mysterious about determinacy became fully understood through this analysis. Whatever one's position on the concept of large cardinals or strong axioms of infinity, we cannot understand determinacy of the projective sets, say, without it. The 'reversals' alluded to above prove this. Even if one dislikes arguments from 'extrinsic evidence' to the existence of large cardinals as actual sets, there is no denying their explanatory, and so instrumental, power. Probably for many set theorists that instrumental power alone is a considerable motive for their adoption. Explanatory power is a strong conceptual motivator in other areas of mathematics: mathematicians would have no hesitation in embedding what they were doing in some larger or new kind of space, if it led to understanding. If we regard set theory purely as a mathematical activity, for a moment imagining it shorn of any foundational pretensions, then adoption of such axioms might well be be considered unexceptionable.

Another significant definability property one may ask for is *Uniformization.* Suppose Γ, H are two pointclasses of sets of reals. We say that $\mathrm{Unif}(\Gamma, H)$ holds if whenever $Q \in \Gamma \cap \mathcal{P}(^k\mathbb{R} \times \mathbb{R})$ $(k < \omega)$ then there is $P \subseteq Q, P \in H$ and

$$\forall x[\exists y Q(y, x) \implies \exists! y P(y, x)]$$

(P thus acts as a function *uniformising* the relation Q by choosing a unique element in the relation for each x for which such is possible.) Classically one had (The Novikoff-Kondo-Addison Theorem) $\mathrm{Unif}(\Pi_1^1, \Pi_1^1)$. We let PU (Projective Uniformisation) abbreviate the statement that every projective relation can be projectively uniformised. It is fundamental outcome of Moschovakis's analysis of the scale property in the projective hierarchy, that projective determinacy implies the existence of projectively definable scales, and thence that PU holds (again cf [**23**]).

At a first encounter it seems an unusual position to take, to posit winning strategies for infinite games as plausible set-theoretical axioms. Why should such strategies even exist? However we see now that the regularity properties of sets is actually *equivalent* to determinacy, *i.e.* to the existence of such strategies, for all sets definable from $\mathbb{R}$ and the ordinals (*cf* Theorem 1.9). The regularity properties have been the bread and butter of analysts for decades and are innately uncontroversial. That the early analysts were unable to establish these properties beyond the co-analytic, was indeed because they were trammelled by working, albeit implicitly, only with ZFC-provable tools.

Games.

Definition 3.1 *For $A \subseteq \omega^\omega$ (or more generally X^ω) the infinite perfect information game G_A is defined between two players I, II, alternately playing elements $n_i, m_i \in X$:*

I	n_0		n_1		n_2		$\cdots$		n_k			$\cdots$

| *II* | m_0 | m_1 | $\cdots$ | m_k | $\cdots$ |

together construed as constructing $\qquad x = (n_0, m_0, \dots, n_k, m_k, \dots) \in X^\omega$.

We say that *I wins* iff $x \in A$; otherwise *II* wins. Notions of *strategy* and *winning strategy* are defined in the obvious fashion. Notice that for $A \subseteq \omega^\omega$ a strategy σ for, *e.g.* *I* is a map from $\bigcup_k {}^{2k}\omega \longrightarrow \omega$. Since there is a recursive bijection ${}^{<\omega}\omega \leftrightarrow \omega$, we can think of σ as essentially a subset of ω, or again, as a real number.

We speak of a topology of open and closed sets of a space X^ω by letting a typical *basic open set* to be a neighbourhood N_s where $s \in \mathrm{Seq}_X$ is a finite sequence of elements of X:

$$N_s =_{\mathrm{df}} \{x \in X^\omega \mid \exists k < \omega(x \restriction k = s\}.$$

Notice this is suitable for talking about payoff sets in such games: if in the above A is an N_s then essentially if *I* is to win, then he will have done so after $|s|$ many rounds of the game (where $|s|$ is the length of the sequence s). An *open set* is then a countable union of basic open neighbourhoods; a *closed set* is a complement of an open *etc.* When $X = \omega$ and an open set, U, is given as a union of basic open neighbourhoods defined by a *recursively* given set of sequences numbers s coding elements of ${}^{<\omega}\omega$, then we say that U is *semi-recursive*, or Σ^0_1. More generally we speak of open sets and Σ^0_1 sets in *parameter codes* which are themselves subsets of ω or again real numbers. If A is an open set in this topology, then *I* will win by essentially winning at some finite stage of the game; for *II* to win, then she must be vigilant to the end, to play into the complementary closed set.

The basic theorem here is:

Theorem 3.2 (Gale-Stewart) (ZF) *Let $A \subseteq X^\omega$ be an open set. Then G_A is determined, that is one of the players has a winning strategy.*

Proof: Note that if *I* wins it is because he has manœuvred the play so that there is a finite stage $(n_0, m_0, \dots, n_k)$ so that $N_{(n_0, m_0, \dots, n_k)} \subseteq A$. Essentially he has won by this stage as it matters not what n_l he plays for $l > k$. *II* however, playing into the closed set which is $X^\omega \backslash A$ must be vigilant to the end if she is to win. Suppose then *I* has no winning strategy. Then for every n_0 *II* has a reply m_0 so that *I* has no winning strategy in the game $G_{A/(n_0, m_0)}$ where $A/(n_0, m_0) =_{\mathrm{df}} \{x \in A \mid x(0) = n_0, x(1) = m_0\}$. For, if there was an n_0 so that *I* did always have a winning strategy in this latter game for whichever m_0 *II* played, $\sigma(m_0)$ say, then this would amount to a winning strategy for him in G_A: first play n_0, wait for m_0 and then use $\sigma(m_0)$. Thus given n_0 *II* should play m_0 so that *I* has no such strategy. However if she continues in this way, *this* is a winning strategy for *II*: always respond so that *I* has no winning strategy from that point on. The resulting play x cannot be in A. $\qquad$ Q.E.D.

The maxim here is that we can always in ZFC, prove $\mathrm{Det}(\mathrm{Open})$ for any space X^ω and so by taking complements $\mathrm{Det}(\mathrm{Closed})$ too. (AC is needed only to wellorder X if need be.) Many proofs of determinacy of complicated sets in ω^ω or X^ω, involve reducing the game to a closed game in some larger space Y^ω. The latter are determined by the Gale-Stewart theorem. The difficulty arises in showing that the player with the winning strategy for the open set on the space Y^ω also has one for the related, but much more, complicated set in X^ω.

Definition 3.3 *(i) A tree T on $\omega \times X$ (for $X \neq \varnothing$) is a set of sequences in $\bigcup_k {}^k\omega \times {}^kX$ where if (σ, u) and $k \leq |\sigma| = |u|$ then $(\sigma \restriction k, u \restriction k) \in T$. Similarly defines trees on ${}^n\omega \times X$.*
(ii) For such a tree T we set

$$T_\sigma =_{\mathrm{df}} \{u \mid (\sigma, u) \in T\} \; ; T^\subseteq_\sigma =_{\mathrm{df}} \bigcup_{k \leq |\sigma|} T_{\sigma \restriction k} \; ; \text{and } T_x = \bigcup_k T_{x \restriction k}.$$

Definition 3.4 *For T a tree:*

(i) $[T] =_{\text{df}} \{(x, f) \mid \forall k(x \restriction k, f \restriction k) \in T\}$ - *is the set of* branches through T.

(ii) $p[T] =_{\text{df}} \{x \in {}^{\omega}\omega(\text{or } {}^{k}({}^{\omega}\omega)) \mid \exists f(x, f) \in [T]\}$ - *the* projection *of T.*

(iii) A set $A \subseteq {}^{k}({}^{\omega}\omega)$ is κ-Suslin (for $\kappa \geq \omega$, $\kappa \in \text{Card}$) if $A = p[T]$ for some tree on $\omega \times \kappa$.

Clearly a tree is wellfounded (under $\supseteq$) if $[T] = \varnothing$. For a $C \subseteq \omega \times X$ a closed set, there is a tree T on $\omega \times X$ with $C = [T]$. In particular for $X = \omega$. If C has a recursively open complement, then we may take T as recursive set of sequences from $\bigcup_{k}({}^{k}\omega \times {}^{k}\omega)$.

If $A \subseteq \omega^{\omega}$ is Σ_1^1 then $A = p[T]$ for a tree on $\omega \times \omega$ - thus such sets are projections of closed sets, and conforms to the idea that $x \in A \leftrightarrow \exists y \in \omega^{\omega}(x, y) \in [T]$. This classical result (due to Suslin) is sometimes stated that analytic ($= \Sigma_1^1$) sets are "ω-Suslin". Many of the classical properties of analytic sets as studied by analysts can be attributed to this (and similar) representations.

Given this last fact, we can represent Π_1^1 sets, being complements of Σ_1^1's, as those sets where the tree T for the complement is wellfounded. We may thus define *rank functions* for such trees, by finding functions that map the nodes into the ordinals in a tree-order preserving way. We do this next.

Let then $A \in \Pi_1^1$ be a (lightface) co-analytic set. Then by the above discussion there is a recursive tree T (meaning there is a recursive set of sequence numbers) on $\omega \times \omega$ with:

$$\forall x(x \in A \leftrightarrow T_x \text{ is wellfounded}).$$

T_x being countable, we have that $\text{rk}(T_x) < \omega_1$. Consequently:

$$x \in A \Leftrightarrow \exists g(g : T_x \longrightarrow \omega_1 \text{ in an order preserving way}).$$

That is we may define a tree $\hat{T}$ on $\omega \times \omega_1$ as follows:

$$\hat{T} = \{(\tau, u) \mid \forall i, j < |\tau| : \tau_i \supset \tau_j \wedge (\tau \restriction |\tau_i|, \tau_i) \in T \longrightarrow u(i) < u(j)\}.$$

Then one can see that

$$x \in A \leftrightarrow \exists g \in {}^{\omega}\omega_1((x, g) \in [\hat{T}]) \leftrightarrow x \in p[\hat{T}].$$

The above reasoning thus shows that any Π_1^1 set A is ω_1-Suslin. Further:

(i) $\hat{T} \in L$, and by the absoluteness of well founded relations on ω between ZF$^-$-models containing all countable ordinals $[\hat{T}] \neq \varnothing \leftrightarrow ([\hat{T}] \neq \varnothing)^L$.

(ii) If the underlying set is a $\Pi_1^1(a)$ set for some real parameter a then $\hat{T} \in L[a]$.

Now the argument can be stepped up to Σ_2^1 sets: suppose $x \in B \leftrightarrow \exists y(x, y) \in A$ (we write '$B = pA$') with $A \in \Pi_1^1$ and hence there is a tree $\hat{T}_A$ on $(\omega \times \omega) \times \omega_1$ with

$$(x, y) \in A \leftrightarrow \exists g \in {}^{\omega}\omega_1\big(((x, y), g) \in [\hat{T}_A]\big).$$

However there is a Δ_0^{ZF} definable bijection $\omega \times (\omega \times \omega_1) \leftrightarrow (\omega \times \omega) \times \omega_1$ thus re-defining the tree $\hat{T}_A$ as $\overline{T}$ but on different sequences, so that for any x, y we have

$$(x, y) \in p[\hat{T}_A] \leftrightarrow \exists g \in {}^{\omega}\omega_1\big(((x, y), g) \in [\hat{T}_A]\big) \leftrightarrow \exists g \in {}^{\omega}\omega_1\big((x, (y, g)) \in [\overline{T}]\big).$$

However now we again use a Δ_0^{ZF} bijection $\omega \times \omega_1 \leftrightarrow \omega_1$ to recast $\overline{T}$ as a tree $\overline{\overline{T}}$ on ω_1 alone, then:

$$x \in B \leftrightarrow x \in pA \leftrightarrow x \in p[\overline{\overline{T}}].$$

We have:

Theorem 3.5 (Shoenfield) (ZF) *Any Σ_2^1 set is ω_1-Suslin, as a projection of a tree $T \in L$.*

Corollary 3.6 (ZF) *Let B be any Σ^1_2 relation. The $\exists x B(x) \leftrightarrow (\exists x B(x))^L$. In particular Σ^1_2 sentences are absolute between L and the universe V. Moreover if $A \subseteq \mathbb{N}$ is Σ^1_2 then $A \in L$.*

Theorem 3.7 (Levy) (ZF) *There is an ordinal $\sigma_1 < \omega_1^L$ such that $(L_{\sigma_1}, \in) \prec_{\Sigma_1} (V, \in)$.*

Proof: Let A be a Σ_1 sentence in $\mathcal{L}$ true in V. By Löwenheim-Skolem, the following holds:

"*There is a countable wellfounded transitive model $(M, \in)$ with $(A)^M$*".

The latter can be expressed as a Σ^1_2 assertion about a real number x (coding such an $(M, \in)$). By the last corollary there is such an x, and $M_x \in L_{\omega_1^L}$. By the upward persistence of Σ_1 sentences $(A)^{M_x} \implies (A)^L$. Hence $L_{\omega_1^L} \prec_{\Sigma_1} V$.

<div align="right">Q.E.D.</div>

We sometimes wish to be more specific about the tree for Π^1_1 and for this we use the Kleene-Brouwer ordering: $<_{\mathrm{KB}}$ is a linear ordering, and it restricts to linear orderings on $^{<\omega}\kappa$ for cardinals $\kappa \geq \omega$. One may check:

If T is a tree on κ then T is wellfounded iff T is wellordered by $<_{\mathrm{KB}}$. In particular we have for a Π^1_1 set A that there is a tree T on $\omega \times \omega$ so that

$$x \in A \leftrightarrow T_x \text{ is wellfounded } \leftrightarrow T_x \text{ is wellordered by } <_{\mathrm{KB}}.$$

For a tree T on $\omega \times \kappa$ we then have a linear ordering $<_x$ corresponding to the KB ordering of T_x with the following definition and properties:

$$i <_x j \longleftrightarrow_{\mathrm{df}} \quad (\tau_i, \tau_j \notin T_x \wedge i < j) \vee (\tau_i \notin T_x \wedge \tau_j \in T_x) \vee (\tau_i, \tau_j \in T_x \wedge \tau_i <_{\mathrm{KB}} \tau_j).$$

In fact $<_x$ is the union of orderings given by initial segments of x: for $\tau \in {}^{<\omega}\omega$ define

$$i <_\tau j \longleftrightarrow_{\mathrm{df}} i < j < |\tau| \wedge (\tau_i, \tau_j \notin T_\tau^{\subseteq} \wedge i < j) \vee (\tau_i \notin T_\tau^{\subseteq} \wedge \tau_j \in T_\tau^{\subseteq}) \vee (\tau_i, \tau_j \in T_\tau^{\subseteq} \wedge \tau_i <_{\mathrm{KB}} \tau_j).$$

Then:

(i) $<_x = \bigcup_k <_{x \restriction k}$.

(ii) Define T_A^* on $\omega \times \kappa$ by $(\tau, u) \in T_A^* \longleftrightarrow_{\mathrm{df}} \forall i, j < |\tau| (i <_\tau j \leftrightarrow u(i) < u(j))$. Then

$$x \in A \leftrightarrow \exists g \in {}^\omega\kappa ((x, g) \in [T_A^*]).$$

In the above $\kappa \geq \omega$ may be any cardinal of course. After the basic definitions were introduced $\mathrm{Det}(\Sigma^0_3)$ (M. Davies [1]) was the extent of provable determinacy (indeed provable within Z_2). Remarkably H. Friedman ([5]) showed that for Borel Determinacy roughly, order type α iterations of the power set operation (together with some Replacement) would be needed to establish $\mathrm{Det}(\Sigma^0_\alpha)$, thus establishing that $Z \nvdash \mathrm{Det}(\Delta^0_1)$. This phenomenon would start at Σ^0_4. Paris then showed (using ZFC) $\mathrm{Det}(\Sigma^0_4)$ but there matters languished until:

Theorem 3.8 (Martin [20]) (ZF) $\mathrm{Det}(\mathrm{Borel})$ *for Borel subsets of ω^ω.*

Theorem 3.9 (Martin-Harrington) (ZF) $\exists j : L \longrightarrow_e L, j \neq \mathrm{id} \Longleftrightarrow \mathrm{Det}(\Pi^1_1)$.

Proof: ($\longrightarrow$) (Martin) We assume the left hand side, and then by the work of Silver, Kunen, we have a c.u.b. set $C \subseteq \omega_1$ of indiscernibles for $(L, \in)$.

Let $A \in \Pi^1_1$. The usual game G_A involves integer moves but has a complicated payoff set. We replace the space ω^ω with a larger one X^ω for some X to be defined, and relate A to some *closed* $A^* \subseteq X^\omega$. By the Gale-Stewart theorem, this is determined. We have to show that winning strategies in G_{A^*} can be translated to winning strategies for the same player in G_A.

Definition 3.10 G_{A^*} *is defined between two players I, II playing as follows:*

I	(n_0, ξ_0)		(n_1, ξ_1)	(n_2, ξ_2)	$\cdots$	(n_k, ξ_k)		$\cdots$
II		m_0	m_1		$\cdots$		m_k	$\cdots$

together constructing

$$x = ((n_0, \xi_0), m_0, \ (n_1, \xi_1), m_1, (n_2, \xi_2), m_2, \cdots (n_k, \xi_k), , \ldots).$$

The *Rules* are that $n_i, m_i \in \omega$, $\xi_i \in \omega_1$. We think of the ξ_i as laying out a function $g : \omega \longrightarrow \omega_1$ with $g(i) = \xi_i$, and the integers yielding $x = (n_0, m_0, n_1, \dots)$

Winning Conditions: I wins iff $(x, g) \in T_A^*$.
Then g witnesses that T_x is wellfounded, by giving an order preserving map from $<_x$ into ω_1. Note that the game is closed in the sense that if *I* loses he does so at some finite stage (by messing up his ordering, which cannot be redeemed at a later stage). Equivalently, *II* is playing into an open set and if she wins, does so at a finite round. Note that T_A^* is defined in L, and so by the Gale-Stewart theorem, is determined in L.

Claim: If σ^* is a winning strategy for *I* (respectively *II*) in G_{A^*} in L, then there is a winning strategy for *I* (respectively *II*) in G_A.

Proof of *Claim*. If σ^* is a winning strategy for *I* he can use it to play in G_A by suppressing the ordinal moves, and clearly wins (both in L and V). Suppose then $\sigma^* \in L$ is, in L, a winning strategy for *II*.

Idea: *II* simulates a run of G_{A^*} by using indiscernibles from $C_0 =_{\mathrm{df}} C \cap \omega_1$ as 'typical' ordinal moves for *I*. She defines a strategy $\sigma : \bigcup_n {}^{2n+1}\omega \longrightarrow \omega$ as follows; fix $n < \omega$ and consider the formula $\varphi(\omega_1, T^*, \sigma^*, \tau, \vec{\xi}, m)$ which defines a term $t(\omega_1, T^*, \sigma^*, \tau, \vec{\xi}) = m$:

$$\vec{\xi} \in {}^{n+1}\omega_1 \wedge \sigma^*(\tau \upharpoonright n+1, \vec{\xi}) \in T^* \wedge \sigma^*(\tau, \vec{\xi}) = m$$

As $\sigma^*, T^* \in L$ the term t will have a fixed value $m \in \mathbb{N}$ for any $\vec{\xi} \in {}^{n+1}C_0$ she chooses, since the latter are indiscernibles for the formula φ. So she sets (in V where C_0 lives):

$$\sigma(\tau) = m = t(\omega_1, T^*, \sigma^*, \tau, \vec{\xi}) \text{ for any } \vec{\xi} \in {}^{n+1}C_0.$$

Now argue that were $x \in A$ even though *II* followed this strategy, then we'd have that there is an order preserving embedding $g : (\omega, <_x) \longrightarrow (C_0, <)$ (as C_0 is uncountable). However that corresponds to a run of the game G_{A^*} (with ordinal moves delivered by g), where *II* has used σ^*, which was supposed to be winning for her! Contradiction!   Q.E.D.

It is an exercise to show that if a G_A for $A \subseteq {}^\omega\omega$ is determined, A is countable, or else contains a perfect subset and hence is the size of the continuum. However in L there is an uncountable Π_1^1 set which has no perfect subset. Conclusion: $\mathrm{Det}(\Pi_1^1)$ is false in L and so $\mathrm{ZF} \nvdash \mathrm{Det}(\Pi_1^1)$. In the above we constructed a winning strategy for *II* in V. However the statement "$\exists \sigma[\sigma$ is a winning strategy for *II* in $G_A]$" is a strictly Σ_3^1 statement, and such is not in general absolute to L.

We should like to establish $\mathrm{Det}(\Pi_n^1)$ for other $n > 1$ but this requires considerably more ingenuity. The key is to find representations of projective sets as projections of trees enjoying so-called 'homogenity properties' which entail that there are measures (and more) on the trees. This tree representation was implicit in the proofs of $\mathrm{Det}(\Pi_1^1)$ of Martin and in Martin-Solovay

[18] where such trees were used to analyse Π^1_2 sets. However the notion was only made explicit in Kechris [11]. For the rest of this article we shall work in ZFC unless otherwise stated.

Definition 3.11 *(Homogeneous Trees and Sets) Let T be a tree on $\omega \times X$ and $\kappa > \omega$ be a cardinal.*
(A) Then T is a κ-homogeneously Suslin tree iff $\exists \langle U_\tau \mid \tau \in \mathrm{Seq}\rangle$ where
    (i) Each U_τ is a κ-complete measure on T_τ;
    (ii) For any $\tau \supset \sigma$ U_τ projects to U_σ: i.e. $u \in U_\sigma \leftrightarrow \{v \in T_\tau : v \restriction |\sigma| \in u\} \in U_\tau$.
    (iii) The U_τ form a countably complete tower: if $\sigma_i \in \mathrm{Seq}$, $Z_i \in U_{\sigma_i}$ are such that $i < j \longrightarrow \sigma_i \subset \sigma_j$ then there is a $g \in {}^\omega X$ so that: $\forall i (g \restriction i \in Z_i)$.
(B) We say $A \subseteq \omega^\omega$ is κ-homogeneously Suslin if $A = p[T]$ for a κ-homogeneously Suslin tree T.

Notice that one way to phrase *(iii)* is to say that if $x \in p[T]$ then $\langle U_{x \restriction i} : i < \omega\rangle$ form a countably complete tower. Although the notion had to be isolated in fact Martin had used essentially the fact that if there exist a measurable cardinal κ then any analytic set is $p[T]$ for a κ-homogenously Suslin tree on $\omega \times \kappa$.

Theorem 3.12 *Suppose A is κ-homogeneously Suslin for some κ. Then G_A is determined.*

Proof: Suppose $A = p[T]$ with T κ-homogeneously Suslin on $\omega \times \lambda$ (some $\lambda \geq \kappa$.

Definition 3.13 G_{A^*} *is defined between two players I, II playing as follows:*

I	(n_0, ξ_0)		(n_1, ξ_1)		(n_2, ξ_2)		$\cdots$		(n_k, ξ_k)		$\cdots$
II		m_0		m_1		$\cdots$				m_k		$\cdots$

together constructing

$$x = ((n_0, \xi_0), m_0, \ (n_1, \xi_1), m_1, (n_2, \xi_2), m_2, \cdots).$$

Rules: $n_i, m_i \in \omega$, $\xi_i \in \lambda$. *Winning Conditions:* as before *I* wins iff $(x, g) \in [T]$ where $g(i) = \xi_i$.

Again this is a closed game, and so there is a winning strategy for one of the players in $L[T]$. Now to show that if *II* has a winning strategy in G_A if she has a winning strategy σ^* in G_{A^*}, we use the ω_1-completeness of the measures rather than indiscernibility.

She defines a strategy $\sigma : \bigcup_n {}^{2n+1}\omega \longrightarrow \omega$ as follows; for $n < \omega$ and then for $\tau \in {}^{2n+1}\omega, u \in {}^{n+1}\lambda$, define $\overline{\sigma}(\tau, u) = \sigma^*(\tau(0), u(0), \cdots, \tau(2n), u(2n)) \in \omega$.

Define $Z_{\tau,k} =_{\mathrm{df}} \{u \in T(\tau \restriction n + 1) : \overline{\sigma}(\tau, u) = k\}$. Since $U_{\tau \restriction n+1}$ is a measure on $T(\tau \restriction n + 1)$, by its ω_1-completeness, for precisely one value of k is $Z_{\tau,k} \in U_{\tau \restriction n+1}$. So let that value of k, k_0 say, be the response given by the strategy:

$$\sigma(\tau) = k_0 \text{ where } k_0 \text{ is the unique value of } k_0 \text{ with } Z_{\tau,k_0} \in U_{\tau \restriction n+1}.$$

Again we check that σ is a winning strategy for *II* in G_A. If x is a result of a play with *II* using σ, but nevertheless $x \in A$, then by *(iii)* in the definition of κ-homogeneously Suslin $\exists g \in {}^\omega X$ so that for every n $g \restriction n + 1 \in Z_{x \restriction 2n+1, \sigma(x \restriction 2n+1)}$. This implies $(x, g) \in [T]$ and is then a losing outcome of the game where *II* in fact uses σ^*, thus contradicting that the latter is a winning strategy for *II*. Hence $x \in A$.   Q.E.D.

Hence determinacy would follow if we could establish homogeneity properties for trees.

Definition 3.14 $A \subseteq \omega^\omega$ *is* weakly homogeneously Suslin *if* $A = pB$ *where* $B \subseteq (\omega^\omega)^2$ *is homogeneously Suslin.*

In fact this is not the official definition of weakly homogeneously Suslin which defines "*weakly homogenous trees*" and is in terms of towers of measures, but nevertheless has this equivalence. These were also studied by Kechris [11] and Martin.

Weakly homogeneously Suslin sets (being the projections of weakly homogeneous trees) can be thought of as generalisations of analytic sets and are of great interest in their own right: we have seen that if there is a measurable cardinal, then Σ^1_2 sets are weakly homogeneously Suslin.

Theorem 3.15 *If a set A is weakly homogeneously Suslin, then it has the regularity properties (Lebesgue measurability, the Baire and Perfect subset properties ...).*

It thus has the *consequences* of being determined, without actually being so. Unfortunately A being weakly homogeneously Suslin does not imply that it is homogeneously Suslin. However one important feature is that they do have complements defined as projections of trees $\tilde{T}$ with the latter definable from their weakly homogenously tree T:

Lemma 3.16 *Let A be $p[T]$ with T weakly homogeneously Suslin. Then there is a tree $\tilde{T}$ with $\omega^\omega \backslash A = p[\tilde{T}]$.*

However on its own this is no help. The breakthrough was:

Theorem 3.17 (Martin-Steel [19]) *Suppose that λ is a Woodin cardinal and T is a λ^+ weakly homogeneously Suslin tree. Then for $\gamma < \lambda$, the $\tilde{T}$ above is γ-homogeneously Suslin.*

There is thus some trade off: the completeness of the measures drops. However we now have:

Theorem 3.18 (Martin-Steel) *Suppose λ_0 is a Woodin cardinal, and $\kappa > \lambda_0$ is measurable. Then $\mathrm{Det}(\Pi^1_2)$. Further if $\lambda_{n-1} < \lambda_{n-2} < \cdots < \lambda_0$ are $n-1$ further Woodin cardinals, then $\mathrm{Det}(\Pi^1_{n+1})$. Thus:*

$$\mathrm{ZFC} + \text{``there exist infinitely many Woodin cardinals''} \vdash \mathrm{PD}.$$

Proof: Let $A \subseteq \omega^\omega$ be Π^1_2. Then A is the complement of a Σ^1_2 set on ω^ω which is itself the projection of a Π^1_1 set $B \subseteq (\omega^\omega)^2$. B is κ-homogeneously Suslin for some homogeneously tree T, as κ is measurable, and *a fortiori* is also λ_0^+-homogeneously Suslin. By Theorem 3.18 we have a γ-homogeneously Suslin tree $\tilde{T}$ projecting to A with γ-complete measures. By Theorem 3.12 we have G_A is determined, and we are done. The last two sentences follow by repetition of the argument.   Q.E.D.

The exact consistency strength of the assumption above is slightly stronger, with the ($\Longleftarrow$) being very involved:

Theorem 3.19 (Woodin)
$\mathrm{Con}(\mathrm{ZFC} + \text{``there exist infinitely many Woodin cardinals''}) \Longleftrightarrow \mathrm{Con}(\mathrm{ZFC} + \mathrm{AD}^{L(\mathbb{R})})$.

The final conclusion of Theorem 3.18 yields the boundary of the *provability* of PD from large cardinal axioms; this is not a relative consistency result, PD holds outright. Set forcing can change the character of the universe *locally*, but, roughly speaking, does not destroy large cardinal properties beyond the rank of the partial order. Consequently if V has a proper class of Woodin cardinals, then PD is not only provable outright but is *absolute* into set forcing extensions. Much more is possible:

Theorem 3.20 (Woodin) *Suppose there is a proper class of Woodin cardinals; then* $\mathrm{Th}(L(\mathbb{R}\})$ *is absolute with respect to set forcing. Thus if* $V[G]$ *is a set generic extension*

$$(L(\mathbb{R}))^V \equiv (L(\mathbb{R}))^{V[G]}.$$

The larger cardinals that we look at next will again all prove PD and in fact $\mathrm{AD}^{L(\mathbb{R})}$ outright. We have seemingly reached a particular stage beyond which these determinacy properties are simply unavoidable.

4 Large Cardinals

Much large cardinal theory is about which ultrafilters can or do exist on (large) sets; in particular when those large sets are the power set of some cardinal of an inner model M (which might be V itself), then there is usually an equivalent formulation in terms of elementary embeddings, of that inner model M. In the past large cardinals, or the axioms asserting their existence, were often claimed as justified by a variety of differing arguments. (An example being the paper of [**29**].) Such arguments tended to give a somewhat patchwork view of large cardinals: some justified "by analogy with ω", others by reflection principles of one kind or another. Contemporary set theory now tends to see large cardinals in a linear, indeed well founded, hierarchy of consistency strength, but moreover as providing a lined up picture of *embedding properties* ordered by the degree that the embedding "pulls in" to its range model, more, or less, of the V_α-hierarchy. There is some over-simplification here, as pulling in some V_λ into a range model M will be weaker than ensuring that such an M has more closure properties, such as being closed under λ'-*sequences* for a $\lambda' < \lambda$. However the overall slogan holds true: the more closure properties the range model enjoys, the stronger the embedding and the stronger the large cardinal needed to justify it. Again the unificatory impulse which sees large cardinals as positioned within a somewhat continuous spectrum of embeddability relations, can be said to lead to a greater understanding of possible embedding phenomena. The very possibility of mathematically defining an embedding of L to L immediately then impresses us with the need to study the whole such hierarchy: if the universe of constructible sets can embed to itself then why not the model K^{DJ}? However if this occurred, then there is an inner model with a measurable cardinal (see Theorem 2.14 above). In short, there seem to be no lines to be drawn here.

Such embeddings are almost all expressible in terms of the existence of ultrafilters, or in other words, measures (or sequences of such) on certain sets. We have seen that a measure U on $\mathcal{P}(\kappa)$ in V yields an ultrapower $(\mathrm{Ult}(V, U), E)$ which is wellfounded and hence isomorphic to a transitive inner model $(M, \in)$ of ZFC. The following facts hold:

- $V_{\kappa+1} = (V_{\kappa+1})^M$
- $(j(\kappa)$ is measurable with measure $j(U))^M$;
- $U \notin M$ and thus $V_{\kappa+2} \neq (V_{\kappa+2})^M$.
- κ may, or may not, be measurable in M because of the presence in M of some other measure $\overline{U}$). (If κ was the least measurable of V then it cannot contain such a $\overline{U}$, by the Scott argument).

Thus given an elementary embedding $j : V \longrightarrow_e M$, the above shows precisely which initial part of V can be expected to be in M in general, namely $V_{\kappa+1}$. The first generalisations of this concept consider sequences of such measures concentrating on different subsets of the same cardinal κ. (There are potentially then 2^{2^κ} of those for a given measurable cardinal κ.) Beyond that we start to ask for sets of higher rank to be in the range model, and this motivates the next definition.

Definition 4.1 *A cardinal κ is α-strong if there is an embedding $j : V \longrightarrow_e M$ with $V_a = (V_\alpha)^M$, with $\mathrm{cp}(j) = \kappa$, and $j(\kappa) \geq \alpha$.*

Thus a measurable cardinal is $\kappa + 1$ strong. The larger the α, the stronger the embedding, as more of the initial V hierarchy is preserved by the identity map into the inner model M.
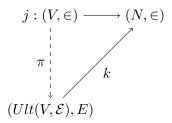
Definition 4.2 *A cardinal κ is* strong *if it is α-strong for all α.*

(Note the order of quantifiers: for every α there is an embedding j, depending on α ...) One may wonder about the first order formalizability of the above notions in ZF. However just as the first statement of the existence of a measure on $\mathcal{P}(\kappa)$ is equivalent to the existence of a class embedding j (which we might formulate in $\mathrm{ZFC}_{\dot{j}}$) it is possible to give an *extender representation* of such embeddings. Thus: for an α-strong embedding j we may find a generalisation of a measure on $\mathcal{P}(\kappa)$ called an *extender* which we may think of as given by a *sequence* of measures $\langle E_a : a \in [\alpha]^{<\omega} \rangle$ with each E_a itself a measure on $\mathcal{P}([\kappa]^{|a|})$.

Given an α-strong embedding $j; V \longrightarrow_e N$ we define an α-*extender at κ* generalising what we did for measures.

$$X \in E_a \leftrightarrow_{\mathrm{df}} X \in \mathcal{P}([\kappa]^{|a|}) \wedge a \in j(X).$$

The sequence $\mathcal{E} = \langle E_a : a \in [\alpha]^{<\omega} \rangle$ then has satisfactory coherence properties, in fact enough so that we can define an *extender ultrapower* $(\mathrm{Ult}(V, \mathcal{E}), E)$ from it. In the situation described, this ultrapower has a wellfounded E-relation, and is again isomorphic to some $(M, \in)$ which in general is not necessarily equal to $(N, \in)$ but which can be elementarily embedded back into $(N, \in)$ and we thus shall have again:

$$j : (V, \in) \longrightarrow (N, \in)$$

$$\pi \qquad \qquad k$$

$$(Ult(V, \mathcal{E}), E)$$

It is possible to view the $\mathrm{Ult}(V, \mathcal{E})$ as a direct limit of the 'ordinary' ultrapowers by the measures E_a. It is part of the flexibility of the approach that this is inessential though.

Having thus generalised the notion of measure ultrapower to extender ultrapower we can use these to give us first order formulations of α-strong etc. A simplified statement is:

Lemma 4.3 *Let α be a strong limit cardinal; then κ is α-strong iff there is an α-extender sequence $\mathcal{E} = \langle E_a : a \in [\alpha]^{<\omega} \rangle$ at κ, with $V_{\kappa+\alpha} \subseteq \mathrm{Ult}(V, \mathcal{E}) \wedge j(\kappa) > \alpha$.*

The notion of κ being strong is then also first order (although involving a quantifier over On). One may note the following easily proven fact: κ strong implies that $V_\kappa \prec_{\Sigma_2} V$.

Definition 4.4 *κ is* superstrong *if there is $j : V \longrightarrow_e M$ with $V_{j(\kappa)} \subseteq M$.*

Note that α-strong only asked for $V_\alpha \subseteq M$ whilst $j(\kappa) > \alpha$. This seemingly innocuous extension is in fact a powerful strengthening. Again it has a first order formalisation. We proceed to a definition of Woodin cardinal. First we define a strengthening of the concept of strong.

Definition 4.5 *Let $A \subseteq V$. We say that κ is A-strong in V if for every α there are $M, B \subseteq V$ with $IM(M)$ and an $\mathcal{L}_{\dot{\in}, \dot{A}}$ - elementary embedding*

$$j : \langle V, A \rangle \longrightarrow_e \langle M, B \rangle \text{ such that } \mathrm{cp}(j) = \kappa, V_\alpha \subseteq M \text{ and } V_\alpha \cap A = V_\alpha \cap B.$$

This is not a first order formalisation, but now consider an inaccessible λ and relativise the notion from V down to V_λ

Theorem 4.6 *An inaccessible cardinal λ is called* Woodin *if for every $A \subseteq V_\lambda$ there is a $\kappa < \lambda$ which is A-strong in V_λ.*

There are many other equivalent formulations (*cf.* Kanamori [**10**] Sect 26.). A Woodin cardinal is necessarily Mahlo, but may fail to be weakly compact. It turned out that this is precisely the right concept to analyse the various determinacy properties. Subsequently it turned out to be also precisely the right concept to gauge a whole host of other set theoretical phenomena, and thus it has become one of the central notions of modern set theory.

Lemma 4.7 *(i) If κ is superstrong then it is a Woodin limit of Woodins.*
(ii) If λ is Woodin then (“there are arbitrarily large strong cardinals”)$^{V_\lambda}$.

A particular constellation of cardinals is also of interest for determinacy of infinite games played with reals, rather than integers. The assertion “$AD_\mathbb{R}$” is that for every $\mathcal{A} \subseteq {}^\omega\mathbb{R}$, the game $G_\mathcal{A}$ is determined, where in the game now, players play a complete real number at each round instead of a single integer. The following conjecture emerged.

Conjecture (The “$AD_\mathbb{R}$ hypothesis”) The consistency strength of $AD_\mathbb{R}$ is that of a cardinal μ that is simultaneously a limit of infinitely many Woodins $\lambda_n < \lambda_{n+1} \cdots < \mu$ and of μ-strong cardinals $\kappa_n < \lambda_n < \kappa_{n+1}$.

The hypothesis is also interesting since (given a mild strengthening of AD), the $AD_\mathbb{R}$-hypothesis is equivalent to the assertion that every set of reals is Suslin. The following is another connection between the worlds of large cardinals and determinacy of games.

Theorem 4.8 (Woodin ($\Leftarrow$), Neeman-Steel ($\Rightarrow$))

$$Con(\text{ZF} + AD_\mathbb{R}) \Leftrightarrow Con(\text{ZFC} + AD_\mathbb{R}\text{-}hypothesis)$$

We continue our cataloguing of some more large cardinals through elementary embeddings.

Definition 4.9 *(i) A cardinal κ is α-supercompact if there is a $j : V \longrightarrow_e M$ with ${}^\alpha M \subseteq M$.*
(ii) κ is supercompact if it is α-supercompact for all α.

An embedding arising from a measurable cardinal κ in general only implies the closure of M under κ sequences, so this definition only starts to have bite once $\alpha > \kappa$. The closure under all α-sequences is again a considerable strengthening over the “strong” hierarchy of principles. It masks also a telling difference in the kind of embeddings: up to this point the embeddings j have always been “continuous at κ^+” meaning that $\sup(j“\kappa^+) = j(\kappa^+)$. If j is a 2^κ-supercompact embedding, this fails with $\sup(j“\kappa^+) < j(\kappa^+)$ occurring. Whilst looking a rather technical difference this in fact introduces a wide variety of new phenomena. Current inner model theory has a target of producing a good fine structural inner model of a supercompact cardinal, but this target is not yet met. It can do so for strong cardinals, Woodin cardinal, Woodin limit of Woodin cardinals, ..., but although we know what the inner models should look like for example, for measurably Woodin cardinals, we are unable to prove they exist. The chief difficulty being the inability to prove that sufficiently many ultrapowers of the models are wellfounded in order for their construction to get off the ground. This has been dubbed the “iterability problem”.

Woodin has recently proposed versions of inner models constructed from a single supercompact cardinal (dubbed ‘Ultimate L’) that would allow the importation to the model of witnesses

for all large cardinals, including even larger ones, that may exist in V, in one stroke. However establishing these models with the relevant properties is still a work-in-progress.

Continuing onwards we come to such larger cardinals, the *extendible* cardinals of Reinhardt and Silver (see [**29**]):

Definition 4.10 *(i) A cardinal κ is α-extendible if there are λ, j with $j : V_{\kappa+\alpha} \longrightarrow_e V_{\lambda+\alpha}$*
$\wedge \operatorname{cp}(j) = \kappa$.
(ii) κ is extendible *if it is α-extendible for all α.*

Even 1-extendibility is a strong concept:

Lemma 4.11 *If κ is 1-extendible, then it is superstrong (and there are many such below it.)*

Lemma 4.12 *If κ is extendible, it is supercompact; κ extendible implies $V_\kappa \prec_{\Sigma_3} V$.*

Ascending further:

Definition 4.13 *A cardinal κ is* huge *if there is $j : V \longrightarrow_e M$ with $\operatorname{cp}(j) = \kappa \wedge {}^{j(\kappa)}M \subseteq M$.*

However if we try to maximise the extendibility properties we run into inconsistency. Let $\mathcal{L}_{\dot{\in},\dot{j}}$ be the usual language with an additional class predicate $\dot{j}$; let ZFC_j be a formulation of the axioms with the additional predicate symbol allowed into all instances of Separation and Collection.

Theorem 4.14 (Kunen [**15**]) $(\mathrm{ZFC}_{\dot{j}})$
There is no non-trivial $\mathcal{L}_{\dot{\in}}$-elementary embedding $j : V \longrightarrow_e V$.

It is unknown whether AC is necessary for this theorem (see below). The proof of Kunen's theorem is actually a direct ZFC result about sets:

Theorem 4.15 (Kunen) (ZFC) *There is no non-trivial elementary embedding $j : V_{\lambda+2} \longrightarrow_e V_{\lambda+2}$.*

The "2" is an essential artefact of the argument. That there may be a non-trivial $j : V_{\lambda+1} \longrightarrow_e V_{\lambda+1}$ is not known to be inconsistent; if this is to be the case, then $\kappa_0 = \operatorname{cp}(j) < \lambda$ and it can be shown that λ has cofinality ω, being $\sup\{\kappa_0, j(\kappa_0), jj(k_0), \ldots\}$. There are now several proofs of Kunen's theorem - see Kanamori, [**10**].

For such $V_{\lambda+1}$ the model $L(V_{\lambda+1})$ and its possible elementary embeddings has become an object of study, and is likely to be significant. Likewise recent work of Woodin has focussed attention on the possibility of an embedding of V to itself being consistent with ZF without choice. Generalisations of such principles are thus beyond the consistency strength of large cardinals considered with AC. It is perhaps then the case, that there is a choiceless realm of strong axioms or large cardinals, lying in a region where AC fails, thus between the AC-world and inconsistency. It is an interesting observation that there are no deep proofs of the inconsistency of any putative large cardinal axiom yet proposed: usually a half a page at most is needed to see off any inconsistent candidate. So perhaps instead of this picture of all inconsistency proofs being of a simple nature, the $(\mathrm{ZF}_{\dot{j}})$-version of Kunen's theorem, if true, may be the first example where hard work has to be done.

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