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Post-quantum steering

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The discovery of post-quantum nonlocality, i.e. the existence of nonlocal correlations stronger than any quantum correlations but nevertheless consistent with the no-signaling principle, has deepened our understanding of the foundations quantum theory. In this work, we investigate whether the phenomenon of Einstein-Podolsky-Rosen steering, a different form of quantum nonlocality, can also be generalized beyond quantum theory. While post-quantum steering does not exist in the bipartite case, we prove its existence in the case of three observers. Importantly, we show that post-quantum steering is a genuinely new phenomenon, fundamentally different from post-quantum nonlocality. Our results provide new insight into the nonlocal correlations of multipartite quantum systems.

Quantum mechanics allows for distant systems to be entangled, that is, correlated in a way that admits no equivalent in classical physics. The strongest demonstration of this phenomena is quantum nonlocality [1, 2]. Performing well-chosen local measurements on separated entangled quantum systems, allows one to observe correlations stronger than in any physical theory satisfying a natural notion of locality, as discovered by Bell. A third form of quantum inseparability is Einstein-Podolsky-Rosen (EPR) steering, which captures the fact that by making a measurement on half of an entangled pair, it is possible to remotely 'steer' the state of the other half. First discussed by Schrodinger [3], this notion was extensively studied in the context of quantum optics [4]. Following a quantum information approach, the concept was put on firm grounds only a few years ago [5], and has attracted growing attention since then. The detection [6, 7] and quantification [8, 9] of steering have been discussed. The concept was also shown to be relevant in quantum information [10, 11], and related to fundamental aspects of quantum theory such as incompatibility of measurements [12, 13].

These phenomena are today viewed as fundamental aspects of quantum theory. Hence a deeper understanding of them provides a fresh perspective on the foundations of quantum theory. In particular, the development of a generalized theory of nonlocality, independent of quantum theory, has brought substantial progress. In a seminal paper, Popescu and Rohrlich discovered the existence of correlations that are stronger than those of quantum theory, but nevertheless satisfying the no-signaling principle, hence avoiding a direct conflict with relativity [14]. This naturally raised the question of whether there exist physical principles (stronger than no-signaling) from which the limits of quantum nonlocality can be recovered. Significant progress has been reported [22], notably the discovery of simple information-theoretic and physical principles partly capturing quantum correlations [15–21], and novel derivations of quantum theory based on alternative (arguably more physical) axioms [23]. In parallel, this research has led to the device-independent approach, a novel paradigm for "black-box" quantum information processing [24, 25].

In the present work, motivated by the insight that the study of post-quantum nonlocality has brought, we ask whether the phenomenon of steering can be generalized beyond quantum theory (like nonlocality can), but nevertheless in accordance with the no-signaling principle. We start by discussing the case of two observers (where one party, Bob, steers the other, Alice). Here a celebrated theorem by Gisin [26] and Hughston, Josza and Wootters [27] implies that post-quantum steering does not exist. We then move to the multipartite case, and show explicitly that post-quantum nonlocality implies the existence of post-quantum steering when three observers are involved. This brings us to the main question and result of the paper, whether post-quantum steering that is fundamentally distinct from post-quantum nonlocality exists. We discuss what precisely would constitute such a phenomenon, and show that indeed post-quantum nonlocality and post-quantum steering are genuinely distinct phenomena.

Our results motivate the study of the latter as a new way to study the structure and limitations of quantum correlations. Indeed, the use of the concept of steering allows us to investigate quantum correlations while keeping the local structure of quantum theory. Notably, our results highlight the fact that the structure of the Hilbert space describing tripartite quantum systems is fundamentally different compared to the bipartite case, in accordance with previous work [28, 29].

Steering in bipartite scenario.–We start by discussing steering in quantum theory, considering two distant observers, Alice and Bob, sharing a quantum state ρ_{AB}. Bob wants to convince Alice (who does not trust him) that ρ_{AB} is entangled. In order to be convinced, Alice asks Bob to perform various measurements on his system, and to announce the result. Alice can then characterize the state in which her system is steered to, for each measurement of Bob. Although Alice does not know which measurement Bob really performed, she can nevertheless convince herself of the presence of entanglement [5].

The conditional (unnormalised) states of Alice's subsystem (prepared by Bob's measurement) are given by

$$\sigma_{b|y} = \mathrm{tr}_{\mathrm{B}}\left[\rho_{\mathrm{AB}}\left(\mathbb{1}_{\mathrm{A}} \otimes E_{b|y}\right)\right], \tag{1}$$

where $E_{b|y}$ denotes the POVM element (effect operator) of Bob corresponding to the outcome b of the measurement setting y. Note that $\mathrm{tr}\left[\sigma_{b|y}\right]$ gives the conditional probability for Bob to obtain outcome b given that he measured y, i.e. $p(b|y)$. The set of unnormalised conditional states $\{\sigma_{b|y}\}_{by}$ is called an *assemblage*. Since any valid POVM satisfies $\sum_b E_{b|y} = \mathbb{1}$, we have that $\sum_b \sigma_{b|y} = \mathrm{tr}_{\mathrm{B}}(\rho_{\mathrm{AB}}) = \rho_{\mathrm{A}}$. This can be seen as a statement of the no-signalling principle, since without the knowledge of Bob's outcome b, Alice's state is independent of the choice of measurement y, being equal simply to the reduced state ρ_{A}.

In this work we would like to extend steering beyond quantum theory, and will thus not assume its entire structure. We consider that Alice's system is quantum. Moreover, we assume that the no-signalling principle holds. Thus we are interested in the class of 'no-signaling assemblages' which satisfy

$$\sigma_{b|y} \geq 0 \qquad \forall b, y \tag{2a}$$

$$\sum_b \sigma_{b|y} = \sum_b \sigma_{b|y'} = \rho_{\mathrm{A}} \qquad \forall y, y' \tag{2b}$$

$$\mathrm{tr}(\rho_{\mathrm{A}}) = 1. \tag{2c}$$

The first constraint says that Alice's systems is described by (unnormalised) quantum states, i.e. positive semidefinite matrices, the second says that the assemblage should satisfy the no-signalling constraint, and the last that the reduced state of Alice should be normalized.

The question we are interested in is whether every no-signaling assemblage admits a quantum realisation. That is, for any $\{\sigma_{b|y}\}_{by}$ satisfying the constraints (2), can we find a set of POVMs $E_{b|y}$ and a quantum state ρ_{AB} such that $\sigma_{b|y} = \mathrm{tr}_{\mathrm{B}}\left[\rho_{\mathrm{AB}}\, \mathbb{1}_{\mathrm{A}} \otimes E_{b|y}\right]$. In other words, can Alice test whether Bob is using post-quantum resources to prepare the assemblage.

It turns out that in the bipartite case, every no-signaling assemblage admits a quantum realization. Hence, there is no post-quantum steering in this case. This follows from the GHJW theorem [26, 27], which gives an explicit quantum realization. Given a no-signaling assemblage, condition (2a) implies that ρ_A is positive semidefinite, and hence can be diagonalised: $\rho_{\mathrm{A}} = \sum_k \mu_k |k\rangle\langle k|$. Now define the quantum state $|\Psi\rangle_{\mathrm{AB}} = \sum_k \sqrt{\mu_k}|k\rangle_{\mathrm{A}}|k\rangle_{\mathrm{B}}$, (in the Schmidt form) and POVM element for Bob $E_{b|y} = \sqrt{\rho_{\mathrm{A}}^{-1}}\, \sigma_{b|y}^{\mathrm{T}} \sqrt{\rho_{\mathrm{A}}^{-1}}$, where $\sqrt{\rho_{\mathrm{A}}^{-1}} = \sum_k 1/\sqrt{\mu_k}|k\rangle_{\mathrm{A}}\langle k|$. It can be checked that $|\Psi\rangle_{\mathrm{AB}}$ is a normalised state, that $\{E_{b|y}\}_b$ is a well defined POVM for each y, and that the assemblage is recovered, *i.e.* $\sigma_{b|y} = \mathrm{tr}_{\mathrm{B}}\left(|\Psi\rangle\langle\Psi|_{\mathrm{AB}} \mathbb{1}_{\mathrm{A}} \otimes E_{b|y}\right)$, which finishes the proof.

Below we will show that the situation is completely different in the multipartite case. Specifically, there exist tripartite assemblages which satisfy the no-signaling principle yet admit no quantum realisation.

Steering in the tripartite scenario.– Quantum steering has been recently discussed in the multipartite case [30, 31]. Following the approach of Ref. [31], we discuss a tripartite steering scenario where only one observer (Alice) is trusted (characterised). Consider a tripartite quantum state ρ_{ABC} shared between Alice, Bob and Charlie, and let Bob and Charlie perform (uncharacterized) POVMs $E_{b|y}$ and $E_{c|z}$ on their subsystems. In this case, the assemblage (i.e. the set of unnormalised states for Alice's system) is given by [32]

$$\sigma_{bc|yz} = \mathrm{tr}_{\mathrm{BC}}\left[\rho_{\mathrm{ABC}}\left(\mathbb{1}_{\mathrm{A}} \otimes E_{b|y} \otimes E_{c|z}\right)\right] \quad \forall b, c, y, z. \tag{3}$$

Similarly to above, we have that $p(bc|yz) = \mathrm{tr}\left(\sigma_{bc|yz}\right)$. Moreover, no-signalling is ensured, since $\sum_b \sigma_{bc|yz} = \sum_b \sigma_{bc|y'z}$ and $\sum_c \sigma_{bc|yz} = \sum_c \sigma_{bc|yz'}$ $\forall y, y', z, z'$. Finally, Alice's reduced state is $\sum_{bc} \sigma_{bc|yz} = \rho_{\mathrm{A}}$.

Again, we would like to extend steering beyond quantum theory, and consider assemblages limited only by the no-signaling principle. Thus, we are interested in the set of no-signaling assemblages $\sigma_{bc|yz}$ that satisfy

$$\sigma_{bc|yz} \geq 0 \qquad \forall b, c, y, z \tag{4a}$$

$$\sum_b \sigma_{bc|yz} = \sum_b \sigma_{bc|y'z} = \sigma^{\mathrm{C}}_{c|z} \qquad \forall y, y', c, z \tag{4b}$$

$$\sum_c \sigma_{bc|yz} = \sum_c \sigma_{bc|yz'} = \sigma^{\mathrm{B}}_{b|y} \qquad \forall b, y, z, z' \tag{4c}$$

$$\mathrm{tr} \sum_{bc} \sigma_{bc|yz} = \mathrm{tr}(\rho_{\mathrm{A}}) = 1 \tag{4d}$$

where the first constraint imposes positivity, the second no-signalling from Bob to Alice-Charlie, the third no-signalling from Charlie to Alice-Bob, and the fourth normalisation.

We will now show that, contrary to the bipartite case, there exist no-signaling assemblages (i.e. satisfying conditions (4)) which do not admit a quantum realisation (i.e. cannot be written in the form (3)). Hence post-quantum steering is possible in the tripartite case. We will first present a simple example which demonstrates that post-quantum steering is trivially implied by the existence of post-quantum nonlocality. We will then move on to the much more interesting question, namely the existence of post-quantum steering that is not implied by post-quantum non-locality, i.e. that is fundamentally different from it.

Consider first an assemblage for which the behaviour of Bob and Charlie, $p(bc|yz)$, is not realizable in quantum theory [33], for instance the PR-box correlations

[14]: $p(bc|yz) = 1/2$ if $b \oplus c = yz$ and 0 otherwise, with uniform marginals, and where $y, z, b, c = 0, 1$. Then take any normalised positive semidefinite operator ρ_{A} and define $\sigma_{bc|yz} = p(bc|yz)\,\rho_{\mathrm{A}}$. Clearly, this assemblage is no-signaling, but cannot be realized in quantum theory. This is thus an example of post-quantum steering. However, in this (extreme) example Alice is completely factorised from Bob and Charlie, and the post-quantumness follows only from the untrusted parties – i.e. it follows already at the level of nonlocality. Thus examples of this type are not insightful, since they don't rely on the fact that one party is trusted (i.e. that we are in a steering scenario).

Post-quantum steering without post-quantum nonlocality.–We are now ready to discuss our main result, namely the existence of post-quantum steering which does not reduce to post-quantum nonlocality. At this point it is useful to discuss what exactly would constitute a non-trivial example of post-quantum steering. In the previous example we saw that the post-quantumness of the assemblage involving a PR-box could be certified directly from the nonlocal behaviour of the untrusted devices, i.e. by tracing out the trusted party. One possibility this suggests is therefore to look for those assemblages $\sigma_{bc|yz}$ such that $p(bc|yz) = \mathrm{tr}[\sigma_{bc|yz}]$ are quantum. This however is not the strongest requirement we could ask for, since it neglects the trusted party altogether. We could still ask the trusted party to measure it's assemblage, using a set of POVMs $E_{a|x}$, to produce the tripartite behaviour $p(abc|xyz) = \mathrm{tr}[E_{a|x}\sigma_{bc|yz}]$. If this behaviour is post-quantum for some well chosen set of $E_{a|x}$ then the post-quantumness of the assemblage can be witnessed at the level of the nonlocal behaviour it produces. Therefore what we are looking for is an assemblage such that no matter what set of measurements Alice performs she will always produce behaviours explainable within quantum mechanics, yet which is nevertheless post-quantum at the level of the assemblage itself (i.e. with full tomography on the trusted party).

Here we will provide a non-trivial example of a post-quantum steering by finding an assemblage which leads to quantum-realizable behaviours for all dichotomic measurements on Alice. Since our example assemblage will consist of qubits for Alice, this includes all projective measurements, a very natural set of measurements. More generally one may like to find an example which produces quantum-realizable behaviours for all POVMs. However, this appears to be a very difficult task – the related problem of finding local models for POVMs on quantum states being one of the major challenges in the field of nonlocality [**?**]

In summary, in what follows we will outline a method to find an assemblage $\sigma_{bc|yz}$ which: (i) is provably post-quantum (ii) for all dichotomic measurements $\Pi_{a|x}$ (with x now a continuous label), the resulting tripartite behaviour $p(abc|xyz) = \mathrm{tr}_{\mathrm{A}}(\Pi_{a|x}\sigma_{bc|yz})$ admits a quantum realization. The example assemblage with these properties will be a collection of (real) qubit states.

Outline of method.— The first ingredient we need is a test for certifying that a given assemblage is post-quantum, i.e. cannot be written in the form (3). To do so we can use so-called *Tsirelson bounds* [35] for steering inequalities. Consider a linear steering functional given by

$$\beta = \mathrm{tr}\Big(\sum_{bcyz} F_{bcyz}\,\sigma_{bc|yz}\Big). \tag{5}$$

for given operators $\{F_{bcyz}\}_{bcyz}$ [8]. The Tsirelson bound $\beta_{\mathcal{Q}}$ for this functional is the minimum possible value that can be obtained by assemblages which arise from measurements on quantum states. Hence if a given assemblage $\sigma_{bc|yz}$ is such that $\mathrm{tr}(\sum_{bcyz} F_{bcyz}\sigma_{bc|yz}) = \beta < \beta_{\mathcal{Q}}$, we can conclude that $\sigma_{bc|yz}$ is post-quantum. The problem with this is that calculating the Tsirelson bound β_Q of a steering functional is in general a hard problem, since there is no efficient characterisation of the set of quantum assemblages [36]. However, it is possible to lower bound the Tsirelson bound, $\beta_{\tilde{\mathcal{Q}}} \leq \beta_{\mathcal{Q}}$, in a computationally feasible way, inspired from methods used in the context of quantum nonlocality [37]. Full details of how this can be done can be found in the Supplementary Material.

The second ingredient needed is a method for constructing assemblages $\sigma_{bc|yz}$ that always produce behaviours which admit quantum realizations, that is such that $p(abc|xyz) = \mathrm{tr}_{\mathrm{A}}(\Pi_{a|x}\sigma_{bc|yz})$ admits a quantum realization for all possible dichotomic measurements $\Pi_{a|x}$ performed by Alice. Here the challenge arises from the fact that x runs over a continuous set. Nevertheless, inspired by [38], the problem can be reduced to finding a quantum realization for only a finite set of fixed POVMs. First of all, we will make our requirement even more stringent: that the behaviours arising from the assemblage admit a *local* model, and not just a quantum realization, since the set of local behaviours is contained inside the set of quantum behaviours and is easier to characterise [2]. Second, we will use two observations: (i) that noisy measurements of the form $\Pi_{a|x}(\mu) = \mu\Pi_{a|x} + (1-\mu)\mathbb{1}/2$ produce the same behaviour on the assemblage $\sigma_{bc|yz}$ as noise-free measurements do on the noisy assemblage $\sigma_{bc|yz}(\mu) = \mu\sigma_{bc|yz} + (1-\mu)\mathrm{tr}[\sigma_{bc|yz}]\mathbb{1}/2$,

$$\mathrm{tr}[\Pi_{a|x}(\mu)\sigma_{bc|yz}] = \mathrm{tr}[\Pi_{a|x}\sigma_{bc|yz}(\mu)]. \tag{6}$$

That is, the simulation of noisy measurements on a noise-free assemblage is equivalent to the simulation of noise-free measurements on a noisy assemblage. (ii) the set of noisy dichotomic measurements such that $\mu < 1$ can be simulated by a finite set of projective measurements $\mathcal{E}$ [39]. Thus, if we find an assemblage $\sigma_{bc|yz}$ that produces a local behaviour for the set of measurements $\mathcal{E}$, then it also produces a local behaviour for all noisy projective measurements $\Pi_{a|x}(\mu')$, with $\mu' \leq \mu$. This in turn implies that the noisy assemblage $\sigma_{bc|yz}(\mu)$ produces local behaviours for *all* dichotomic measurements $\Pi_{a|x}$. All details of this outline can be found in the Supplementary information.

Putting both ingredients together, if the noisy assemblage $\sigma_{bc|yz}(\mu)$ violates the Tsirelson bound of a steering functional, it is a post-quantum assemblage which produces a local (hence quantum) behaviour for all dichotomic measurements. We have used standard iterative optimisation techniques to find such an inequality and assemblage. Again, full details of the approach, including the computational tractability, can be found in the Supplementary Information.

Example. Implementing the above procedure we were able to construct examples of post-quantum steering without post-quantum nonlocality for dichotomic measurements. In Fig. 1 we represent graphically one of these assemblages, denoted $\sigma^*_{bc|yz}$. This assemblage is symmetric under permutation of Bob and Charlie, i.e. $\sigma^*_{bc|yz} = \sigma^*_{cb|zy}$. Moreover, it is post-quantum: it achieves $\beta = -0.520495$ for an inequality with almost-quantum bound $\beta_{\tilde{Q}} = -0.508417$. Note that the operators F_{bcyz} characterizing the inequality (5), and more details about $\sigma^*_{bc|yz}$ can be found in the Supplementary Information.

Discussion.–Motivated by the development and success of post-quantum nonlocality, we have investigated the possibility of extending steering beyond quantum theory. While such an extension is not possible in the bipartite case, we showed explicitly the existence of post-quantum steering in the multipartite case. Notably, this represents a genuinely new effect, since post-quantum steering does not imply post-quantum nonlocality. Hence the use of post-quantum resources can only be witnessed by looking at the assemblage, but is not apparent at the level of the probability distribution.

An interesting aspect of this work is to highlight a fundamental difference between bipartite and multipartite quantum correlations. This goes alongside previous findings [28, 29]. For instance, in the case of nonlocality it was shown that a natural extension of Gleason's theorem is possible in the bipartite case, but fails for multipartite systems [28]. In the context of entanglement theory, every pure bipartite entangled state admits a canonical form (Schmidt decomposition), however the situation turns out to be more complex in the multipartite case [29]. It would be very interesting to understand whether the above observations are intimately related to each other and to the existence of post-quantum steering.

Our work raises several questions. First, while our example of post-quantum steering was shown not to give rise to post-quantum nonlocality for arbitrary dichotomic measurements, it is natural to see if this is also the case when arbitrary POVMs are considered. In fact, a negative answer would also be an interesting outcome, as it would provide an example where POVMs and projective measurements differ. Also, notice that although we restricted the measurements to be dichotomic (which in principle enlarges the set of examples we can obtain), we also imposed the behaviours to be local instead of just quantum (which certainly restricts the set of examples we can obtain).

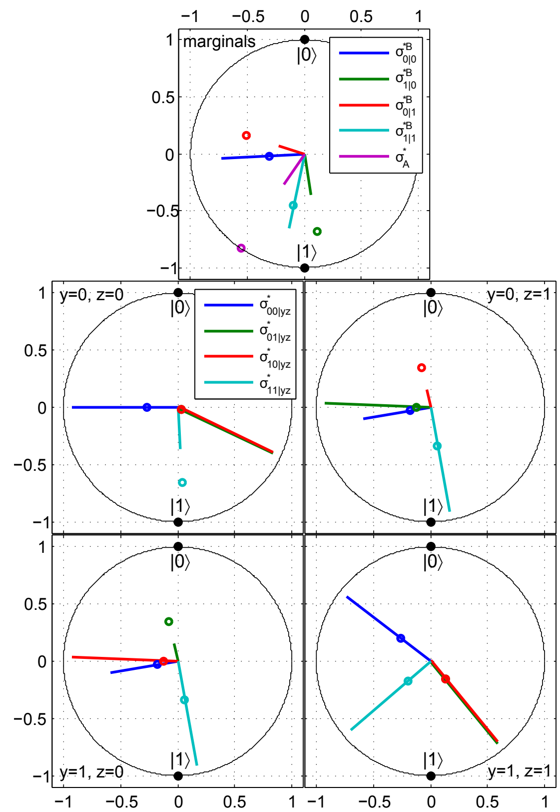

FIG. 1. Bloch sphere representation of the post-quantum assemblage $\sigma^*_{bc|yz}$. For each pair of settings y, z we represent the four conditional real qubit states $\sigma^*_{bc|yz}$ in an equator of the Bloch sphere. The normalized state is given by its Bloch vector, while the normalization is indicated by the corresponding circle; more precisely, the distance to the origin corresponds to $p(bc|yz) = \mathrm{tr}(\sigma^*_{bc|yz})$. The upper figure indicates the marginal states $\sigma^{B\,*}_{b|y}$, as well as the reduced state ρ^*_A.

Finally, it would also be interesting to find further examples of post-quantum steering, and understand how generic the phenomenon is. Moreover, given the strong information-theoretic power of certain post-quantum nonlocal correlations, it would be relevant to investigate what can be achieved using post-quantum steering. In particular, can post-quantum steering enhance protocols involving quantum information, for instance better quantum teleportation or remote state preparation?

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Appendix A: Relaxations of the quantum set: a hierarchy of SDPs

In this appendix we present the details of a relaxation of the set of quantum assemblages, which we call 'almost quantum' and denote by $\widetilde{\mathcal{Q}}$ [36]. The name comes from its close relation to the definition of set of almost quantum correlations, which is also characterised by an SDP [37].

Similarly to the NPA hierarchy of Ref. [37], consider a moment matrix Γ whose rows and columns are labelled by the 'words' from the following set:

$$\mathcal{S} := \{\emptyset\} \cup \{(b|y)\}_{\substack{b=1:k_B-1\\ y=1:m_b}} \cup \{(c|z)\}_{\substack{c=1:k_C-1\\ z=1:m_c}} \cup \{(bc|yz)\}_{\substack{b=1:k_B-1,\, c=1:k_C-1\\ y=1:m_b,\, z=1:m_c}}\,, \tag{7}$$

where m_b denotes the possible measurement choices by Bob, each with k_B number of outcomes (and similarly for Charlie). In the NPA hierarchy, a matrix with such labels is the object of study for the 1+AB level, where some elements of the matrix are related to the values of a conditional probability distribution $p(bc|yz)$ and its marginals. In our case, however, each of the elements of Γ corresponds to a conditional state prepared on Alice's side, in a way that we make explicit below.

The elements of the first row of Γ are set as follows:

$$\Gamma_{\emptyset,\emptyset} := \rho_{\mathrm{A}}, \tag{8}$$
$$\Gamma_{\emptyset,b|y} := \sigma_{b|y}, \tag{9}$$
$$\Gamma_{\emptyset,c|z} := \sigma_{c|z}, \tag{10}$$
$$\Gamma_{\emptyset,bc|yz} := \sigma_{bc|yz}, \tag{11}$$

where the reduced states are as in Eq. (4b) to (4c).

Once such an identification is done, further constraints are imposed between the elements of Γ to enforce some quantum-like properties on the assemblage. In order to make it clearer to the reader, we present first the relation between Γ and quantum assemblages, from which the extra constraints on the moment matrix will hopefully arise naturally.

A quantum assemblage arises by Bob and Charlie performing measurements on their share of a tripartite quantum system ρ_{ABC}. Let $M_{b|y}$ and $M_{c|z}$ be the measurement elements. Note that we can assume them to be projectors, since in principle we do not impose any constraints on the dimensions of Bob and Charlie's subsystems. The assemblage then arises as:

$$\rho_{\mathrm{A}} = \mathrm{tr}\left(\rho_{\mathrm{ABC}}\right), \tag{12}$$
$$\sigma_{b|y} = \mathrm{tr}\left(\mathbb{1}_{\mathrm{A}}\, M_{b|y}\, \mathbb{1}_{\mathrm{C}}\, \rho_{\mathrm{ABC}}\right), \tag{13}$$
$$\sigma_{c|z} = \mathrm{tr}\left(\mathbb{1}_{\mathrm{A}}\, \mathbb{1}_{\mathrm{B}}\, M_{c|z}\, \rho_{\mathrm{ABC}}\right), \tag{14}$$
$$\sigma_{bc|yz} = \mathrm{tr}\left(\mathbb{1}_{\mathrm{A}}\, M_{b|y}\, M_{c|z}\, \rho_{\mathrm{ABC}}\right). \tag{15}$$

Note that we are using the commutativity paradigm, where we do not require that the measurements be of the form $\mathbb{1}_{\mathrm{A}} \otimes M_{b|y} \otimes M_{c|z}$, but rather demand that $\left[M_{b|y}, M_{c|z}\right] = 0$ for all b, c, y, z. The tensor product type of measurements is just a particular case of the general form of the latter.

Now consider the moment matrix again. To each of its entries we associate the following element:

$$\Gamma(v,w) = \mathrm{tr}\left(\mathbb{O}_v^\dagger\, \mathbb{O}_w\, \rho_{\mathrm{ABC}}\right), \tag{16}$$
$$\text{where} \quad \mathbb{O}_\emptyset = \mathbb{1} \tag{17}$$
$$\mathbb{O}_{b|y} = \mathbb{1}_{\mathrm{A}}\, M_{b|y}\, \mathbb{1}_{\mathrm{C}}, \tag{18}$$
$$\mathbb{O}_{c|z} = \mathbb{1}_{\mathrm{A}}\, \mathbb{1}_{\mathrm{B}}\, M_{c|z}, \tag{19}$$
$$\mathbb{O}_{bc|yz} = \mathbb{1}_{\mathrm{A}}\, M_{b|y}\, M_{c|z}. \tag{20}$$

It is clear to see that the elements of the first row $\Gamma(\emptyset, v)$ satisfy eq. (12) to (15) for all v. In addition, the commutation relations between the measurement operators of Bob and Charlie also impose that:

$$\Gamma(v,v) = \Gamma(\emptyset, v), \tag{21}$$
$$\Gamma(v,w) = \Gamma(w,v), \quad \text{whenever } [\mathbb{O}_v, \mathbb{O}_w] = 0, \tag{22}$$

and constraints of the type:

$$\Gamma(b|y, bc|yz) = \Gamma(\emptyset, bc|yz), \tag{23}$$
$$\Gamma(b|y, bc|yz) = \Gamma(b|y, c|z), \tag{24}$$
$$\Gamma(bc|yz, bc'|yz') = \Gamma(bc|yz, c'|z'). \tag{25}$$

Note that these constraints are the ones imposed on the matrix moment of the 1+AB level of the NPA hierarchy. In our case, however, the elements of Γ are matrices instead of numbers, and hence some specific properties also arise. These are of the type:

$$\Gamma(b|y, b'|y') = \Gamma(b'|y', b|y)^\dagger, \tag{26}$$
$$\Gamma(bc|yz, b'c|y'z) = \Gamma(b'c|y'z, bc|yz)^\dagger, \tag{27}$$
$$\Gamma(bc|yz, b'|y') = \Gamma(b'c|y'z, b|y)^\dagger. \tag{28}$$

Finally, note that such a Γ is hermitian and positive semidefinite.

The idea now is, given a general assemblage $\left\{\sigma_{bc|yz}\right\}_{bcyz}$ check whether there exists a PSD moment matrix Γ whose first row relates to the assemblage via eq. (12) to (15), and that satisfies properties (21) to (28). This is a well defined semidefinite program, and when it is feasible the assemblage belongs to $\widetilde{\mathcal{Q}}$. Since every quantum assemblage satisfies the properties, such an SDP is always feasible for quantum inputs, hence every quantum assemblage belongs to $\widetilde{\mathcal{Q}}$. Note that the converse may not always be true.

Throughout the manuscript we have used this set $\widetilde{\mathcal{Q}}$ to find bounds on the quantum violation of a steering inequality. Since $\widetilde{\mathcal{Q}}$ may contain post- quantum assemblages, a lower bound on $\beta_{\mathcal{Q}}$ is obtained by finding the minimum value of the inequality over $\widetilde{\mathcal{Q}}$, which is itself an SDP:

$$\text{minimise} \quad \mathrm{tr}\left(\sum_{bcyz} F_{bcyz}\, \sigma_{bc|yz}\right) \tag{29}$$
$$\text{such that} \quad \left\{\sigma_{bc|yz}\right\}_{bcyz} \in \widetilde{\mathcal{Q}}. \tag{30}$$

For the scope of this work we only need a bound on $\beta_{\mathcal{Q}}$, whose violation ensures that the assemblage is post-quantum. We do not need to study different optimal bounds on $\beta_{\mathcal{Q}}$ or other relaxations of the quantum set of assemblages. For the reader interested in SDPs, however, that may also be a valid question and we comment on it in what follows.

A natural step towards studying different relaxations of the quantum set goes in spirit with the NPA hierarchy, similar to the idea by Pusey for bipartite steering scenarios. One could consider then a hierarchy of moment matrices Γ_n, where n relates to length of the words in the set (7), which is now allowed to contain elements of the form $(b_1 \dots b_j c_1 \dots c_k | y_1 \dots y_j z_1 \dots z_k)$. In the case of quantum assemblages, such indices would relate to the following:

$$\Gamma(b_1 \dots b_{j_1} c_1 \dots c_{k_1} | y_1 \dots y_{j_1} z_1 \dots z_{k_1}, b'_1 \dots b'_{j_2} c'_1 \dots c'_{k_2} | y'_1 \dots y'_{j_2} z'_1 \dots z'_{k_2}) = \tag{31}$$
$$\mathrm{tr}\left(\mathbb{1}_{\mathrm{A}}\, M^\dagger_{b_{j_1}|y_{j_1}} \dots M^\dagger_{b_1|y_1}\, M^\dagger_{c_{k_1}|z_{k_1}} \dots M^\dagger_{c_1|z_1}\, M_{b'_1|y'_1} \dots M_{b'_{j_2}|y'_{j_2}}\, M_{c'_1|z'_1} \dots M_{c'_{k_2}|z'_{k_2}}\, \rho_{\mathrm{ABC}} \right).$$

From the commutations relations between Bob and Charlie's measurements arise different constrains that Γ_n is asked to satisfy. Note that the longer the words in $\mathcal{S}_n$ are, the more the properties that the moment matrix should satisfy. For each n, testing whether those properties are satisfied when some elements of the first row are set to be the conditional states on Alice's side (eq. (8) to (11)) is an SDP, and feasibility of level n implies feasibility of level $m < n$. This last statement follows from the fact that every word in $\mathcal{S}_m$ is a word on $\mathcal{S}_n$, hence the constraints imposed in level $m < n$ are just a subset of those imposed in level n. Note also that when the input is a quantum assemblage, the SDP is feasible for any level n by definition.

Denote by $\mathcal{Q}_n$ the set of assemblages which satisfy the conditions of the level n SDP. Then, the following SDPs define a sequence of lower bounds to the quantum bound of a steering inequality:

$$\text{minimise} \quad \beta_{\mathcal{Q}_n} = \mathrm{tr}\left(\sum_{bcyz} F_{bcyz}\, \sigma_{bc|yz} \right) \tag{32}$$
$$\text{such that} \quad \left\{\sigma_{bc|yz}\right\}_{bcyz} \in \mathcal{Q}_n. \tag{33}$$

By definition, these lower bounds satisfy $\beta_{\mathcal{Q}_m} \leq \beta_{\mathcal{Q}_n}$ whenever $m < n$.

Appendix B: Details about example of post-quantum steering

We give here all details concerning the example of post-quantum steering without post-quantum nonlocality. Specifically, the assemblage $\sigma^*_{bc|yz}$ is given explicitly in Table I; in the main text, we represented graphically the assemblage in Fig. 1. Note that we present $\sigma^*_{bc|yz}$ in a minimal representation, using the no-signalling and normalization conditions (4), and symmetry under permutation of Bob and Charlie, i.e. $\sigma^*_{bc|yz} = \sigma^*_{cb|zy}$.

Moreover, in Table II, we give the operators F_{bcyz} for constructing the steering inequality of Eq. (5). These operators are also given in minimal representation, where $F_A = \sum_{yz} F_{11yz}$, $F^{\mathrm{B}}_y = \sum_{bz} (-1)^b F_{b1yz}$, $F^{\mathrm{C}}_z = \sum_{cy} (-1)^c F_{1cyz}$, and $F_{yz} = \sum_{bc} (-1)^{b+c} F_{bcyz}$. The quantity in Eq. (5) is then calculated as follows:

$$\beta = \mathrm{tr}\left(F_A \rho_A + \sum_y F^{\mathrm{B}}_y \sigma^{\mathrm{B}}_{0|y} + \sum_z F^{\mathrm{C}}_z \sigma^{\mathrm{C}}_{0|z} + \sum_{yz} F_{yz} \sigma_{00|yz} \right). \tag{34}$$

$$\rho_{\mathrm{A}}^{*} = \begin{pmatrix} 0.3666 & -0.0896 \\ -0.0896 & 0.6334 \end{pmatrix} \quad \sigma_{00|00}^{*} = \begin{pmatrix} 0.1360 & -0.1257 \\ -0.1257 & 0.1360 \end{pmatrix}$$

$$\sigma_{0|0}^{\mathrm{B}\,*} = \begin{pmatrix} 0.1464 & -0.1114 \\ -0.1114 & 0.1600 \end{pmatrix} \quad \sigma_{00|10}^{*} = \begin{pmatrix} 0.0803 & -0.0523 \\ -0.0523 & 0.0982 \end{pmatrix}$$

$$\sigma_{0|1}^{\mathrm{B}\,*} = \begin{pmatrix} 0.2851 & -0.0586 \\ -0.0586 & 0.2473 \end{pmatrix} \quad \sigma_{00|11}^{*} = \begin{pmatrix} 0.2555 & -0.1192 \\ -0.1192 & 0.0709 \end{pmatrix}$$

TABLE I. Example of post-quantum assemblage that cannot lead to post-quantum nonlocality (for arbitrary projective measurements).

$$F_{A} = \begin{pmatrix} 1.4622 & 0.1773 \\ 0.1773 & -0.4622 \end{pmatrix} \quad F_{00} = \begin{pmatrix} -0.1948 & 0.5653 \\ 0.5653 & -0.7229 \end{pmatrix}$$

$$F_{0}^{\mathrm{B}} = \begin{pmatrix} -0.2894 & 0.2468 \\ 0.2468 & 0.9767 \end{pmatrix} \quad F_{10} = \begin{pmatrix} 0.5482 & -0.4270 \\ -0.4270 & -0.8690 \end{pmatrix}$$

$$F_{1}^{\mathrm{B}} = \begin{pmatrix} -1.0943 & -0.4673 \\ -0.4673 & 0.0648 \end{pmatrix} \quad F_{11} = \begin{pmatrix} 0.2875 & 1.0320 \\ 1.0320 & 0.9182 \end{pmatrix}$$

TABLE II. Operators defining an inequality of the form (5) which witnesses the fact that the assemblage given in Table I is post-quantum.