Irreducible A_1 Subgroups of Exceptional Algebraic Groups

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Abstract

A closed subgroup of a semisimple algebraic group is called irreducible if it lies in no proper parabolic subgroup. In this paper we classify all irreducible A_1 subgroups of exceptional algebraic groups G. Consequences are given concerning the representations of such subgroups on various G-modules: for example, the conjugacy classes of irreducible A_1 subgroups are determined by their composition factors on the adjoint module of G.

1 Introduction

Let G be a reductive connected algebraic group over an algebraically closed field K of characteristic p. A subgroup X of G is called *G-irreducible* (or just *irreducible* if G is clear from the context) if it is closed and not contained in any proper parabolic subgroup of G. This definition, as given by Serre in [26], generalises the standard notion of an irreducible subgroup of $\mathrm{GL}(V)$. Indeed, if $G = \mathrm{GL}(V)$, a subgroup X is G-irreducible if and only if X acts irreducibly on V. Similarly, the notion of complete reducibility can be generalised (see [26]): a subgroup X of G is said to be *G-completely reducible* (or *G-cr* for short) if, whenever it is contained in a parabolic subgroup P of G, it is contained in a Levi subgroup of P.

Now let G be a simple algebraic group of exceptional type. In [30], the author classified the simple, G-irreducible connected subgroups of rank at least 2. In this paper we classify the G-irreducible A_1 subgroups, completing the classification of simple, G-irreducible connected subgroups. We note that the G-irreducible A_1 subgroups can be deduced from [12, Theorem 1] when $p > N(A_1, G)$ (see the preceding table to Theorem 3.9 for the definition), in particular when $p > 7$. In low characteristics there are fewer classes of irreducible A_1 subgroups but the existence of non-G-cr subgroups complicates the proof. We also note that if $G \neq E_8$ then partial results can be found in [1]; we require a set of conjugacy class representatives without repeat for the G-irreducible A_1 subgroups for the E_8 case and therefore classify the irreducible A_1 subgroups independently. The following theorem summarises the individual cases for each exceptional algebraic group G; it classifies the G-irreducible A_1 subgroups of G.

Theorem 1. *Suppose X is a G-irreducible subgroup A_1 of a simple exceptional algebraic group G. Then X is conjugate to exactly one subgroup of Tables 4 to 8, found in Sections 5 to 9, respectively and each subgroup in the tables is G-irreducible.*

The validity of Theorem 1 will be established by proving Theorems 2 to 6 found below, which classify the G-irreducible A_1 subgroups of G when G is of type G_2 through E_8. Each subgroup in Tables 4 to 8 is described by its embedding in some reductive, maximal connected subgroup, given in Theorem 3.1. When we say a reductive, maximal connected subgroup we mean a subgroup that is maximal among all closed connected subgroups and is reductive. We note that the case $p = 0$ can be recovered by simply removing any subgroup in the tables for which a non-zero field twist is necessary and assuming $p = \infty$ when reading inequalities; this yields only finitely many classes of irreducible A_1 subgroups when $p = 0$.

A natural question to ask is whether G-irreducible subgroups of a certain type exist, especially in small characteristics. As an immediate consequence of Theorems 2 to 6, we reprove the following corollary, first proved by Liebeck and Testerman in [21] with a correction by Amende in [1], showing that G-irreducible A_1 subgroups almost always exist.

Corollary 1. *Let G be an exceptional algebraic group. Then G contains an irreducible subgroup A_1, unless $G = E_6$ and $p = 2$.*

Given the existence of irreducible A_1 subgroups, we can use the proofs of Theorems 2 to 6 to study their overgroups. The next result shows the existence of a reductive, maximal connected subgroup that contains representatives of each conjugacy class of G-irreducible A_1 subgroups in small characteristics, with one exception.

Corollary 2. *Let G be an exceptional algebraic group and $p = 2$ or 3. Then there exists a reductive, maximal connected subgroup M containing representatives of every G-conjugacy class of G-irreducible A_1 subgroups, unless $G = F_4$ and $p = 3$ (in which case two reductive, maximal connected subgroups are required). The following table lists such subgroups M.*

Table 1: Maximal connected overgroups for G-irreducible A_1 subgroups.

G	$p=3$	$p=2$
G_2	$A_1\tilde{A}_1$	$A_1\tilde{A}_1$
F_4	B_4 and $\bar{A}_1C_3$	B_4
E_6	C_4	—
E_7	$\bar{A}_1D_6$	$\bar{A}_1D_6$
E_8	D_8	D_8

The choice of M is not unique; for example, if $G = F_4$ and $p = 2$ then C_4 also contains a representative of every G-conjugacy class of G-irreducible subgroup A_1. We also note that in larger characteristics more reductive, maximal subgroups are required. In particular, when $p \geq 19$ we need seven such subgroups for $G = E_7$.

The next corollary shows that the G-conjugacy class of a G-irreducible subgroup A_1 is determined by its composition factors on the adjoint module for G. This is similar to [15, Theorems 4, 6] and extends part of Theorem 3.9 to low characteristics for irreducible A_1 subgroups.

We must first explain a definition we will use throughout the paper. Let X and Y be semisimple subgroups of a semisimple algebraic group G and let V be a G-module. Then we say that X and Y *have the same composition factors on* V if there exists an isomorphism from X to Y sending the set of composition factors of $V \downarrow X$ to the set of composition factors $V \downarrow Y$ (counted with multiplicity).

Corollary 3. *Let G be an exceptional algebraic group and X and Y be irreducible A_1 subgroups. If X and Y have the same composition factors on $L(G)$ then X is conjugate to Y.*

We also deduce that the G-conjugacy class of a simple connected subgroup of G is determined by its composition factors on a smallest dimensional non-trivial module for G, which we will abbreviate to "minimal module" throughout. The dimensions of such a module are 7 (6 if $p = 2$), 26 (25 if $p = 3$), 27, 56 and 248 for $G = G_2$, F_4, E_6, E_7 and E_8, respectively.

Corollary 4. *Let G be an exceptional algebraic group and X and Y be irreducible A_1 subgroups. If X and Y have the same composition factors on a minimal module for G then X is conjugate to Y.*

The next corollary lists some of the interesting A_1 subgroups that are M-irreducible but not G-irreducible for some reductive, maximal connected subgroup M. Here "interesting" means that the M-irreducible subgroup is not obviously G-reducible, i.e. M'-reducible for some other reductive, maximal connected subgroup M' or contained in a proper Levi subgroup.

To describe one of the subgroups we first define a piece of notation from [30]. Suppose $G = E_8$ and $p = 2$. There are two D_8-conjugacy classes of B_4 subgroups in D_8 acting irreducibly on the natural module for D_8. Since $p = 2$, one is E_8-irreducible (by [30, Lemma 7.5]) and denoted by $B_4(\dagger)$ and the other is E_8-reducible (by [30, Lemma 7.4]) and denoted by $B_4(\ddagger)$. Furthermore, there are two D_8-conjugacy classes of A_1 subgroups acting irreducibly on the natural module for D_8 and one of these is contained in $B_4(\dagger)$ and the other in $B_4(\ddagger)$; thus we can differentiate them by giving the B_4 overgroup they are contained in.

Corollary 5. *Let G be an exceptional algebraic group and X be a subgroup A_1 of G. Suppose that whenever X is contained in a reductive, maximal connected subgroup M it is M-irreducible and assume that such an overgroup M exists. Assume further that X is not contained in a proper Levi subgroup of G. Then either:*

(1) *X is G-irreducible, or*

(2) *X is conjugate to a subgroup in Table 2 below. Such X are non-G-cr and satisfy the hypothesis.*

Table 2: Non-G-cr subgroups that are irreducible in every (and at least one) maximal, reductive overgroup

G	Max. sub. M	p	M-irreducible subgroup X
G_2	$A_1\tilde{A}_1$	$p = 2$	$A_1 \hookrightarrow A_1\tilde{A}_1$ via $(1, 1)$
E_7	A_7	$p = 2$	$V_{A_7}(\lambda_1) \downarrow A_1 = 1 \otimes 1^{[r]} \otimes 1^{[s]}$ $(0 < r < s)$
	A_1G_2	$p = 7$	$A_1 \hookrightarrow A_1A_1 < A_1G_2$ via $(1, 1)$ where $A_1 < G_2$ is maximal
E_8	D_8	$p = 2$	$A_1 < B_4(\ddagger)$ where $V_{D_8}(\lambda_1) \downarrow A_1 = 1 \otimes 1^{[r]} \otimes 1^{[s]} \otimes 1^{[t]}$ $(0 < r < s < t)$ and $V_{B_4}(\lambda_1) \downarrow A_1 = 2 \oplus 2^{[r]} \oplus 2^{[s]} \oplus 2^{[t]}$

The notation in the fourth column of Table 2 is explained in Section 2.

2 Notation

Let G be a simple algebraic group over an algebraically closed field K of characteristic p. Let Φ be the root system of G and Φ^+ be the set of positive roots in Φ. Write $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ for the simple roots of G and $\lambda_1, \ldots, \lambda_l$ for the fundamental dominant weights of G, both with respect to the ordering of the Dynkin diagram as given in [6, p. 250]. We sometimes use $a_1a_2 \ldots a_l$ to denote a dominant weight $a_1\lambda_1 + a_2\lambda_2 + \cdots + a_l\lambda_l$. We denote by $V_G(\lambda)$ (or just λ) the irreducible G-module of dominant high weight λ. Similarly, the Weyl module of high weight λ is denoted $W(\lambda) = W_G(\lambda)$ and the tilting module of highest weight λ is denoted by $T(\lambda)$. Another module we refer to frequently is the adjoint module for G, which we denote $L(G)$. We let $V_7 := W_{G_2}(10)$, $V_{26} := W_{F_4}(0001)$, $V_{27} := V_{E_6}(\lambda_1)$ and $V_{56} := V_{E_7}(\lambda_7)$. We note that

V_7 (V_{26}) is irreducible unless $p = 2$ ($p = 3$). For G-modules V, W we write $V + W$ for the module $V \oplus W$ and let V^* denote the dual module of V. If $Y = Y_1 Y_2 \dots Y_k$, a commuting product of simple algebraic groups, then $(V_1, \dots, V_k)$ denotes the Y-module $V_1 \otimes \dots \otimes V_k$ where each V_i is an irreducible Y_i-module. The notation $\bar{X}$ denotes a subgroup of Y that is generated by long root subgroups of Y. If Y has short root elements then $\tilde{X}$ means $\tilde{X}$ is generated by short root subgroups.

Now suppose $p > 0$. Let $F : G \to G$ be the standard Frobenius endomorphism (acting on root groups $U_\alpha = \{u_\alpha(c) | c \in K\}$ by $u_\alpha(c) \mapsto u_\alpha(c^p)$) and V be a G-module afforded by a representation $\rho : G \to \mathrm{GL}(V)$. Then $V^{[r]}$ denotes the module afforded by the representation $\rho^{[r]} := \rho \circ F^r$. Let $M_1, \dots, M_k$ be G-modules and $n_1, \dots, n_k$ be positive integers. Then $M_1^{n_1} / \dots / M_k^{n_k}$ denotes a G-module having the same composition factors as $M_1^{n_1} \oplus \dots \oplus M_k^{n_k}$. Furthermore, $V = M_1 | \dots | M_k$ denotes a G-module with a socle series as follows: $M_k \cong \mathrm{Soc}(V) = \mathrm{Soc}^1(V)$ and for $i > 0$, we have M_{k-i} is $\mathrm{Soc}^{i+1}(V) = \mathrm{Soc}(V/N_i)$ where N_i is the inverse image in V of $\mathrm{Soc}^i(V)$ under the quotient mapping $V \to V/N_{i-1}$ (so $N_0 = 0$ and $N_1 = M_k$). Sometimes, to make things clearer, we will use a tower of modules

$$\frac{\dfrac{M_1}{M_2}}{M_3}$$

to mean the same as $M_1 | M_2 | M_3$.

We need a notation for diagonal subgroups of $Y = H_1 H_2 \dots H_k$, a commuting product of subgroups of type A_1. Let H be a simply connected subgroup A_1 and $\hat{Y} = H \times H \dots \times H$, the direct product of k copies of H. Then we may regard Y as $\hat{Y}/Z$ where Z is a subgroup of the centre of $\hat{Y}$ and H_i is the image of the ith projection map. A diagonal subgroup of $\hat{Y}$ is a subgroup $\hat{X} \cong H$ of the following form: $\hat{X} = \{(\phi_1(h), \dots, \phi_k(h)) | h \in H\}$ where each ϕ_i is a non-trivial endomorphism of H. A diagonal subgroup X of Y is the image of a diagonal subgroup of $\hat{Y}$ under the natural map $\hat{Y} \to Y$. To describe such a subgroup it therefore suffices to give a non-trivial endomorphism, ϕ_i, of H for each i. By [27, Chapter 12], $\phi_i = \alpha \theta_i F^{r_i}$ where α is an inner automorphism, θ_i is a graph morphism and F^{r_i} is a power of the standard Frobenius endomorphism. We only wish to distinguish these diagonal subgroups up to conjugacy and therefore assume α is trivial. Moreover, there are no non-trivial graph automorphisms of A_1. It therefore suffices to give a non-negative integer r_i for each i. Such a diagonal subgroup X is denoted "$X \hookrightarrow H_1 H_2 \dots H_k$ via $(1^{[r_1]}, 1^{[r_2]}, \dots, 1^{[r_k]})$". We often abbreviate this to "X via $(1^{[r_1]}, \dots, 1^{[r_k]})$" if the group Y is clear. We note that to avoid any redundancies we always take the minimum of the $r_1, \dots, r_k$ to be zero.

Now let G be a simple exceptional algebraic group. In Tables 4 to 8 we give an ID number to each of the conjugacy classes of G-irreducible A_1 subgroups in Theorems 2 to 6. The notation $G(\#a)$ (or simply a if G is clear from the context) means the G-irreducible subgroup corresponding to the ID number a. Sometimes $G(\#a)$ will refer to infinitely many conjugacy classes of G-irreducible subgroups. This only occurs for diagonal subgroups and the conjugacy class will depend on field twists $r_1, \dots, r_k$. Sometimes we need to refer to a subset of the conjugacy classes that $G(\#a)$ represents, described by an ordered set of field twists $s_1, \dots, s_k$ and this will be denoted by $G(\#a^{\{s_1, \dots, s_k\}})$. Let us give a concrete example to make this clearer. Consider $G_2(\#1)$, the conjugacy classes of diagonal subgroups $A_1 \hookrightarrow A_1 \tilde{A}_1$ via $(1^{[r]}, 1^{[s]})$ ($rs = 0$; $r \neq s$) (see Table 4). Then the notation $G_2(\#1^{\{r,0\}})$ refers to the conjugacy classes with $s = 0$ and the notation $G_2(\#1^{\{1,0\}})$ refers to the single conjugacy class $A_1 \hookrightarrow A_1 \tilde{A}_1$ via $(1^{[1]}, 1)$.

In Tables 4 to 8 we also need a notation to be able to describe M-irreducible A_1 subgroups X of reductive, maximal connected subgroups M of G. Suppose $M = M_1 M_2 \dots M_r$, with each M_i simple. If all of the factors are simple classical algebraic groups then we define

$$V_M := V_{M_1}(\lambda_1) \otimes V_{M_2}(\lambda_1) \otimes \cdots \otimes V_{M_r}(\lambda_1)$$

and let $V_M \downarrow X$ be the usual restriction of the M-module V_M to X. If $M = F_4$ then we define $V_M \downarrow X$ to be $F_4(\#a)$, where a is the ID number of the subgroup X of F_4. The final case we need to describe the notation for is $M = M_1M_2$ where M is one of A_1G_2, A_2G_2, G_2C_3, A_1F_4, G_2F_4 or A_1E_7. The projection of X to both M_1 and M_2 is an M_i-irreducible subgroup A_1, say X_i (by Lemma 3.3) and therefore X is a diagonal subgroup of X_1X_2 via $(1^{[r_1]}, 1^{[r_2]})$. We need to give X_1, X_2 and the field twists r_1, r_2. If M_i is classical then write $V_{M_i}(\lambda_1) \downarrow X_i$ for X_i and otherwise write $M_i(\#a)$ for X_i, where a is the ID number of X_i in M_i. Then we define $V_M \downarrow X = (X_1^{[r]}, X_2^{[s]})$. We make a slight modification if X_1 (or similarly X_2 but not both) is a diagonal subgroup of some subgroup Y of M_1, which is of exceptional type, so $X_1 = M_1(\#a) < Y$ via $(1^{[s_1]}, \ldots 1^{[s_k]})$. In this case, X is a diagonal subgroup of YX_2 and we define $V_M \downarrow X = (M_1(\#a^{\{s_1,\ldots,s_k\}}), X_2^{[s_{k+1}]})$.

Again, let us give a concrete example to make this clearer. Suppose $G = E_7$ and $M = A_1F_4$. Then F_4 has a maximal subgroup A_1 when $p \geq 13$, which is of course F_4-irreducible and denoted by $F_4(\#10)$. Letting the factor A_1 of M be X_1 and the maximal subgroup of F_4 be X_2 we have an M-irreducible subgroup $X \hookrightarrow X_1X_2$ via $(1^{[r]}, 1^{[s]})$ $(rs = 0)$. Then $V_M \downarrow X = (1^{[r]}, F_4(\#10)^{[s]})$. The notation changes slightly when we consider another F_4-irreducible subgroup. Let X_1 be as before but this time let X_2 be the subgroup $F_4(\#8)$, i.e. $A_1 \hookrightarrow A_1A_1 < A_1C_3$ via $(1^{[u]}, 1^{[v]})$ $(p \geq 7;\ uv = 0)$ where the second A_1 factor is maximal in C_3. Then $X \hookrightarrow X_1X_2$ via $(1^{[r]}, 1^{[w]})$ $(rw = 0)$ is M-irreducible and represents $X \hookrightarrow X_1A_1A_1$ via $(1^{[r]}, 1^{[s]}, 1^{[t]})$ $(rst = 0)$ where $s = u + w$ and $t = v + w$. We then write $V_M \downarrow X = (1^{[r]}, F_4(\#8^{\{s,t\}}))$ $(rst = 0)$.

Let $J = \{\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}\} \subseteq \Pi$ and define $\Phi_J = \Phi \cap \mathbb{Z}J$. Then the standard parabolic subgroup corresponding to J is the subgroup $P = \langle T, U_\alpha : \alpha \in \Phi_J \cup \Phi^+ \rangle$. The Levi decomposition of P is $P = QL$ where $Q = R_u(P) = \langle U_\alpha \,|\, \alpha \in \Phi^+ \setminus \Phi_J \rangle$, and $L = \langle T, U_\alpha \,|\, \alpha \in \Phi_J \rangle$. For $i \geq 1$ we define

$$Q(i) = \left\langle U_\alpha \;\middle|\; \alpha = \sum_{j \in \Pi} c_j\alpha_j \text{ where } \sum_{j \in \Pi \setminus J} c_j \geq i \right\rangle,$$

which is a subgroup of Q. The *ith level* of Q is $Q(i)/Q(i+1)$, and this is central in $Q/Q(i+1)$.

3 Preliminaries

Let G be a simple algebraic group over an algebraically closed field of characteristic p. The first result needed for the proofs of Theorems 2 to 6 is the classification of reductive, maximal connected subgroups of exceptional algebraic groups.

Theorem 3.1 ([19, Corollary 2])**.** *The following tables give the conjugacy classes of reductive, maximal connected subgroups M for G a simple exceptional algebraic group. We also give the composition factors of the restrictions to M of V_7, V_{26}, V_{27}, V_{56} and $L(G)$.*

$G = G_2$

M	Comp. factors of $V_7 \downarrow M$	Comp. factors of $L(G_2) \downarrow M$
A_2	10/01/00	$W(11)/10/01$
$\tilde{A}_2$ $(p = 3)$	11	11/30/03/00
$A_1\tilde{A}_1$	$(1, 1)/(0, W(2))$	$(W(2), 0)/(0, W(2))/(1, W(3))$
A_1 $(p \geq 7)$	6	$W(10)/2$

$G = F_4$

M	Comp. factors of $V_{26} \downarrow M$	Comp. factors of $L(F_4) \downarrow M$
B_4	$W(1000)/0001/0000$	$W(0100)/0001$
C_4 $(p = 2)$	0100	$2000/0100/0001/0000^2$
$\bar{A}_1C_3$ $(p \neq 2)$	$(1, 100)/(0, W(010))$	$(2, 000)/(0, 200)/(1, 001)$
A_1G_2 $(p \neq 2)$	$(2, 10)/(W(4), 00)$	$(2, 00)/(0, W(01))/(W(4), 10)$
$A_2\tilde{A}_2$	$(10, 10)/(01, 01)/(00, W(11))$	$(W(11), 00)/(00, W(11))/(10, W(02))/(01, W(20))$
G_2 $(p = 7)$	20	01/11
A_1 $(p \geq 13)$	$W(16)/8$	$W(22)/W(14)/10/2$

$G = E_6$

M	Comp. factors of $V_{27} \downarrow M$	Comp. factors of $L(E_6) \downarrow M$
$\bar{A}_1A_5$	$(1, \lambda_1)/(0, \lambda_4)$	$(W(2), 0)/(0, W(\lambda_1 + \lambda_5))/(1, \lambda_3)$
F_4	$W(0001)/0000$	$W(1000)/W(0001)$
C_4 $(p \neq 2)$	0100	$2000/W(0001)$
A_2^3	$(10, 01, 00)/(00, 10, 01)/$ $(01, 00, 10)$	$(W(11), 00, 00)/(00, W(11), 00)/(00, 00, W(11))/$ $(10, 10, 10)/(01, 01, 01)$
A_2G_2	$(10, W(10))/(W(02), 00)$	$(W(11), W(10))/(W(11), 00)/(00, W(01))$
G_2 (2 classes; $p \neq 7$)	$W(20)$	$W(01)/W(11)$
A_2 (2 classes; $p \geq 5$)	$W(22)$	11/41/14

$G = E_7$

M	Comp. factors of $V_{56} \downarrow M$	Comp. factors of $L(E_7) \downarrow M$
$\bar{A}_1D_6$	$(1, \lambda_1)/(0, \lambda_5)$	$(W(2), 0)/(0, W(\lambda_2))/(1, \lambda_6)$
A_7	λ_2/λ_6	$W(\lambda_1 + \lambda_7)/\lambda_4$
A_2A_5	$(10, \lambda_1)/(01, \lambda_5)/(00, \lambda_3)$	$(W(11), 0)/(00, W(\lambda_1 + \lambda_5))/(10, \lambda_4)/(01, \lambda_2)$
G_2C_3	$(W(10), 100)/(00, W(001))$	$(W(10), W(010))/(W(01), 000)/(00, W(200))$
A_1G_2 $(p \neq 2)$	$(1, W(01))/(W(3), 10)$	$(W(2), 00)/(0, W(01))/(W(4), 10)/(2, W(20))$
A_1F_4	$(1, W(0001))/(W(3), 0000)$	$(W(2), 0000)/(0, W(1000))/(W(2), W(0001))$
A_2 $(p \geq 5)$	$W(60)/W(06)$	$W(44)/11$
A_1A_1 $(p \geq 5)$	$(W(6), 3)/(4, 1)/(2, W(5))$	$(2, 0)/(0, 2)/(2, W(8))/(W(6), 4)/(4, W(6))/(4, 2)/$ $(2, 4)$

A_1 $(p \geq 17)$	$W(21)/15/11/5$	$W(26)/W(22)/W(18)/16/14/10^2/6/2$
A_1 $(p \geq 19)$	$W(27)/17/9$	$W(34)/W(26)/W(22)/18/14/10/2$

$G = E_8$

M	Comp. factors of $L(E_8) \downarrow M$
D_8	$W(\lambda_2)/\lambda_7$
A_8	$W(\lambda_1 + \lambda_8)/\lambda_3/\lambda_5$
$\bar{A}_1 E_7$	$(W(2), 0)/(0, \lambda_1)/(1, \lambda_7)$
$A_2 E_6$	$(W(11), 0)/(00, W(\lambda_2))/(10, \lambda_6)/(01, \lambda_1)$
A_4^2	$(W(1001), 0000)/(0000, W(1001))/(1000, 0100)/(0001, 0010)/(0100, 0001)/(0010, 1000)$
$G_2 F_4$	$(W(10), W(0001))/(W(01), 0000)/(00, W(1000))$
B_2 $(p \geq 5)$	$02/W(06)/W(32)$
$A_1 A_2$ $(p \geq 5)$	$(W(6), 11)/(W(2), W(22))/(4, 30)/(4, 03)/(2, 00)/(0, 11)$
A_1 $(p \geq 23)$	$W(38)/W(34)/W(28)/W(26)/22^2/18/16/14/10/6/2$
A_1 $(p \geq 29)$	$W(46)/W(38)/W(34)/28/26/22/18/14/10/2$
A_1 $(p \geq 31)$	$W(58)/W(46)/W(38)/W(34)/26/22/14/2$

Note that in the cases in Theorem 3.1 where M is of maximal rank, the composition factors are not given in [19] but can be found in [20, Lemmas 11.2(iii), 11.8, 11.10, 11.11, 11.12(ii)]; moreover, for $(G, M) = (E_6, A_2^3)$, $(E_7, A_2 A_5)$, (E_8, D_8) and (E_8, A_4^2) we have made a choice of simple system within each factor.

We next recall the algorithm of Borel and de Siebenthal, described in [23, 14.2]. The algorithm describes a way of using the extended Dynkin diagram to find subsystem subgroups of G. When G is exceptional it finds all such subsystem subgroups unless $(G, p) = (G_2, 3)$ or $(F_4, 2)$, in which case certain subgroups containing short root subgroups need to be added. The subgroups of maximal rank in Theorem 3.1 come from this algorithm, although a maximal rank subsystem subgroup need not be maximal amongst connected subgroups. For example, we have the following inclusion of maximal rank subsystem subgroups $A_1^8 < A_1^4 D_4 < D_4^2 < D_8 < E_8$. Throughout the proofs of Theorems 2 to 6 we will implicitly make use of this algorithm to describe maximal rank subsystem subgroups.

For the following lemmas let G be a semisimple connected algebraic group. We describe some elementary results about G-irreducible subgroups.

Lemma 3.2 ([21, Lemma 2.1]). *If X is a G-irreducible connected subgroup of G, then X is semisimple and $C_G(X)$ is a finite subgroup.*

Lemma 3.3 ([30, Lemma 3.6]). *Suppose a G-irreducible connected subgroup X is contained in $K_1 K_2$, a commuting product of connected non-trivial subgroups K_1, K_2 of G. Then X has a non-trivial projection to both K_1 and K_2. Moreover, each projection is a K_i-irreducible subgroup.*

We now need some results that allow us to deduce whether or not a subgroup A_1 of G is G-irreducible. The first result allows us to do that when G is a classical simple group. Recall that if G is not of type A_n, then G has a natural non-degenerate bilinear form on $V_G(\lambda_1)$ (noting that we are factoring out the radical when (G, p) is $(B_n, 2)$) and the notation $V = V_1 \perp V_2$ denotes an orthogonal decomposition with respect to this form.

Lemma 3.4 ([21, Lemma 2.2]). *Suppose G is a classical simple algebraic group, with natural module $V = V_G(\lambda_1)$. Let X be a semisimple connected closed subgroup of G. If X is G-irreducible then one of the following holds:*

(i) *$G = A_n$ and X is irreducible on V.*

(ii) *$G = B_n, C_n$ or D_n and $V \downarrow X = V_1 \perp \ldots \perp V_k$ with the V_i all non-degenerate, irreducible and inequivalent as X-modules.*

(iii) *$G = D_n$, $p = 2$, X fixes a non-singular vector $v \in V$, and X is a G_v-irreducible subgroup of $G_v = B_{n-1}$.*

Applications of this lemma often implicitly invoke some facts about the representation theory of X. For instance, suppose that X is of type A_1 and that $n < p$. Then we implicitly use that the dimension of $V = V_X(n)$ is $n+1$; and that X preserves a symplectic form on V when n is odd and preserves an orthogonal form when n is even.

The next lemma and corollary are used heavily in the proofs of Theorems 2 to 6 to show a subgroup is G-irreducible for a simple exceptional algebraic group G.

Lemma 3.5 ([30, Lemma 3.8]). *Let X be a semisimple connected subgroup of G and let V be a G-module. Suppose that X does not have the same composition factors as any semisimple connected subgroup H of the same type as X with $H \leq L'$ and L-irreducible, for some proper Levi subgroup L of G. If X is of type B_n and $p = 2$ then assume further that there is no subgroup H of type C_n with $H \leq L'$ and L-irreducible, for some Levi subgroup L of G, such that there is an isogeny $\phi : X \to H$ inducing a mapping which takes the composition factors of $V \downarrow X$ to those of $V \downarrow H$. Then X is G-irreducible.*

Corollary 3.6 ([30, Corollary 3.9]). *Suppose $X < G$ is semisimple and $L(G) \downarrow X$ has no trivial composition factors. Then X is G-irreducible.*

Proof. Suppose X is G-reducible. Then by Lemma 3.5 (with $V = L(G)$) there exists a subgroup H of some Levi subgroup L_1 such that the composition factors of $L(G) \downarrow H$ are the same as $L(G) \downarrow X$. But $H < L_1$, so $L(G) \downarrow H$ has trivial composition factors coming from $L(Z(L_1))$, a contradiction. $\square$

The next result is [16, Prop. 1.4] with S allowed to be any closed subgroup of X; the proof is the same.

Lemma 3.7. *Let X be a linear algebraic group over K and let S be a closed subgroup of X. Suppose V is a finite-dimensional X-module satisfying the following conditions:*

(i) *every X-composition factor of V is an irreducible S-module;*

(ii) *for any X-composition factors M, N of V, the restriction map $\mathrm{Ext}^1_X(M,N) \to \mathrm{Ext}^1_S(M,N)$ is injective;*

(iii) *for any X-composition factors M, N of V, if $M \downarrow S \cong N \downarrow S$, then $M \cong N$ as X-modules.*

Then X and S fix precisely the same subspaces of V.

The following well-known result will be used implicitly to prove certain extensions of A_1-modules exist.

Lemma 3.8 ([2, Corollary 3.9]). *Suppose X is an algebraic group of type A_1 and that M is an irreducible X-module such that $H^1(X,M) \neq 0$. Then M is a Frobenius twist of $(p-2) \otimes 1^{[1]}$ and $H^1(X,M) \cong K$.*

When considering the conjugacy class of a subgroup A_1 in an exceptional algebraic group, the following result is useful. We define a prime number $N(A_1, G)$ for each exceptional algebraic group G as in the table below.

G	G_2	F_4	E_6	E_7	E_8
$N(A_1, G)$	3	3	5	7	7

Theorem 3.9 ([12, Theorem 4]). *Let G be an exceptional algebraic group in characteristic $p > N(A_1, G)$ and X_1 and X_2 be A_1 subgroups of G that have the same composition factors on $L(G)$. Then X_1 and X_2 are G-conjugate.*

In the proofs of Theorems 5 and 6, we use Lemma 3.5 to prove A_1 subgroups are G-irreducible when $p = 2$ for $G = E_7$ and E_8. To do this, we need to know the L'-irreducible A_1 subgroups of Levi subgroups L of G when $p = 2$, which we list in the following lemma. For a Levi subgroup L such that all factors of $L' = L_1 \dots L_m$ are classical, we define V_L to be the module $V_{L_1}(\lambda_1) \otimes \cdots \otimes V_{L_m}(\lambda_1)$.

Lemma 3.10. *Let $G = E_7$ or E_8 with $p = 2$ and let L be a proper Levi subgroup of G. Then the following table contains each L-irreducible subgroup A_1.*

Table 3: L-irreducible A_1 subgroups of Levi factors L of E_7 and E_8 when $p = 2$

Levi L'	$V_L \downarrow A_1$
E_7	see Theorem 5
D_7	$0\|(2 + 2^{[r]} + 2^{[s]} + 2^{[t]} + 2^{[u]} + 2^{[v]})\|0$ $(0 < r < s < t < u < v)$
	$0\|(2^{[r]} + 2^{[s]})\|0 + 1^{[t]} \otimes 1^{[u]} \otimes 1^{[v]}$ $(rt = 0;\ r < s;\ t < u < v)$
	$0\|(2^{[r]} + 2^{[s]})\|0 + 1^{[t]} \otimes 1^{[u]} + 1^{[v]} \otimes 1^{[w]}$ $(rt = 0;\ r < s;\ t < u;\ t \leq v;\ v < w)$
D_6	$0\|(2 + 2^{[r]} + 2^{[s]} + 2^{[t]} + 2^{[u]})\|0$ $(0 < r < s < t < u)$
	$0\|(2^{[r]} + 2^{[s]} + 2^{[t]})\|0 + 1^{[u]} \otimes 1^{[v]}$ $(ru = 0;\ r < s < t;\ u < v)$
	$1^{[r]} \otimes 1^{[s]} \otimes 1^{[t]} + 1^{[u]} \otimes 1^{[v]}$ $(ru = 0;\ r < s < t;\ u \leq v)$
	$1 \otimes 1^{[r]} + 1^{[s]} \otimes 1^{[t]} + 1^{[u]} \otimes 1^{[v]}$ $(s \leq \min\{t, u, v\};\ \|\{0, r\}\| + \|\{s, t\}\| + \|\{u, v\}\| \geq 5;$ $\|\{\{0, r\}, \{s, t\}, \{u, v\}\}\| = 3)$
A_1D_5	$(1^{[r]}, 0\|(2^{[s]} + 2^{[t]} + 2^{[u]} + 2^{[v]})\|0)$ $(rs = 0;\ s < t < u < v)$
D_5	$0\|(2 + 2^{[r]} + 2^{[s]} + 2^{[t]})\|0$ $(0 < r < s < t)$
A_1D_4	$(1^{[r]}, 0\|(2^{[s]} + 2^{[t]} + 2^{[u]})\|0)$ $(rs = 0;\ s < t < u)$
	$(1^{[r]}, 1^{[s]} \otimes 1^{[t]} \otimes 1^{[u]})$ $(rs = 0;\ s < t < u)$
	$(1^{[r]}, 1^{[s]} \otimes 1^{[t]} + 1^{[u]} \otimes 1^{[v]})$ $(rs = 0;\ s < t;\ u \leq v;$ if $s = u$ then $t < v)$
D_4	$0\|(2 + 2^{[r]} + 2^{[s]})\|0$ $(0 < r < s)$
	$1 \otimes 1^{[r]} \otimes 1^{[s]}$ (2 classes; $0 < r < s$)
	$1 \otimes 1^{[r]} + 1^{[s]} \otimes 1^{[t]}$ $(0 < r;\ s \leq t;$ if $s = 0$ then $r < t)$
A_7	$1 \otimes 1^{[r]} \otimes 1^{[s]}$ $(0 < r < s)$
A_3^2	$(1^{[r]} \otimes 1^{[s]}, 1^{[t]} \otimes 1^{[u]})$ $(rt = 0;\ r < s;\ t < u)$
$A_1^2A_3$	$(1^{[r]}, 1^{[s]}, 1^{[t]} \otimes 1^{[u]})$ $(rst = 0;\ t < u)$
A_1A_3	$(1^{[r]}, 1^{[s]} \otimes 1^{[t]})$ $(s < t)$

A_1^4	$(1^{[r]}, 1^{[s]}, 1^{[t]}, 1^{[u]})$ $(rstu = 0)$
A_3	$1 \otimes 1^{[r]}$ $(r \neq 0)$
A_1^3	$(1^{[r]}, 1^{[s]}, 1^{[t]})$ $(rst = 0)$
A_1^2	$(1^{[r]}, 1^{[s]})$ $(rs = 0)$
A_1	1

Proof. Let L be a Levi subgroup of G. Write $L' = L_1 \dots L_m$ where each L_i is a simple factor. Given the L_i-irreducible subgroups of type A_1, then all L'-irreducible subgroups of type A_1 follow, since they are just diagonal subgroups (by Lemma 3.3). We therefore give a brief description of the L_i-irreducible subgroups of type A_1 to conclude the proof.

Suppose L_i is of classical type. We use Lemma 3.4 to find all L_i-irreducible A_1 subgroups. First let $L_i \cong A_n$, then $V_n := V_{A_n}(\lambda_1) \downarrow A_1$ is irreducible. By Steinberg's Tensor Product Theorem, it follows that $V_n \downarrow A_1 = 1^{[r_1]} \otimes 1^{[r_2]} \otimes \cdots \otimes 1^{[r_l]}$ for distinct $r_1, \dots, r_l$, since $p = 2$. Therefore, A_n has an A_n-irreducible subgroup A_1 if and only if $n+1 = 2^{[k]}$ for some $k \geq 1$. Now let $L_i \cong D_n$ $(4 \leq n \leq 7)$. In all cases L_i has an L_i-irreducible subgroup A_1, acting as $0|(2^{[r_1]} + \cdots + 2^{[r_{n-1}]})|0$ on $V_{D_n}(\lambda_1)$, coming from part (iii) of Lemma 3.4. If $n = 4$ or 6 then L_i has an L_i-irreducible subgroup A_1 acting as $1^{[r_1]} \otimes 1^{[r_2]} + \cdots + 1^{[r_{n-1}]} \otimes 1^{[r_n]}$ on $V_{D_n}(\lambda_1)$. Finally, if $n = 4$ then there are two further classes of L_i-irreducible A_1 subgroups, acting as $1 \otimes 1^{[r]} \otimes 1^{[s]}$ on $V_{D_4}(\lambda_1)$.

Now suppose L_i is of exceptional type and hence isomorphic to E_6 or E_7. We use Theorem 4 and 5, respectively, to find the L_i-irreducible A_1 subgroups. We note that we are permitted to do this since we prove Theorems 4 to 6 successively and so are only using Theorem 4 and 5 after they have been proved. In particular, there are no E_6-irreducible A_1 subgroups when $p = 2$. All E_7-irreducible A_1 subgroups are contained in $\bar{A}_1 D_6$ when $p = 2$ and are listed in Table 8. $\square$

4 Strategy for the proof of Theorem 1

To prove Theorem 1 we prove Theorems 2 to 6 in Sections 5 to 9, respectively and successively. In this section we describe the strategy used in proving Theorems 2 to 6. Let G be an exceptional algebraic group over an algebraically closed field of characteristic p. Suppose X is a G-irreducible subgroup A_1 of G. Then X is contained in a maximal connected subgroup M of G. Since X is G-irreducible, M is reductive. Furthermore, X is M-irreducible as any parabolic subgroup of M is contained in a parabolic subgroup of G by the Borel-Tits Theorem [4]. Therefore, X is contained M-irreducibly in some reductive, maximal connected subgroup M of G and the following strategy finds all such X.

Take a reductive, maximal connected subgroup M from Theorem 3.1 and find all M-irreducible A_1 subgroups of M, up to M-conjugacy. To do this we use Lemma 3.4 for classical simple components of M, and Theorems 2 to 5 for exceptional simple components of M of smaller rank than $G = F_4, E_6, E_7, E_8$. For each class of M-irreducible A_1 subgroups X we then check whether there exists another reductive, maximal connected subgroup containing X that we have already considered. If there is, then we have already considered X and are done. If not, we must then decide whether X is G-irreducible or not. To do this we heavily use Lemma 3.5 and Corollary 3.6. To apply these results we must find the composition factors of the action of X on the minimal or adjoint module. These can be found by restricting the composition factors of M to X. This can be done for all M-irreducible A_1 subgroups and the composition factors for the G-irreducible ones can be found in Section 11, Tables 9 to 13. To apply Lemma 3.5 we also need the composition factors for the action of the Levi subgroups of G on the minimal and adjoint modules.

These can be found in Appendix A, Tables 18 to 22. In most cases, an M-irreducible subgroup A_1 is G-irreducible: Corollary 5 lists the subgroups which are irreducible in every reductive, maximal connected overgroup yet G-reducible. To prove an M-irreducible subgroup X is G-reducible can be difficult and can require precise knowledge of the action of X on the minimal or adjoint module for G as well as computation in Magma [5].

5 Proof of Theorem 2: G_2-irreducible A_1 subgroups

In this section we find the irreducible A_1 subgroups of G_2, proving Theorem 2 below. We note that Theorem 2 is [1, Theorem 5.4] and can also be deduced from [28, Theorem 1]. We give a proof for completeness and also to show how the strategy described in Section 4 works. Recall the notation V_M from Section 2.

Theorem 2. *Suppose X is an irreducible subgroup A_1 of G_2. Then X is conjugate to exactly one subgroup of Table 4 and each subgroup in Table 4 is irreducible.*

Table 4: The G_2-irreducible A_1 subgroups of G_2

ID	M	$V_M \downarrow X$	p
1	$A_1\tilde{A}_1$	$(1^{[r]}, 1^{[s]})$ $(rs = 0; r \neq s)$	any
2		$(1, 1)$	≥ 3
3	A_1	1	≥ 7

We note that Table 9 gives the composition factors of V_7 and $L(G_2)$ restricted to each irreducible subgroup A_1 in Table 4.

Proof. The conjugacy classes of reductive, maximal connected subgroups M of G_2 are listed in Theorem 3.1. They are $\bar{A}_2$, $\tilde{A}_2$ $(p = 3)$, $A_1\tilde{A}_1$ and A_1 $(p \geq 7)$. Let X be an M-irreducible subgroup A_1 of M.

First suppose $M = A_1\tilde{A}_1$. By Lemma 3.3, X has non-trivial projection to both A_1 and $\tilde{A}_1$. Hence X is a diagonal subgroup of M and we have $X \hookrightarrow A_1\tilde{A}_1$ via $(1^{[r]}, 1^{[s]})$ $(rs = 0;\ r \neq s)$ or $(1, 1)$. The diagonal subgroups with distinct field twists are G_2-irreducible for all p. Indeed, to show X via $(1^{[r]}, 1^{[s]})$ $(rs = 0;\ r \neq s)$ is G_2-irreducible we use Lemma 3.5. From Table 9, the restriction of $V_{G_2}(10)$ to X has a 4-dimensional composition factor, namely $1^{[r]} \otimes 1^{[s]}$ (since $r \neq s$). Neither a Levi subgroup A_1 nor a Levi subgroup $\tilde{A}_1$ has a 4-dimensional composition factor on $V_{G_2}(10)$ (the composition factors are listed in Table 18) and hence X does not have the same composition factors as either Levi subgroup on $V_{G_2}(10)$. Therefore, X is G_2-irreducible by Lemma 3.5.

Now consider X via $(1, 1)$. When $p > 3$, we see from Table 9 that X has no trivial composition factors on $L(G_2)$ and hence X is G_2-irreducible by Corollary 3.6. When $p = 3$, we see that X has two 3-dimensional composition factors on $V_{G_2}(10)$. Neither A_1 nor $\tilde{A}_1$ has two 3-dimensional composition factors on $V_{G_2}(10)$ and hence X is G_2-irreducible by Lemma 3.5. Finally, let $p = 2$. Then $V_{G_2}(10) \downarrow A_1\tilde{A}_1 = (1, 1) + (0, 2)$ and so $V_{G_2}(10) \downarrow X = (0|2|0) + 2$ and X fixes a non-zero vector of $V_{G_2}(10)$. The stabiliser of this non-zero vector in G_2 is a parabolic subgroup. Indeed, G_2 is transitive on 1-spaces of $V_{G_2}(10)$ by [13, Theorem B] and the stabiliser of some 1-space is a parabolic subgroup. Hence X is G_2-reducible.

Now suppose $M = \bar{A}_2$. The only irreducible subgroup A_1 of $\bar{A}_2$ is embedded via the representation with high weight 2, when $p \neq 2$, by Lemma 3.4 and therefore X is such a subgroup A_1. By [19, Table 10.3], we have $\bar{A}_2.2$ is contained in G_2. The subgroup $\bar{A}_2.2$ contains an involution t such that X is the centraliser of t in $\bar{A}_2.2$. The centraliser in G_2 of t is $A_1\tilde{A}_1$, by [10, Table 4.3.1]. Therefore $X < A_1\tilde{A}_1$ and has already been considered. In fact, X is conjugate to $G_2(\#2)$.

Now let $M = \tilde{A}_2$ ($p = 3$). Then as before, X is embedded in $\tilde{A}_2$ via the representation of high weight 2. The same argument as for $M = \bar{A}_2$ shows that X is contained in $A_1\tilde{A}_1$. In particular, X is G_2-irreducible and conjugate to $G_2(\#1^{\{0,1\}})$.

Finally, when $M = A_1$ ($p \geq 7$) we have $X = M$ and hence X is G_2-irreducible.

To finish the proof of Theorem 2, we use the composition factors in Table 9 to check that $G_2(\#1)$, $G_2(\#2)$ and $G_2(\#3)$ are pairwise non-conjugate. □

6 Proof of Theorem 3: F_4-irreducible A_1 subgroups

In this section, we classify all F_4-irreducible A_1 subgroups of F_4, proving Theorem 3.

Theorem 3. *Suppose X is an irreducible subgroup A_1 of F_4. Then X is conjugate to exactly one subgroup of Table 5 and each subgroup in Table 5 is irreducible.*

Table 5: The F_4-irreducible A_1 subgroups of F_4

ID	M	$V_M \downarrow X$	p
1	B_4	$1 \otimes 1^{[r]} + 1^{[s]} \otimes 1^{[t]} + 0$ $(0 < r < s < t)$	any
2		$2^{[r]} + 2^{[s]} + 1^{[t]} \otimes 1^{[u]}$ $(rt = 0;\ r < s;\ t < u)$	$= 2$
3		$2 + 2^{[r]} + 2^{[s]} + 2^{[t]}$ $(0 < r < s < t)$	$= 2$
4		$2 + 2^{[r]} + 2^{[s]}$ $(0 < r < s)$	≥ 3
5		$2 \otimes 2^{[r]}$ $(r \neq 0)$	≥ 3
6		$4^{[r]} + 1^{[s]} \otimes 1^{[t]}$ $(rs = 0;\ s \leq t)$	≥ 5
7		8	≥ 11
8	$\bar{A}_1C_3$ $(p \neq 2)$	$(1^{[r]}, 5^{[s]})$ $(rs = 0)$	≥ 7
9		$(1^{[r]}, 2^{[s]} \otimes 1^{[t]})$ $(rst = 0; s \neq t)$	≥ 3
10	A_1	1	≥ 13
11	A_1G_2	$(1^{[r]}, G_2(\#3)^{[s]})$ $(rs = 0;\ r \neq s)$	≥ 7

The composition factors of V_{26} and $L(F_4)$ restricted to each irreducible subgroup A_1 in Table 5 are found in Table 10.

Proof. The conjugacy classes of reductive, maximal connected subgroups M of F_4 are listed in Theorem 3.1. They are B_4, C_4 $(p = 2)$, $\bar{A}_1C_3$ $(p \neq 2)$, A_1G_2 $(p \neq 2)$, $A_2\tilde{A}_2$, G_2 $(p = 7)$ and A_1 $(p \geq 13)$. Let X be an M-irreducible subgroup A_1 of M.

Firstly, let $M = B_4$. The M-irreducible A_1 subgroups are straightforward to find, using Lemma 3.4. They are the subgroups $F_4(\#1)$–$F_4(\#7)$ listed in Table 5 (without the constraints imposed on the field twists) as well as the subgroups Y_1 and Y_2 acting as $3^{[r]} \otimes 1^{[s]} + 0$ $(p \geq 5)$ and $1 \otimes 1^{[r]} \otimes 1^{[s]}$ $(p = 2; 0 < r < s)$ on $V_{B_4}(\lambda_1)$, respectively. Note that $Y_1, Y_2 < D_4 < B_4$ and are conjugate by a triality automorphism to subgroups acting on $V_{B_4}(\lambda_1)$ as $4^{[r]}+2^{[s]}+0$ or $0|(2+2^{[r]}+2^{[s]})|0$, respectively. The first is $F_4(\#6^{\{r,s,s\}})$ and the second is B_4-reducible, by Lemma 3.4. Now consider $F_4(\#1)$ and $F_4(\#3)$. These are diagonal subgroups of the maximal rank subsystem subgroups $\bar{A}_1^4$ and $\tilde{A}_1^4$, respectively. Indeed, the subgroup $\bar{A}_1^4$ is a maximal subgroup of $D_4 < B_4$, corresponding to the chain $\mathrm{SO}_4\mathrm{SO}_4 < \mathrm{SO}_8 < \mathrm{SO}_9$. The subgroup $\tilde{A}_1^4$ only exists when $p=2$ and is the image of $\bar{A}_1^4$ under the graph automoprhism of F_4. The Weyl group of F_4 induces an action of S_4 on both $\bar{A}_1^4$ and $\tilde{A}_1^4$. The field twists in the embeddings of $F_4(\#1)$ and $F_4(\#3)$ can hence be chosen such that $0<r<s<t$, as in Table 5. The constraints on the field twists in the remaining subgroups in Table 5 all come from considering the M-conjugacy classes of the M-irreducible A_1 subgroups.

We must now prove that $F_4(\#1)$–$F_4(\#7)$ are F_4-irreducible. We first treat the cases when $p>2$.

Let X be $F_4(\#1)$ $(p\neq 2)$ or $F_4(\#6)$, so X is contained in $\bar{A}_1^2B_2$. By restricting the composition factors of $L(F_4)\downarrow M$ (from Theorem 3.1) to $\bar{A}_1^2B_2$ we have

$$L(F_4)\downarrow \bar{A}_1^2B_2 = (2,0,00)/(0,2,00)/(0,0,02)/(1,1,10)/(1,0,01)/(0,1,01).$$

Let $X=F_4(\#1)$ $(p\neq 2)$. Then X is contained in $\bar{A}_1^4$ and we have

$$\begin{aligned}L(F_4)\downarrow \bar{A}_1^4 =&(2,0,0,0)/(0,2,0,0)/(0,0,2,0)/(0,0,0,2)/(1,1,1,1)/(1,1,0,0)/\\&(1,0,1,0)/(1,0,0,1)/(0,1,1,0)/(0,1,0,1)/(0,0,1,1).\end{aligned}$$

Since $0<r<s<t$, there are no trivial composition factors occurring in $L(F_4)\downarrow X$ and thus X is F_4-irreducible by Corollary 3.6. Now let $X=F_4(\#6)$, so the projection of X to B_2 is a maximal subgroup A_1 $(p\geq 5)$ and X is a subgroup of $\bar{A}_1^2A_1$. The composition factors of $L(F_4)$ restricted to $\bar{A}_1^2A_1$ are then as follows:

$$L(F_4)\downarrow \bar{A}_1^2A_1 = (2,0,0)/(0,2,0)/(0,0,2)/(0,0,W(6))/(1,1,4)/(1,0,3)/(0,1,3).$$

Therefore, Corollary 3.6 shows that X is F_4-irreducible unless $p=5$ and $X=F_4(\#6^{\{0,0,1\}})$, in which case $L(F_4)\downarrow X = 10^2/8^3/6/4/2^4/0$. To prove X is F_4-irreducible in this case we use Lemma 3.5. Suppose Y is an L'-irreducible subgroup A_1 of a Levi subgroup L having the same composition factors as X on $L(F_4)$. Then using Table 19, we see that $L'=B_3$, $A_2\tilde{A}_1$ and $\tilde{A}_2A_1$ are the only possibilities since X and hence Y, by definition, has only one trivial composition factor on $L(F_4)$. Suppose $L'=B_3$. Then from Table 19, we see that $V_{B_3}(100)$ occurs as a multiplicity two composition factor of $L(F_4)\downarrow B_3$. But it is impossible to construct two isomorphic 7-dimensional modules from the composition factors of $L(F_4)\downarrow Y$, hence Y is not contained in B_3. Now suppose $L'=A_2\tilde{A}_1$ or $\tilde{A}_2A_1$. Then from Table 19, we have that $(V_{A_2}(00),V_{A_1}(1))$ occurs as a composition factor of $L(F_4)\downarrow L'$ and hence Y has a 2-dimensional composition factor on $L(F_4)$. This is a contradiction. Hence we conclude that Y does not exist and X is F_4-irreducible by Lemma 3.5.

Next, if $X=F_4(\#4)$ or $F_4(\#7)$ then X is F_4-irreducible by Corollary 3.6, with the composition factors of $L(F_4)\downarrow X$ given in Table 10.

The final case when $p\neq 2$ is when $X=F_4(\#5)$, acting on $V_{B_4}(\lambda_1)$ as $2\otimes 2^{[r]}$ $(r\neq 0)$. From Table 10, if $p>3$ then X has no trivial composition factors on $L(F_4)$. Hence Corollary 3.6 applies and X is F_4-irreducible. If $p=3$ then the composition factors of $V_{26}\downarrow X$ have dimensions $9,4^4,1$ or $9,4^3,3,1^2$ (if $r=1$). From Table 19, we see that the only Levi subgroup with a composition factor of dimension at least 9 on V_{26} is $L'=C_3$. Moreover, the composition factors of C_3 acting on V_{26} have dimensions $13,6^2,1$. Therefore, no subgroup A_1 of C_3 has the same composition factors as X on V_{26}. Hence X is F_4-irreducible by Lemma 3.5.

We now assume $p=2$. If X is either $F_4(\#1)$ or $F_4(\#3)$, acting as $1\otimes 1^{[r]}+1^{[s]}\otimes 1^{[t]}$ or $2+2^{[r]}+2^{[s]}+2^{[t]}$ $(0<r<s<t$ in both cases), respectively, we use Lemma 3.5. From Table 10, we see that $1\otimes 1^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}$

occurs as a composition factor of $L(F_4) \downarrow X$. Table 19 shows that there are no Levi subgroups with $L(F_4) \downarrow L'$ having a composition factor of dimension at least 16 when $p = 2$. Hence X is F_4-irreducible by Lemma 3.5.

Finally, suppose $X = F_4(\#2)$ and so contained in $\bar{A}_1^2 \tilde{A}_1^2$, a maximal rank connected subgroup of F_4, corresponding, for example, to the chain $\mathrm{SO}_4\mathrm{Sp}_2\mathrm{Sp}_2 < \mathrm{Sp}_4\mathrm{Sp}_4 < \mathrm{Sp}_8$. From $L(F_4) \downarrow B_4$ we have

$$L(F_4) \downarrow \bar{A}_1^2 \tilde{A}_1^2 = \begin{array}{l}(2,0,0,0)/(0,2,0,0)/(0,0,2,0)/(0,0,0,2)/(1,1,2,0)/(1,1,0,2)/(1,1,0,0)/\\ (1,0,1,1)/(0,1,1,1)/(0,0,2,2)/(0,0,0,0)^4.\end{array}$$

It follows that X has at most 5 trivial composition factors on $L(F_4)$ with the possible extra one coming from $(1,1,2,0)$, $(1,1,0,2)$, $(1,0,1,1)$ or $(0,1,1,1)$. Using Table 19, we see that only $L' = B_3$, C_3 can have an irreducible subgroup A_1 with the same composition factors as X on $L(F_4)$. The only L'-irreducible A_1 subgroups of B_3 or C_3 when $p = 2$ are diagonal subgroups of $\bar{A}_1^3$, $\bar{A}_1^2\tilde{A}_1$, $\bar{A}_1\tilde{A}_1^2$ or $\tilde{A}_1^3$. Any such subgroup has at least 6 trivial composition factors on $L(F_4)$. Hence X is F_4-irreducible by Lemma 3.5. This completes the analysis of the M-irreducible A_1 subgroups contained in $M = B_4$.

Now let $M = C_4$ $(p = 2)$. Then another application of Lemma 3.4 shows that X acts as $1 \otimes 1^{[r]} + 1^{[s]} \otimes 1^{[t]}$ $(r \neq 0;\ s \neq t;\ \{0,r\} \neq \{s,t\})$, $1^{[r]} \otimes 1^{[s]} + 1^{[t]} + 1^{[u]}$ $(r < s;\ t < u)$, $1 + 1^{[r]} + 1^{[s]} + 1^{[t]}$ $(0 < r < s < t)$ or $1 \otimes 1^{[r]} \otimes 1^{[s]}$ $(0 < r < s)$. The first three are contained in the subsystem subgroup C_2^2 and the latter is contained in the subsystem subgroup $\tilde{D}_4$. Suppose X is contained in C_2^2. There is only one F_4-conjugacy class of subgroups C_2^2 in F_4 because $C_{F_4}(C_2)^\circ = C_2$ ([14, p.333, Table 2]). Therefore X is contained in B_4 and has already been considered. Now suppose X is contained in $\tilde{D}_4$ acting on $V_{C_4}(\lambda_1)$ as $1 \otimes 1^{[r]} \otimes 1^{[s]}$ $(0 < r < s)$. In this case X is contained in A_1B_2 which is contained in a subgroup B_3 since $p = 2$. By [7, Table 8], we have $N_{F_4}(\bar{D}_4)/\bar{D}_4 \cong S_3$ and hence applying the graph automorphism of F_4, we have $N_{F_4}(\tilde{D}_4)/\tilde{D}_4 \cong S_3$. It follows that all three $\tilde{D}_4$-conjugacy classes of B_3 are conjugate in F_4. Therefore X is contained in a C_4-reducible B_3 acting as $000|100|000$ on $V_{C_4}(\lambda_1)$ and hence X is F_4-reducible.

Next, we consider the case $M = \bar{A}_1C_3$ $(p \neq 2)$. Using Lemma 3.4, we see that the projection of X to C_3 acts as 5 $(p \geq 5)$, $3^{[r]} + 1^{[s]}$ $(p \geq 7)$, $2^{[r]} \otimes 1^{[s]}$ $(p \neq 2; r \neq s)$, or $1 + 1^{[r]} + 1^{[s]}$. In the second and fourth cases, the projection of X is contained in $\bar{A}_1C_2$ and so X is contained in $\bar{A}_1^2C_2$, which is also a subgroup of B_4. Therefore we have already considered them. Now consider the first case, where the projection of X to C_3 is a maximal subgroup A_1 $(p \geq 7)$ and so X is a diagonal subgroup of $\bar{A}_1A_1$. This gives the conjugacy classes $F_4(\#8)$ in Table 5. From Table 10, we see that X has no trivial composition factors on $L(F_4)$. Hence Corollary 3.6 applies and X is F_4-irreducible.

The final possibility is that the projection of X to C_3 is contained in a maximal subgroup A_1A_1 $(p \neq 2)$, which acts as $(2,1)$ on $V_{C_3}(100)$. In this case $X \hookrightarrow \bar{A}_1A_1A_1$ via $(1^{[r]}, 1^{[s]}, 1^{[t]})$ $(rst = 0)$. First, we suppose $s = t$. Then if $p \geq 5$ we have X is contained in $\bar{A}_1^2B_2$ and hence B_4 because $2^{[r]} \otimes 1^{[r]} = 3^{[r]} + 1^{[r]}$. If $p = 3$ then X is M-reducible by Lemma 3.4 because $2^{[r]} \otimes 1^{[r]} = 1^{[r]}|3^{[r]}|1^{[r]}$. Therefore, we only need to consider the case $s \neq t$, yielding the conjugacy classes $F_4(\#9)$ in Table 5. We now prove that they are all F_4-irreducible. From the composition factors of $L(F_4) \downarrow \bar{A}_1C_3$ in Theorem 3.1, we find that

$$L(F_4) \downarrow \bar{A}_1A_1A_1 = (2,0,0)/(0,2,0)/(0,0,2)/(1,W(4),1)/(1,0,W(3))/(0,W(4),2).$$

If $p > 5$ then X has no trivial composition factors on $L(F_4)$ and so Corollary 3.6 shows that X is F_4-irreducible. If $p = 5$ then Corollary 3.6 applies unless $X = F_4(\#9^{\{0,0,1\}})$, which is embedded via $(1,1,1^{[1]})$. In this case $L(F_4) \downarrow X = 14/10^2/8^3/2^2/0$. To show that X is F_4-irreducible we use Lemma 3.5. Suppose Y is an L-irreducible subgroup A_1 of a Levi factor L' having the same composition factors as X on $L(F_4)$. Then using Table 19, we find that $L' = B_3$ since X, and hence Y, has a 15-dimensional composition factor and only one trivial composition factor on $L(F_4)$. Further inspection of the dimensions of the composition factors of X on $L(F_4)$ shows that there are three composition factors of dimension 8 as well as the one of dimension 15. The dimensions of the composition factors of $L(F_4) \downarrow B_3$ are $21, 8^2, 7^2, 1$. It follows that Y is not contained in B_3. This contradiction shows that Y does not exist and hence X is F_4-irreducible by Lemma 3.5.

Now let $p = 3$. From the restriction of $V_{26} \downarrow \bar{A}_1 C_3$ in Theorem 3.1, we have

$$V_{26} \downarrow \bar{A}_1 A_1 A_1 = (1,2,1)/(0,4,0)/(0,2,2)/(0,0,0).$$

Using Table 19, we see that $L' = C_3$ is the only Levi factor that can contain a subgroup A_1 with the same composition factors as X on V_{26} because X has a 9-dimensional composition factor, namely $2^{[s]} \otimes 2^{[t]}$ (recalling $s \neq t$). We want to apply Lemma 3.5 to conclude that X is F_4-irreducible. From Table 19, we have $V_{26} \downarrow C_3 = 100^2/010$. If the field twists for the embedding of X in $\bar{A}_1 A_1 A_1$ are all distinct then X has a 12-dimensional composition factor as well as a 9-dimensional one, and hence there is no subgroup A_1 of C_3 with the same composition factors as X on V_{26}. The cases which remain are X embedded via $(1^{[r]}, 1^{[r]}, 1^{[s]})$ $(rs = 0)$ and $(1^{[r]}, 1^{[s]}, 1^{[r]})$ $(rs = 0)$. The dimensions of the composition factors on V_{26} are then $9, 4^4, 1$ (or $9, 4^3, 3, 1^2$ if $s = r+1$) or $9^2, 4, 3, 1$ respectively. None of these are compatible with a subgroup A_1 of C_3 and hence X is F_4-irreducible. This completes the analysis of the M-irreducible A_1 subgroups contained in $M = \bar{A}_1 C_3$.

Now suppose $M = A_1 G_2$ $(p \neq 2)$. By Theorem 2, the projection of X to G_2 is either contained in $A_1 A_1$ or is maximal with $p \geq 7$. In the first case we claim that X is contained in $\bar{A}_1 C_3$. Indeed, since the factor G_2 of M is contained in D_4 by [25, 3.9], it follows that the long root subgroup A_1 of G_2 is a long root subgroup A_1 of D_4 and hence F_4. Therefore X is contained in $\bar{A}_1 C_{F_4}(\bar{A}_1)^\circ = \bar{A}_1 C_3$. Now consider the second case. Then $X \hookrightarrow A_1 A_1 < A_1 G_2$ $(p \geq 7)$ via $(1^{[r]}, 1^{[s]})$ $(rs = 0)$. From Table 10, we see that if $r \neq s$ then X has no trivial composition factors on $L(F_4)$. Hence X is F_4-irreducible by Corollary 3.5, yileding $F_4(\#11)$. Now consider $X \hookrightarrow A_1 A_1$ via $(1,1)$. Then

$$L(F_4) \downarrow X = W(10)^2/W(8)/6/4/2^3.$$

From Table 3.5, we have $Y = F_4(\#8^{\{0,0\}}) < \bar{A}_1 C_3$ has the same composition factors on $L(F_4)$. Since $p \geq 7 > 3 = N(A_1, F_4)$, Theorem 3.9 applies. Hence X is conjugate to Y and has already been considered.

Now suppose $M = A_2 \tilde{A}_2$. Then X is contained in $Y = Y_1 Y_2 = A_1 A_1 < M$ $(p \neq 2)$ (both factor A_1 subgroups are irreducibly embedded in A_2). We claim that Y is contained in $\bar{A}_1 C_3$ and hence so is X, which has therefore already been considered. Indeed, by [7, Table 8], we have that F_4 contains an involution which acts as a graph automorphism on both A_2 factors of M. Therefore, there exists an involution t such that $Y < C_{F_4}(t)^\circ$. One calculates that $C_{F_4}(t)^\circ = \bar{A}_1 C_3$, as required.

Now let $M = G_2$ $(p = 7)$. By Theorem 2, up to M-conjugacy, X is contained in $A_1 A_1$ or is a maximal subgroup. Consider the first case. By [10, Table 4.3.1], the subgroup $A_1 A_1 < G_2$ is the centraliser in G_2 of a semisimple element of order 2. By [17, Proposition 1.2] the connected centraliser in F_4 of t is B_4 or $\bar{A}_1 C_3$ with the trace of t on V_{26} being -6 or 2, respectively. We calculate that the trace of t on V_{26} is 2 using $V_{26} \downarrow A_1 A_1 = (2,2)/(1,1)/(1, W(3))/(0, W(4))$ and the fact the element t can be seen as minus the identity in both A_1 factors. Therefore the subgroup $A_1 A_1$ is contained in $\bar{A}_1 C_3$, and in particular $X \hookrightarrow A_1 A_1$ via $(1^{[r]}, 1^{[s]})$ $(rs = 0)$ is conjugate to $F_4(\#9^{\{r,r,s\}})$.

Now consider the second case, when X is a maximal subgroup A_1 of M. By restricting from $L(F_4) \downarrow M$, it follows that $L(F_4) \downarrow X = 16/14/10^3/6/2^3$. Now let $Y = F_4(\#8^{\{1,0\}})$ from Table 5. Then from Table 10, we have $L(F_4) \downarrow Y = 16/14/10^3/6/2^3$. As $p = 7$, Theorem 3.9 applies and hence X is conjugate to Y, and has already been considered.

If $M = A_1$ ($p \geq 13$) then $X = M$ and X is F_4-irreducible and not conjugate to any other subgroup A_1 (this follows immediately from Theorem 3.1).

Finally, we use the composition factors given in Table 10 to show that there are no further conjugacies between the A_1 subgroups in Table 5. □

7 Proof of Theorem 4: E_6-irreducible A_1 subgroups

In this section we prove Theorem 4, which classifies the E_6-irreducible A_1 subgroups of E_6.

Theorem 4. *Suppose X is an irreducible subgroup A_1 of E_6. Then X is conjugate to exactly one subgroup of Table 6 and each subgroup in Table 6 is irreducible.*

Table 6: The E_6-irreducible A_1 subgroups of E_6

ID	M	$V_M \downarrow X$	p
1	$\bar{A}_1 A_5$	$(1^{[r]}, 5^{[s]})$ $(rs = 0)$	≥ 7
2		$(1^{[r]}, 2^{[s]} \otimes 1^{[t]})$ $(rst = 0;\ s \neq t)$	≥ 3
3	A_2^3	$(2, 2^{[r]}, 2^{[s]})$ $(0 < r < s)$	≥ 3
4	$A_2 G_2$	$(2^{[r]}, G_2(\#3)^{[s]})$ $(rs = 0; r \neq s)$	≥ 7
5	F_4	$F_4(\#10)$	≥ 13
6	C_4 $(p \neq 2)$	7	≥ 11

The composition factors of V_{27} and $L(E_6)$ restricted to each irreducible subgroup A_1 in Table 6 are found in Table 11.

Proof. We use the same method as for F_4, taking each reductive, maximal connected subgroup M of E_6 in turn (from Theorem 3.1) and finding all E_6-irreducible A_1 subgroups contained in them, up to E_6-conjugacy. Let X be an M-irreducible subgroup A_1 of M.

First, consider $M = \bar{A}_1 A_5$. Then using Lemma 3.4, we see that the projection of X to A_5 acts on $V_{A_5}(\lambda_1)$ either as 5 ($p \geq 7$) or $2^{[r]} \otimes 1^{[s]}$ ($p \neq 2$; $r \neq s$). Suppose we are in the first case and so $X \hookrightarrow \bar{A}_1 A_1$ ($p \geq 7$) via $(1^{[r]}, 1^{[s]})$, where the second factor A_1 acts as 5 on $V_{A_5}(\lambda_1)$. From Table 11, we see that X has no trivial composition factors on $L(E_6)$ and is thus E_6-irreducible by Corollary 3.6, yielding $E_6(\#1)$.

In the second case $X = E_6(\#2)$, a diagonal subgroup of $\bar{A}_1 A_1 A_1 < \bar{A}_1 A_5$, where $A_1 A_1 < A_5$ acts on $V_{A_5}(\lambda_1)$ as $(2, 1)$. From the restriction of $L(E_6) \downarrow \bar{A}_1 A_5$ in Theorem 3.1, it follows that

$$L(E_6) \downarrow \bar{A}_1 A_1 A_1 = (2,0,0)/(1,W(4),1)/(1,2,1)/(1,0,W(3))/(0,W(4),2)/(0,W(4),0)/ \\ (0,2,0)/(0,2,2)/(0,0,2).$$

If $p > 5$ then X is E_6-irreducible by Corollary 3.6. If $p = 5$ then Corollary 3.6 applies unless $X = E_6(\#2^{\{0,0,1\}})$. In this case $V_{27} \downarrow X = 12/8/6/4/0$ (from Table 11) so the dimensions of the composition factors are $9, 8, 5, 4, 1$. We use Lemma 3.5 to show that X is E_6-irreducible. Suppose not, then there exists a subgroup A_1 with the same composition factors as X on V_{27} contained (not necessarily L-irreducibly) in $L' = D_5, A_1 A_4, A_1 A_2^2$ or A_5. But the dimensions of their composition factors on V_{27} are $16, 10, 1$ for D_5, $10^2, 5, 2$ for $A_1 A_4$, $9, 6^2, 3^2$ for $A_1 A_2^2$ and $15, 6^2$ for A_5. Therefore, no subgroup A_1 of L' has the same composition factors as X, a contradiction. When $p = 3$, we use Lemma 3.5 again. There are four possibilities for the dimensions of the composition factors of X on V_{27}: $12, 9, 4, 1^2$ (r, s, t distinct), $9^2, 4, 3, 1^2$ ($r = t$), $9, 4^4, 1^2$ ($r = s \neq t-1$) and $9, 4^3, 3, 1^3$ ($r = s = t-1$). It follows that only $L' = D_5$ can contain a subgroup A_1 with the same composition factors as X on V_{27}. Further consideration of the dimensions (and recalling that $p = 3$) leads to the only possibility being $Y < D_5$ with $V_{D_5}(\lambda_1) \downarrow Y = 2 \otimes 2^{[a]} + 0$ ($a \neq 0$). Then $V_{27} \downarrow Y = 2 \otimes 2^{[a]}/3 \otimes 1^{[a]}/1 \otimes 3^{[a]}/(1 \otimes 1^{[a]})^2/0^2$. The composition factors of X and Y do not agree on V_{27} regardless of the choice of r, s, t and a. Hence X is E_6-irreducible, completing the analysis of the M-irreducible A_1 subgroups of $M = \bar{A}_1 A_5$.

Now let $M = A_2^3$. Then $p \neq 2$ and X is a diagonal subgroup of $A_1^3 < A_2^3$, where each factor A_1 is irreducibly embedded in A_2. By Theorem 3.1, we have $V_{27} \downarrow A_2^3 = (10, 01, 00)/(00, 10, 01)/(01, 00, 10)$ and hence $V_{27} \downarrow A_1^3 = (2, 2, 0)/(0, 2, 2)/(2, 0, 2)$. First consider the case where all of the field twists in the embedding of X are distinct. Then the action of X on V_{27} has three composition factors, all of dimension 9. Using Table 20, we see that no subgroup A_1 of a Levi subgroup can have the same composition factors as X on V_{27}. Hence X is E_6-irreducible by Lemma 3.5 and this yields $E_6(\#3)$.

If at least two of the field twists in the embedding of X are equal then we claim that X is contained in $\bar{A}_1 A_5$. To prove the claim, we first show that $A_1^3 < A_2^3$ is contained in C_4, acting as $(1, 1, 1)$ on $V_{C_4}(\lambda_1)$. Consider the standard graph automorphism of E_6, call it τ. Then $w_0 = -\tau$ and so $t := \tau w_o$ acts as -1 on a maximal torus of E_6. Therefore t induces a graph automorphism on each A_2 factor of A_2^3. It follows that $A_1^3 < C_{E_6}(t)$ because an irreducible A_1 in a subgroup A_2 is centralised by a graph automorphism of A_2. We check that $\dim(C_{L(E_6)}(t)) = 36$ and so $\dim(C_{E_6}(t)) = 36$ (by [3, 9.1], since t is semisimple). Therefore, $C_{E_6}(t)^\circ = C_4$ by [10, Table 4.3.1] and we have $A_1^3 < C_4$. Considering the composition factors of A_1^3 on V_{27}, it follows that it is conjugate to a subgroup A_1^3 acting as $(1, 1, 1)$ on $V_{C_4}(\lambda_1)$, as required. Therefore, X is contained in $\bar{A}_1 C_3 < C_4$ since $1^{[r]} \otimes 1^{[r]} \otimes 1^{[s]} = 2^{[r]} \otimes 1^{[s]} + 1^{[s]}$ ($p \neq 2$). The factor A_1 of $\bar{A}_1 C_3$ is generated by root subgroups of E_6 and so $X < \bar{A}_1 C_{E_6}(\bar{A}_1)^\circ = \bar{A}_1 A_5$, proving the claim. If only two of the twists are equal then X is E_6-irreducible and conjugate to $E_6(\#2^{\{r,r,s\}})$. If all three twists are equal then X is C_4-reducible and hence E_6-reducible. This completes the case $M = A_2^3$.

Next, we let $M = A_2 G_2$. By Theorem 2, up to M-conjugacy, the projection of X to G_2 is either contained in $A_1 A_1$ or is maximal ($p \geq 7$). Assume the former. Since the G_2 factor of M is contained in D_4 by [25, 3.15], the first A_1 factor of $A_1 A_1$ is generated by root subgroups of E_6. Therefore, $A_2 \bar{A}_1 A_1 < \bar{A}_1 C_{E_6}(\bar{A}_1)^\circ = \bar{A}_1 A_5$. Therefore, X has already been considered in the $\bar{A}_1 A_5$ case. Now assume the projection to G_2 is maximal, so $p \geq 7$. Then $X \hookrightarrow A_1 A_1 < A_2 G_2$ via $(1^{[r]}, 1^{[s]})$ ($rs = 0$) or $(1, 1)$ where each factor A_1 is maximal. If X is embedded via $(1^{[r]}, 1^{[s]})$ then X is E_6-irreducible by Corollary 3.6, yielding $E_6(\#4)$. If X is embedded via $(1, 1)$ then X is conjugate to $Y = E_6(\#1^{\{0,0\}})$, by Theorem 3.9 since $p > 5 = N(A_1, E_6)$ and X and Y have the same composition factors on $L(E_6)$.

Now suppose $M = F_4$. Theorem 3 gives all of the conjugacy classes of F_4-irreducible A_1 subgroups, showing they are all contained in B_4, $\bar{A}_1 C_3$ ($p \neq 2$), $A_1 G_2$ ($p \neq 2$) or A_1 ($p \geq 13$). If X is contained in B_4 then X is E_6-reducible because B_4 is contained in a D_5 Levi subgroup. If X is contained in $\bar{A}_1 C_3$ ($p \neq 2$) then X is also contained in $\bar{A}_1 A_5$ since $C_{E_6}(\bar{A}_1)^\circ = A_5$, as above and has already been considered. If X is contained in $A_1 G_2$ then X is contained in the maximal subgroup $A_2 G_2$ (since $C_{E_6}(G_2)^\circ = A_2$) and has also been considered already. Finally, if X is a maximal subgroup A_1 of F_4 then X is E_6-irreducible by Corollary 3.6, yielding $E_6(\#5)$.

Now let $M = C_4$ ($p \neq 2$). By considering the action of X on $V_{C_4}(\lambda_1)$ and using Lemma 3.4, it follows that X is contained in C_2^2, $\bar{A}_1 C_3$, A_1^3 or A_1 ($p \geq 11$). If X is contained in C_2^2 then X is E_6-reducible because $C_{E_6}(C_2)^\circ = C_2 T_1$, by [14, p.333, Table 3] and so C_2^2 is contained in a Levi subgroup of E_6. If X is contained in $\bar{A}_1 C_3$ then X is also contained in $\bar{A}_1 A_5$, hence considered in the $\bar{A}_1 A_5$ case above. If X is contained in A_1^3, acting as $(1,1,1)$ on $V_{C_4}(\lambda_1)$ then an argument in the A_2^3 case showed that X is contained in A_2^3. The last possibility is $p \geq 11$ and X is maximal in C_4 acting as 7 on $V_{C_4}(\lambda_1)$. From Table 11, we see that X is E_6-irreducible by Corollary 3.6, yielding $E_6(\#6)$ in Table 6.

Now let M be one of the two conjugacy classes of G_2 ($p \neq 7$). By Theorem 2, an M-irreducible subgroup A_1 is contained in A_1A_1 or is maximal with $p > 7$. If X is maximal then X is conjugate to $E_6(\#6)$ by Theorem 3.9, since both subgroups have the same composition factors on $L(E_6)$ and $p > 5 = N(A_1, E_6)$. Now suppose X is contained in A_1A_1. When $p \neq 2$, we claim that A_1A_1 is contained in $\bar{A}_1A_5$ and X has already been considered. In G_2, we have that A_1A_1 is the centraliser of a semisimple element of order 2, and by [10, Table 4.3.1], the centraliser in E_6 of this involution is either $\bar{A}_1A_5$ or T_1D_5. An easy check shows it to be the former, proving the claim.

When $p = 2$, we claim that X is E_6-reducible. To prove this we consider the action of A_1A_1 on $L(E_6)$. By [19, Table 10.1], we have $L(E_6) \downarrow G_2 = 11 + 01$. In Table 9, the composition factors of $V_{G_2}(01) \downarrow A_1A_1$ are given and moreover, $V_{G_2}(01) \downarrow A_1A_1 = ((0,0)|((2,0)+(0,2))|(0,0)) + (1,3)$. Therefore A_1A_1, and hence X, fixes a non-zero vector of $L(E_6)$. By [25, Lemma 1.3], we have X is contained in a parabolic subgroup, $\bar{A}_1A_5$ or A_2^3. The $\bar{A}_1A_5$ and A_2^3 cases show that neither $\bar{A}_1A_5$ nor A_2^3 contain an E_6-irreducible subgroup A_1 when $p = 2$. Therefore X is E_6-reducible, as claimed.

Finally, let M be one of the two conjugacy classes of A_2 ($p \geq 5$). There is just one M-irreducible subgroup X, acting as 2 on $V_{A_2}(10)$. If $p \geq 7$ then Theorem 3.9 shows that X is conjugate to $E_6(\#1^{\{0,0\}})$, which is contained in $\bar{A}_1A_5$. When $p = 5$, we show that X is E_6-reducible. By [19, Table 10.2], we have $V_{27} \downarrow A_2 = W(22) = 22|11$ or $W(22)^*$ and $V_{A_2}(20) \otimes V_{A_2}(02) = (11|22|11) + 00$. Using [9, 1.2], which states that a tensor product of tilting modules is again tilting, we have $4 \otimes 4 = (0|8|0) + (2|6|2) + 4$. Since $V_{A_2}(20) \downarrow X = 4 + 0$ and $V_{A_2}(11) \downarrow X = 4 + 2$, it follows that $V_{27} \downarrow X = (0|8|0) + (6|2) + 4^2$ or $(0|8|0) + (2|6) + 4^2$. This shows that X fixes a 1-space of V_{27}. The dimension of the centraliser in E_6 of this 1-space is at least $51 = 78 - 27$ and hence X is either contained in a parabolic subgroup or F_4. Assume the latter. By [19, Table 10.2], we have $V_{27} \downarrow F_4 = 0001 + 0000$ and so any subgroup of F_4 has a trivial direct summand on V_{27}. Since X does not have such a summand on V_{27}, it is not contained in F_4. Therefore X is E_6-reducible.

Using the composition factors listed in Table 11, we see there are no further conjugacies between the A_1 subgroups in Table 6, which completes the proof. □

8 Proof of Theorem 5: E_7-irreducible A_1 subgroups

In this section we find the E_7-irreducible A_1 subgroups of E_7, proving Theorem 5.

Theorem 5. *Suppose X is an irreducible subgroup A_1 of E_7. Then X is conjugate to exactly one subgroup of Table 7 and each subgroup in Table 7 is irreducible.*

Table 7: The E_7-irreducible A_1 subgroups of E_7

ID	M	$V_M \downarrow X$	p
1	A_1D_6	$(1^{[r]}, 5^{[s]} \otimes 1^{[t]})$ $(rst = 0; s \neq t)$	≥ 7
2		$(1^{[r]}, 5^{[s]} \otimes 1^{[t]})$ $(rst = 0; s \neq t)$	≥ 7
3		$(1^{[r]}, 2^{[s]} \otimes 1^{[t]} \otimes 1^{[u]})$ $(rstu = 0;$ s, t, u distinct$)$	≥ 3
4		$(1^{[r]}, 10^{[s]} + 0)$ $(rs = 0)$	≥ 11

5		$(1^{[r]}, 8^{[s]} + 2^{[t]})$ $(rst = 0)$	≥ 11
6		$(1^{[r]}, 6^{[s]} + 4^{[t]})$ $(rst = 0)$	≥ 7
7		$(1^{[r]}, 6^{[s]} + 1^{[t]} \otimes 1^{[u]} + 0)$ $(rs = 0; r < t < u)$	≥ 7
8		$(1^{[r]}, 4^{[s]} + 2^{[t]} + 1^{[u]} \otimes 1^{[v]})$ $(rstu = 0;\ s \neq t;\ u \leq v;$ if $u = v$ then $t < u)$	≥ 5
9		$(1^{[r]}, 4^{[s]} + 2^{[s]} + 1^{[t]} \otimes 1^{[u]})$ $(rst = 0;\ r < t < u$ or $t = u \neq s)$	≥ 5
10		$(1^{[r]}, 2^{[s]} \otimes 2^{[t]} + 2^{[u]})$ $(rsu = 0; s < t)$	≥ 3
11		$(1^{[r]}, 2^{[s]} + 2^{[t]} + 2^{[u]} + 2^{[v]})$ $(rs = 0; s < t < u < v)$	≥ 3
12		$(1, 1^{[r]} \otimes 1^{[s]} + 1^{[t]} \otimes 1^{[u]} + 1^{[v]} \otimes 1^{[w]})$ (see Table 14 for conditions on $r, \dots, w$)	all
13		$(1^{[r]}, 0\|(2^{[s]} + 2^{[t]} + 2^{[u]})\|0 + 1^{[v]} \otimes 1^{[w]})$ $(rsv = 0; s < t < u; v < w)$	2
14		$(1^{[r]}, 0\|(2^{[s]} + 2^{[t]} + 2^{[u]} + 2^{[v]} + 2^{[w]})\|0)$ $(rs = 0; s < t < u < v < w)$	2
15	G_2C_3	$(G_2(\#3)^{[r]}, 5^{[s]})$ $(rs = 0; r \neq s)$	≥ 7
16		$(G_2(\#3)^{[r]}, 2^{[s]} \otimes 1^{[t]})$ $(rst = 0; r \neq s; s \neq t)$	≥ 7
17	A_1G_2	$(1^{[r]}, G_2(\#3)^{[s]})$ $(rs = 0; r \neq s)$	≥ 7
18	A_1F_4	$(1^{[r]}, F_4(\#10)^{[s]})$ $(rs = 0)$	≥ 13
19	A_1A_1	$(1^{[r]}, 1^{[s]})$ $(rs = 0; r \neq s)$	≥ 5
20	A_1	1	≥ 17
21	A_1	1	≥ 19

The composition factors of the irreducible A_1 subgroups in Table 7 acting on V_{56} and $L(E_7)$ are listed in Table 12.

Proof. We consider each reductive, maximal connected subgroup M of E_7 in turn. By Theorem 3.1, they are A_1D_6, A_7, A_2A_5, G_2C_3, A_1G_2 $(p \neq 2)$, A_1F_4, A_2 $(p \geq 5)$, A_1A_1 $(p \geq 5)$, A_1 $(p \geq 17)$ and A_1 $(p \geq 19)$. Let X be an M-irreducible subgroup A_1.

First let $M = A_1D_6$. First, we need to find the E_7-conjugacy classes of the A_1D_6-irreducible A_1 subgroups contained in A_1D_6. Using Lemma 3.4, we see that the A_1D_6-irreducible A_1 subgroups are the subgroups listed in lines 1 to 14 of Table 7 without the constraints on the field twists, as well as $Y = A_1 < A_1D_6$ acting as $(1^{[r]}, 3^{[s]} \otimes 1^{[t]} + 1^{[u]} \otimes 1^{[v]})$ $(p \geq 5; s \neq t)$ and $Z = A_1 < A_1D_6$ acting as $(1^{[r]}, 1^{[s]} \otimes 1^{[t]} \otimes 1^{[u]} + 1^{[v]} \otimes 1^{[w]})$ $(p = 2;\ s, t, u$ distinct). For most of the subgroups, the E_7-conjugacy classes are just the A_1D_6-conjugacy classes. This is the case for subgroups $E_7(\#1)$–$E_7(\#6)$, $E_7(\#10)$, $E_7(\#11)$, $E_7(\#14)$ (we can check that E_7 does not fuse any of the A_1D_6-conjugacy classes by considering the composition factors on V_{56} given in Table 12).

All of the remaining A_1D_6-irreducible A_1 subgroups are contained in $A_1^3D_4$, a maximal rank connected subgroup. By [7, Table 10], we have $N_{E_7}(A_1^3D_4) = (A_1^3D_4).S_3$ where the S_3 acts simultaneously as the outer automorphism group of A_1^3 and D_4. First, suppose that the projection of X to D_4 acts as $6^{[r]} + 0$ $(p \geq 7)$ or $4^{[r]} + 2^{[r]}$ $(p \geq 5)$ on $V_{D_4}(\lambda_1)$. Then the projection of X is contained in the centraliser of both a triality automorphism (since X is contained in a G_2 or A_2 respectively) and an involutory automorphism of D_4 (since X is contained in a B_3 or A_1B_2 respectively). The conjugacy classes of $E_7(\#7)$ and $E_7(\#9)$ follow. Next, assume the projection of X to D_4 acts as $4^{[r]} + 2^{[s]}$ $(p \geq 3; r \neq s)$ or $3^{[r]} \otimes 1^{[s]}$ $(p \geq 5; r \neq s)$. Then, using the triality automorphism, we assume that X acts as $4^{[r]} + 2^{[s]}$ on $V_{D_4}(\lambda_1)$, hence excluding Y from Table 7. The projection of X to D_4 is contained in A_1B_2 and is therefore fixed by an involution in the outer automorphism group of D_4. By considering composition factors, we see that this involution swaps the two A_1 factors contained in D_6. The conjugacy classes of X now follow, which are $E_7(\#8)$ in Table 7. The same argument applies when the projection to D_4 acts as $0|(2^{[r]} + 2^{[s]} + 2^{[t]})|0$ ($p = 2$; $r < s < t$) or $1^{[r]} \otimes 1^{[s]} \otimes 1^{[t]}$ ($p = 2$; r, s, t distinct). Therefore, we exclude Z from Table 7 and obtain the conjugacy classes $E_7(\#13)$.

The last possibility to consider is when the projection of X to D_4 is contained in $\mathrm{SO}_4\mathrm{SO}_4$ and so X is $E_7(\#12)$. In this case X is contained in A_1^7 and $N_{E_7}(A_1^7) = (A_1^7).\mathrm{PSL}(2,7)$ by [7, Table 10]. The automorphism group of the Fano plane is $\mathrm{PSL}(2,7)$ and this leads to an isomorphism between $\mathrm{PSL}(2,7)$ and the subgroup of S_7 generated by $(1,2,3)(5,6,7)$ and $(2,4)(3,5)$. This subgroup is 2-transitive and has two orbits on sets of 3 points. One orbit is made up of all 3 point sets that form a line in the Fano plane and the other orbit is the 3 point sets that do not form a line.

As $\mathrm{PSL}(2,7)$ is transitive we may assume that the field twist in the first A_1 is zero and we let $X \hookrightarrow A_1^7$ via $(1, 1^{[r]}, 1^{[s]}, 1^{[t]}, 1^{[u]}, 1^{[v]}, 1^{[w]})$. In particular, we have fixed that $(0, r, s)$, $(0, w, v)$ and $(0, t, u)$ form lines in the Fano plane. We claim that the 7-tuples $0, r, \ldots, w$ satisfying the conditions in Table 14 yield a set of representatives of the E_7-conjugacy classes of X without repetition. We will prove the first few lines of Table 14. The others are similar and easier.

Assume $r, \ldots, w$ are all non-zero, so we are in the first seven rows of Table 14. By Lemma 3.4, we have X is A_1D_6-irreducible if and only if the following conditions hold: the sets $\{r, s\}$, $\{t, u\}$ and $\{v, w\}$ are distinct and at most one of the sets has cardinality one. The stabiliser in $\mathrm{PSL}(2,7)$ of a point is isomorphic to S_4.

First suppose $r, \ldots, w$ are all distinct. The action of S_4 on $r, \ldots, w$ is given by the natural action of S_4 on pairs of $\{1, 2, 3, 4\}$. This action of S_4 on $r, \ldots, w$ is transitive and so we may assume that r is the smallest integer in $r, \ldots, w$. Now consider the stabiliser in S_4 of r, a Klein four-group, V_4. Since $0, r, s$ form a line, it follows that s is fixed and so no further conditions can be imposed on s. The action of V_4 on t, u, v, w allows us to assume that t is the smallest integer of t, u, v, w. The stabiliser of t in V_4 is trivial and hence we have the conditions given in the first row of Table 14.

Next, suppose exactly two of $r, \ldots, w$ are the same. The action of S_4 on $r, \ldots, w$ has two orbits on pairs. Hence we assume either $r = s$ or $r = t$. If $r = s$, then the stabiliser of the pair r, s is isomorphic to Dih_8, the dihedral group of order 8. The action of Dih_8 on t, u, v, w is transitive. Therefore we may assume that t is the smallest integer of these. The stabiliser of t in Dih_8 is isomorphic to $\mathbb{Z}_2$, swapping v and w and we therefore assume $v < w$. If $r = t$, then the stabiliser in S_4 of $\{r, t\}$ is $\mathbb{Z}_2$, swapping s and u, and so we assume $s < u$. This yields the second and third rows of Table 14.

For the final example, suppose exactly three of $r, \ldots, w$ are the same. The action of S_4 on $r, \ldots, w$ has two orbits on triples. Hence we assume either $r = s = t$ or $r = t = w$. The stabiliser of r, s, t in S_4 is trivial and so no further conditions can be imposed. The stabiliser of r, t, w is isomorphic to $\mathbb{Z}_3$, acting as a 3-cycle on t, u, v. Therefore, we assume that t is the smallest.

We now need to show that the subgroups $E_7(\#1)$–$E_7(\#14)$ are E_7-irreducible.

The subgroups with ID numbers 1, 2 ($p > 7$), 3 ($p > 5$), 4, 5 ($p > 11$), 6 ($p > 7$), 7 ($p > 7$), 8 ($p > 5$), 9 ($p > 5$), 10 ($p > 5$), 11 and 12 ($p > 3$) are all E_7-irreducible by Corollary 3.6 (the composition factors of $L(E_7) \downarrow X$ are listed in Table 12).

In many of the remaining cases, Corollary 3.6 still applies. We present the cases where we use Lemma 3.5 with Table 21 to prove the remaining subgroups are E_7-irreducible. The arguments are all very similar and so we will omit the details for some of them.

Firstly, consider $X = E_7(\#2)$ when $p = 7$. Then Corollary 3.6 applies unless $r = s = t - 1$ in which case X has one trivial composition factor on $L(E_7)$ and $L(E_7) \downarrow X = 22/18^2/16/14^2/12^2/10^3/8/6/4/2^5/0$. We will use Lemma 3.5 by showing that the composition factors of any irreducible subgroup A_1 of a Levi subgroup on $L(E_7)$ are not the same as those of X. Suppose, for a contradiction, that Y is an L-irreducible subgroup A_1 of a Levi subgroup L, having the same composition factors as X on $L(E_7)$. Since X has only one trivial composition factor on $L(E_7)$, we have L' has only one trivial composition factor on $L(E_7)$. Therefore, using Table 21, we find the possibilities for L' are E_6, A_1D_5, A_6, A_1A_5, A_2A_4 and $A_1A_2A_3$. Since $p = 7$, Lemma 3.4 shows there are no A_6-irreducible A_1 subgroups and so we immediately rule out $L' = A_6$. Suppose Y is contained in E_6. Then by Theorem 4, we see that Y is conjugate to a subgroup in Table 6. Using the composition factors in Table 11, we find that the composition factors of $L(E_7) \downarrow E_6(\#n)$ are not the same as $L(E_7) \downarrow X$ for any n and hence Y is not contained in E_6. Now suppose Y is contained in A_1D_5. From Table 21, we see that $(V_{A_1}(0), V_{D_5}(\lambda_1))$ occurs as a multiplicity two composition factor of $L(E_7) \downarrow A_1D_5$. As there is no combination of composition factors of $L(E_7) \downarrow X$ that form two isomorphic 10-dimensional modules, it follows that Y is not contained in A_1D_5. Using Table 21, we see that A_1A_5 has a 2-dimensional composition factor and hence Y is not contained in A_1A_5. Now suppose Y is contained in A_2A_4. Since Y is A_2A_4-irreducible, it follows that Y acts as $2^{[r]} \otimes 4^{[s]}$ on $(V_{A_2}(10), V_{A_4}(1000))$. Both $(V_{A_2}(00), V_{A_4}(1000))$ and $(V_{A_2}(00), V_{A_4}(0001))$ occur as composition factors of $L(E_7) \downarrow A_2A_4$ and hence Y has two 5-dimensional composition factors on $L(E_7)$, a contradiction. Finally, suppose Y is irreducibly contained in $A_1A_2A_3$. Consider the composition factors of $L(E_7) \downarrow A_1A_2A_3$ given in Table 21. Since all A_1-modules are self-dual it follows that the restriction to Y of the composition factors $(V_{A_1}(2), V_{A_2}(00), V_{A_3}(000))$, $(V_{A_1}(0), V_{A_2}(11), V_{A_3}(000))$ and $(V_{A_1}(0), V_{A_2}(00), V_{A_3}(101))$ yield a copy of each non-trivial odd-multiplicity composition factor of $L(E_7) \downarrow X$. This is a contradiction, because the sum of the dimensions of one copy of each non-trivial odd-multiplicity composition factor is 46, which is greater than 26. Therefore, no such subgroup Y exists and X is indeed E_7-irreducible by Lemma 3.5.

Next we consider $X = E_7(\#3)$. If $p = 5$, then Corollary 3.6 applies unless $r = s = u - 1$, in which case X has one trivial composition factor on $L(E_7)$. A similar argument to the previous one shows that X is E_7-irreducible. If $p = 3$, then there are more cases when Corollary 3.6 does not apply. If $r = u + 1$, $r = s = u - 1$, $r = s = t - 1 = u - 2$ or $r = t = s - 1 = u - 2$ then X has one trivial composition factor on $L(E_7)$ and X is E_7-irreducible by a similar argument to before. The only other case where Corollary 3.6 does not apply is $r = u$ (so r, s, t are distinct), in which case X has two trivial composition factors on $L(E_7)$. The composition factors of X on $L(E_7)$ are $4^{[r]}/2^{[r]} \otimes 4^{[s]}/(2^{[r]} \otimes 2^{[s]} \otimes 2^{[t]})^2/(2^{[r]})^5/4^{[s]} \otimes 2^{[t]}/4^{[s]}/2^{[s]} \otimes 2^{[t]}/2^{[s]}/(2^{[t]})^2/0^2$. Suppose, for a contradiction, that Y is an L-irreducible subgroup A_1 of a Levi subgroup L, having the same composition factors as X on $L(E_7)$. Then using Table 21, we find the possibilities for L' are E_6, A_1D_5, A_6, A_1A_5, A_2A_4, $A_1A_2A_3$ and A_1A_4. Since $p = 3$, neither A_6 nor A_4 contain an irreducible subgroup A_1 and so we immediately rule out A_6, A_2A_4 and A_1A_4. The only E_6-irreducible A_1 subgroups when $p = 3$ are $E_6(\#2)$ and $E_6(\#3)$. Using Tables 11 and 21, we check that neither subgroup has the same composition factors as X on $L(E_7)$ for any r, s, t. Now suppose Y is contained in A_1D_5. Since $p = 3$, the projection of Y to D_5 acts on $V_{D_5}(\lambda_1)$ as $2^{[a]} + 2^{[b]} + 1^{[c]} \otimes 1^{[d]}$ $(a \neq b)$ or $2^{[a]} \otimes 2^{[b]} + 0$ $(a \neq b)$. In the first case when $c = d$ or in the second case, $L(E_7) \downarrow Y$ has at least three trivial composition factors since $(V_{A_1}(0), V_{D_5}(\lambda_1))$ is a multiplicity two composition factor of $L(E_7) \downarrow A_1D_5$, which is a contradiction. Now suppose $c \neq d$. Then by considering the multiplicity of the 3-dimensional composition factors of X on $L(E_7)$, it follows that $\{a, b\} = \{r, t\}$. But then the projection of Y to D_5 will have a $2^{[r]} \otimes 2^{[t]}$ composition factor on $V_{D_5}(\lambda_2)$, a contradiction. We can also rule out Y being contained in A_1A_5, since X and hence Y, has no 2-dimensional composition factors on $L(E_7)$. Finally, we rule out $A_1A_2A_3$ as it has no composition factors of dimension at least 27. Therefore, no such Y exists and X is E_7-irreducible by Lemma 3.5.

Now let $p = 11$, and $X_1 = E_7(\#5^{\{1,0,0\}})$ and $X_2 = E_7(\#5^{\{0,0,1\}})$ (Corollary 3.6 applies for all of the other cases). Then from Table 12, we find that $V_{56} \downarrow X_1 = 19/13/11/9^2/5/3$ and $V_{56} \downarrow X_2 = 23/21/15/9/7$. In particular, neither X_1 nor X_2 have a trivial composition factor on V_{56}. Then by Table 21, if there exists a subgroup A_1 having the same composition factors as X on V_{56} contained in a Levi subgroup, it will be contained in one of the following Levi subgroups: D_6, A_1D_5, A_6, A_1A_5, A_2A_4 or $A_1A_2A_3$. The dimensions of composition factors of X_1 and X_2 on V_{56} are $18, 10^2, 6^2, 4, 2$ and $22, 10^2, 8, 6$, respectively. Using Table 21, we see that this is incompatible with any subgroup of such a Levi subgroup. Hence Lemma 3.5 shows that both X_1 and X_2 are E_7-irreducible.

Similarly, let $p = 7$ and $X = E_7(\#6^{\{1,0,0\}})$. Then $V_{56} \downarrow X = 13/11/9/7/5^2/3^2$ with dimensions $14, 10, 6^3, 4^2, 2$. These dimensions are incompatible with any subgroup of a Levi factor, using Table 21. Hence X is E_7-irreducible by Lemma 3.5. Similar arguments show that $E_7(\#7)$ ($p = 7$), $E_7(\#8)$ and $E_7(\#9)$ (both with $p = 5$) are E_7-irreducible.

Now consider $E_7(\#10)$. Firstly, if $p = 5$ then the only case for which Corollary 3.6 does not apply is $X = E_7(\#10^{\{0,0,1,0\}})$. From Table 12, we have $V_{56} \downarrow X = 17/15/13/11/9/7/3/1$. The dimensions of these composition factors are incompatible with any subgroup of a Levi factor and hence X is E_7-irreducible by Lemma 3.5.

Now suppose $p = 3$. There are many cases where Corollary 3.6 does not apply. Let $X_1 = E_7(\#10^{\{0,0,1,0\}})$ and $X_2 = E_7(\#10^{\{1,0,1,1\}})$. Then both X_1 and X_2 have three trivial composition factors on $L(E_7)$. Suppose Y is a subgroup of a Levi factor L having the same composition factors as X_1 on V_{56} and $L(E_7)$. Using Table 21 and the number of trivial composition factors on $L(E_7)$, it follows that Y is an L'-irreducible subgroup of $L' = E_6, D_6, A_6, A_1D_5, A_1A_5, A_2A_4, A_1A_4$ or $A_1A_2A_3$. From Table 12, we have $V_{56} \downarrow X_1 = 11/9^2/7^3/5^2/3^5/1^3$. Therefore Y is not a subgroup of E_6, A_6, A_2A_4 or A_1A_4 by considering the dimensions of the composition factors on V_{56}. Suppose Y is contained in D_6. Then by Lemma 3.4, we have $V_{D_6}(\lambda_1) \downarrow Y = 2^{[r]} \otimes 2^{[s]} + 2^{[t]}$ ($r \neq s$), $2^{[r]} + 2^{[s]} + 2^{[t]} + 2^{[u]}$ (r, s, t, u distinct) or $1^{[r]} \otimes 1^{[s]} + 1^{[t]} \otimes 1^{[u]} + 1^{[v]} \otimes 1^{[w]}$ (the sets $\{r, s\}$, $\{t, u\}$, $\{v, w\}$ are distinct and at least two of them have cardinality two). But restricting from D_6, we find that Y has a 9-dimensional, 3-dimensional or 4-dimensional composition factor on V_{56}, respectively, which is a contradiction. Now suppose Y is contained in A_1D_5. Then by Lemma 3.4, we have $(V_{A_1}(1), V_{D_5}(\lambda_1)) \downarrow Y = 1^{[r]} \otimes 2^{[s]} \otimes 2^{[t]} + (1^{[r]})^2$ ($s \neq t$) or $1^{[r]} \otimes 1^{[s]} \otimes 1^{[t]} + 1^{[r]} \otimes 2^{[u]} + 1^{[r]} \otimes 2^{[v]}$ ($u \neq v$ and if $s = t$ then s, u, v distinct). This leads to a contradiction as the composition factors of Y do not match those of X_1. Similarly, if Y is contained in $A_1A_2A_3$ then $V_{56} \downarrow Y$ has a 4-dimensional composition factor, a contradiction. Finally, suppose Y is contained in A_1A_5. Then $(V_{A_1}(1), V_{A_5}(\lambda_1)) \downarrow Y = 1^{[r]} \otimes 2^{[s]} \otimes 1^{[t]}$ ($s \neq t$) and $V_{56} \downarrow Y = (1^{[r]} \otimes 2^{[s]} \otimes 1^{[t]})^2/(2^{[s]} \otimes 1^{[t]})^3/(1^{[s+1]} \otimes 1)/1^{[t+1]}/(1^{[t]})^2$. Hence Y has a 4-dimensional composition factor again, a contradiction. Therefore Y does not exist and X_1 is E_7-irreducible. The proof is almost identical for X_2 and is similar and easier for the other cases as they all have fewer trivial composition factors on $L(E_7)$.

Similar arguments show that $E_7(\#11)$ is E_7-irreducible when $p = 3, 5$ and $E_7(\#12)$ is E_7-irreducible when $p = 3$.

Now suppose $p = 2$. First consider $X = E_7(\#14)$. Suppose Y is an L'-irreducible subgroup A_1 of a Levi subgroup L with the same composition factors as X on V_{56} . From Table 12, we see that $V_{56} \downarrow X$ has a 32-dimensional composition factor. Therefore, $L' = D_6$ and from Table 21, we have $V_{56} \downarrow D_6 = \lambda_1^2/\lambda_5$. It follows that the remaining composition factors of $V_{56} \downarrow Y$ have even multiplicity, a contradiction. Therefore X is E_7-irreducible by Lemma 3.5.

Finally, let X be $E_7(\#12)$ or $E_7(\#13)$. Using Lemma 3.10 and Table 21, we find the composition factors on V_{56} of each L-irreducible subgroup A_1 of a Levi factor L'. We carefully check that they do not match the composition factors of X on V_{56}. Thus X is E_7-irreducible by Lemma 3.5. This completes the analysis of the case $M = A_1D_6$.

The next case to consider is $M = A_7$. By Lemma 3.4, it follows that X acts on $V_{A_7}(\lambda_1)$ as 7 ($p \geq 11$), $1 \otimes 1^{[r]} \otimes 1^{[s]}$ ($0 < r < s$) or $3^{[r]} \otimes 1^{[s]}$ ($p \geq 5$; $r \neq s$). In the first two cases X preserves a symplectic form on $V_{A_7}(\lambda_1)$ and hence X is contained in C_4. By [30, Lemma 6.1], this subgroup C_4 is E_7-reducible and hence so is X. In the final case, X acts as $3^{[r]} \otimes 1^{[s]}$ ($p \geq 5$; $r \neq s$) and is hence contained in D_4. The normaliser of D_4 in E_7 contains a triality automorphism of D_4, by [8, Lemma 2.15]. Hence X is E_7-conjugate to an A_7-reducible subgroup A_1 acting as $4^{[r]} + 2^{[s]}$ and there are no E_7-irreducible A_1 subgroups contained in M.

Now let $M = A_2A_5$. Then using Lemma 3.4, we see that the projection of X to A_5 is contained in C_3 and $p \geq 3$. By [15, Table 8.2], the connected centraliser of this subgroup C_3 is G_2 and hence the factor A_2 of M is contained in G_2. Moreover, by Theorem 2 the A_2-irreducible subgroup A_1 is contained in $\bar{A}_1A_1$ and hence X is contained in $\bar{A}_1A_1C_3 < \bar{A}_1D_6$. Therefore X has already been considered.

Now suppose $M = G_2C_3$. We note that the factor G_2 is contained in D_4 and hence subgroups generated by long root subgroups of G_2 are generated by long root subgroups of E_7. By Theorem 2, the projection of X to G_2 is contained in $\bar{A}_1A_1$ or is maximal with $p \geq 7$. If the projection of X to G_2 is contained in $\bar{A}_1A_1$ then X is contained in $\bar{A}_1D_6$ and has already been considered. We therefore assume the projection of X to G_2 is maximal and so $p \geq 7$. Using Lemma 3.4, we find that the projection of X to C_3 is contained in $\bar{A}_1C_2$, A_1A_1 or is maximal. If the projection of X is contained in $\bar{A}_1C_2$ then X is contained in $\bar{A}_1D_6$ and has already been considered. Now suppose the projection is contained in A_1A_1 acting as $(2,1)$ on $V_{C_3}(100)$. Then $X \hookrightarrow A_1A_1A_1 < G_2C_3$ via $(1^{[r]}, 1^{[s]}, 1^{[t]})$ ($rst = 0; s \neq t$). If $r = s$ we claim that X is contained in $\bar{A}_1D_6$. To show this we first note that X is also contained in $A_1A_1G_2 < A_1F_4$, since the factor G_2 of $A_1A_1G_2$ is contained in a D_4 Levi subgroup and is hence conjugate to the factor G_2 of M. It follows that X is conjugate to $Y \hookrightarrow A_1A_1A_1 < A_1A_1G_2 < A_1F_4$ via $(1^{[t]}, 1^{[r]}, 1^{[r]})$. Moreover, by Theorem 3, we have Y is conjugate to a subgroup of $A_1\bar{A}_1C_3 < A_1F_4$ and hence to a subgroup of $\bar{A}_1D_6$. Specifically, X is conjugate to $E_7(\#1^{\{r,r,t\}})$ and is E_7-irreducible. If $r \neq s$ then X is E_7-irreducible by Corollary 3.6, yielding $E_7(\#16)$. Finally, suppose the projection of X to C_3 is maximal, so $X \hookrightarrow A_1A_1 < G_2C_3$ via $(1^{[r]}, 1^{[s]})$ ($rs = 0$). If $r \neq s$ then X is E_7-irreducible by Corollary 3.6, giving $E_7(\#15)$. If $p > 7$ and $r = s = 0$ then Theorem 3.9 shows that X is conjugate to $E_7(\#5^{\{0,0,0\}})$ in $\bar{A}_1D_6$.

When $p = 7$ and $r = s = 0$ then we note a correction to [1, Theorem 8.13] and show that X is E_7-reducible. This is almost shown in the proof of [18, Lemma 4.6] but we provide the full argument here. Assume X is E_7-irreducible. This is almost shown in the proof of [18, Lemma 4.6] but we provide the full argument here. From Theorem 3.1, we have $V_{56} \downarrow G_2C_3 = (10, 100)/(0, 001)$ and hence $V_{56} \downarrow X = 11/9^2/7/5^2/3^4/1^2$. It is easy to check that the following Weyl modules have the indicated structure: $W(11) = 11|1$, $W(9) = 9|3$, $W(7) = 7|5$, $W(5) = 5$, $W(3) = 3$. By [11, II 4.14], only 11 extends 1 and $\mathrm{Ext}^1_{A_1}(11, 1) \cong K$, so X stabilises a module $W \cong 1$. We wish to investigate $N := N_{E_7}(W)^{\circ}$. The variety of all 2-spaces in V_{56} has dimension 108 and so N has dimension at least 25 ($= \dim(E_7) - 108$). Consider a maximal connected subgroup M_1 containing N and hence X. This subgroup is reductive (otherwise X is E_7-reducible, a contradiction) and hence listed in Theorem 3.1. The possibilities for M_1 are A_7, A_1D_6, A_2A_5, A_1F_4 and G_2C_3. Since X is E_7-irreducible and contained in N, it follows that M_1 contains an E_7-irreducible subgroup A_1 with the same composition factors as X on V_{56}. By the previous cases, A_7 does not contain any E_7-irreducible A_1 subgroups and so M_1 is not A_7. Now suppose $M_1 = A_1D_6$. Then X is conjugate to $E_7(\#n)$ where n is one of $1, 2, 3, 6, 7, \ldots, 12$. Using the composition factors given in Table 12 we see this is not possible. Next, suppose $M_1 = A_2A_5$. Then since all A_2A_5-irreducible A_1 subgroups are contained in A_1D_6, this is also impossible. Suppose $M_1 = A_1F_4$. Since $p = 7$, the subgroup A_1F_4 does not fix a 2-space on V_{56} and therefore N is properly contained in A_1F_4. Since N has dimension at least 25, it follows from Theorem 3.1 that N is contained in A_1B_4 or $A_1\bar{A}_1C_3$. In both cases, it follows that N is contained in A_1D_6 (see the $M = A_1F_4$ case below). Hence X is contained in A_1D_6, which is again impossible. Finally, suppose

$M_1 = G_2C_3$. Since G_2C_3 does not fix a 2-space of V_{56}, we have N is contained in a proper reductive, maximal connected subgroup of G_2C_3, which contains X. The only A_1 subgroups of G_2C_3 with the same composition factors as X on V_{56} are all G_2C_3-conjugate to X. It follows that the only reductive, connected proper subgroups of G_2C_3 containing X are A_1C_3, G_2A_1 and A_1A_1, where the factor A_1 subgroups are maximal in their respective factors of G_2C_3. All three subgroups have dimension less than 25. This is a contradiction, proving X is E_7-reducible.

Next, we suppose $M = A_1G_2$ ($p \neq 2$). By Theorem 2, the projection of X to G_2 is contained in A_1A_1 or is maximal with $p \geq 7$. Consider the first case. We claim that X is also contained in A_1D_6 and has already been considered. This follows by calculating the centraliser in E_7 of the involution that the A_1A_1 centralises in G_2, and finding it to be A_1D_6. Now suppose the projection of X to G_2 is maximal, so $p \geq 7$ and $X \hookrightarrow A_1A_1$ via $(1^{[r]}, 1^{[s]})$ ($rs = 0$). If $r \neq s$ then X is E_7-irreducible by Corollary 3.6, yielding $E_7(\#17)$ in Table 12. If $p \geq 11$ and $r = s = 0$ then Theorem 3.9 shows that X is conjugate to $E_7(\#5^{\{0,0,0\}})$, a subgroup of A_1D_6. Another correction to [1, Theorem 8.13] is that if $p = 7$ and $r = s = 0$ then X is E_7-reducible. This follows immediately from the argument given in the case $M = G_2C_3$ because here again we have a subgroup A_1, X, such that $V_{56} \downarrow X = 11/9^2/7/5^2/3^4/1^2$ and the argument only relied upon the composition factors on V_{56}.

Now let $M = A_1F_4$. Theorem 3 shows that the projection of X to F_4 is contained in B_4, $\bar{A}_1C_3$ ($p \neq 2$), A_1G_2 ($p \neq 2$) or A_1 ($p \geq 13$). Any subgroup of A_1B_4 is contained in $\bar{A}_1D_6$. Indeed, B_4 (or its Lie algebra if $p = 2$) has a non-trivial centre and the full centraliser of this centre is $\bar{A}_1D_6$. Similarly, if X is contained in $A_1\bar{A}_1C_3$ then it is contained in $\bar{A}_1D_6$ because the connected centraliser of $\bar{A}_1$ in E_7 is D_6. We saw in the $M = G_2C_3$ case that $A_1A_1G_2$ is contained in G_2C_3 and so X has already been considered when its projection to F_4 is contained in A_1G_2. That leaves us to consider $X \hookrightarrow A_1A_1 < A_1F_4$ via $(1^{[r]}, 1^{[s]})$ ($p \geq 13$; $rs = 0$), where the second factor A_1 is maximal in F_4. In this case Corollary 3.6 shows that X is E_7-irreducible, yielding $E_7(\#18)$.

Now suppose $M = A_2$ ($p \geq 5$). Then X acts on $V_{A_2}(10)$ as 2. First, let $p \geq 11$. By Theorem 3.1, we have $L(E_7) \downarrow M = 44/11$. From this, it follows that $L(E_7) \downarrow X = 16/14/12^2/10^2/8^3/6/4^3/2/0$. By Theorem 3.1, we have $L(E_7) \downarrow A_7 = (\lambda_1 + \lambda_7)/\lambda_4$. Letting $Y = A_1 < A_7$ with $V_{A_7}(\lambda_1) \downarrow Y = 7$, it follows that Y has the same composition factors as X on $L(E_7)$. Since $p \geq 11 > 7 = N(A_1, E_7)$, Theorem 3.9 applies. Hence X is conjugate to Y, which is contained in a parabolic subgroup of E_7. Therefore X is E_7-reducible.

For $p = 5, 7$ we show that X fixes a 1-space of V_{56}. It then follows that X is contained in a parabolic subgroup of E_7 since the dimension of the centraliser of this 1-space is at least 77. From [19, Table 10.2], we see that $V_{56} \downarrow M = 60 + 06$ ($p = 7$) and $V_{56} \downarrow M = 22|(60 + 06)|22$ ($p = 5$). When $p = 7$, we have $V_{A_2}(60) = S^6(V_{A_2}(10))$ and restricting to X yields $S^6(2) = (0|12|0) + (4|8|4)$ (this final calculation follows since $p > 6$ and thus if W is tilting then so is $S^6(W)$). Therefore X fixes a 1-space of V_{56}. When $p = 5$, we have $V_{A_2}(20) \otimes V_{A_2}(02) = (11|22|11) + 00$ and restricting to X gives $(4+0) \otimes (4+0) = (0|8|0) + (2|6|2) + 4^3 + 0$. Since $V_{A_2}(11) \downarrow X = 4 + 2$, it follows that $V_{A_2}(22) \downarrow X = (0|8|0) + 6 + 4$ and X fixes a 1-space of V_{56}. In both cases X fixes a 1-space and is hence E_7-reducible.

Now let $M = A_1A_1$ ($p \geq 5$). Then X is a diagonal subgroup of M embedded via $(1^{[r]}, 1^{[s]})$ ($rs = 0$). If $r \neq s$ then X is E_7-irreducible by Corollary 3.6, yielding $E_7(\#19)$ in Table 7. Now suppose $r = s = 0$. If $p > 7$ then Theorem 3.9 shows that X is conjugate to $E_7(\#6^{\{0,0,0\}})$ and is hence E_7-irreducible. If $p = 7$, we claim that X is also conjugate to $Y = E_7(\#6^{\{0,0,0\}})$. Restricting from M and A_1D_6, we see that X and Y have the same composition factors on $L(E_7)$ and on V_{56}. We note that Y was already shown to be E_7-irreducible when we considered A_1D_6 above, using only the composition factors of Y on $L(E_7)$ as Corollary 3.6 applies. Therefore, X is also E_7-irreducible. To prove X is conjugate to Y we follow the proof of [15, Lemma 6.7]. From Table 12, we see that $L(E_7) \downarrow X$ has no composition factors of the form $5 \otimes c^{[1]}$

where $c > 0$. Since X and Y have the same composition factors on $L(E_7)$ they have the same labelled diagram. Hence the hypothesis of [15, Lemma 6.7] holds and the proof of it shows that $L(X) = L(Y_1)$, where Y_1 is a suitable E_7-conjugate of Y. Since X is E_7-irreducible, we claim that $C := C_{E_7}(L(X))^\circ = 1$. Indeed, X normalises C and so C is reductive, otherwise X would be contained in a parabolic subgroup of E_7. Furthermore, since C is a connected reductive group, the connected group X centralises C and hence by Lemma 3.2, we have $C = 1$. Thus $C_{E_7}(L(X))^\circ = 1$ and $N_{E_7}(L(X))^\circ = X$, showing that $Y_1 = X$ and X is E_7-conjugate to Y.

If $p = 5$ we note a final correction to [1, Theorem 8.13]. We claim that $X \hookrightarrow M$ via $(1,1)$ is E_7-reducible. Suppose, for a contradiction, that X is E_7-irreducible. First, we see that $V_{56} \downarrow X = 9/7^2/5^3/3^6/1^2$ (this follows from $V_{56} \downarrow M$ which is given in Theorem 3.1). We claim the only composition factor that extends 1 is $7 = 2 \otimes 1^{[1]}$ and that $\mathrm{Ext}^1_{A_1}(7,1) \cong K$. This follows from [11, II 4.14] and the structure of the following Weyl modules: $W(9) = 9 = 4 \otimes 1^{[1]}$, $W(7) = 7|1$, $W(5) = 5|3$ and $W(3) = 3$. Since V_{56} is self-dual, $V_{56} \downarrow X$ has a submodule $W \cong 1$. By a previous argument, $N := N_{E_7}(W)^\circ$ is of dimension at least 25 and we may assume that it is contained in a reductive, maximal connected subgroup M_1 of E_7. The possibilities for M_1 are A_7, A_2A_5, A_1D_6, A_1F_4 and G_2C_3. Since X is E_7-irreducible and contained in N, it follows that M_1 contains an E_7-irreducible subgroup A_1 with the same composition factors as X on V_{56}. By the previous cases, A_7 does not contain any E_7-irreducible A_1 subgroups and so M_1 is not A_7. Since $p = 5$, it also follows that every E_7-irreducible subgroup A_1 of A_2A_5, G_2C_3 and A_1F_4 is conjugate to a subgroup of A_1D_6. Therefore, A_1D_6 contains an E_7-irreducible subgroup A_1 with the same composition factors as X on V_{56}. By the $M = A_1D_6$ case, it follows that $E_7(\#n)$, where n is one of $3, 8, 9, 10, 11$ or 12, has the same composition factors as X on V_{56}. Using Table 12, we see that this is a contradiction. Therefore X is E_7-reducible, as claimed.

Now suppose M is one of the two conjugacy classes of maximal A_1 subgroups in E_7. Then $M = X$ and X is E_7-irreducible. This accounts for the subgroups $E_7(\#20)$ and $E_7(\#21)$.

Finally, we check there are no more E_7-conjugacies between any of the irreducible A_1 subgroups by comparing the composition factors in Table 12. $\square$

9 Proof of Theorem 6: E_8-irreducible A_1 subgroups

In this section we classify the E_8-irreducible A_1 subgroups of E_8.

Theorem 6. *Suppose X is an irreducible subgroup A_1 of E_8. Then X is conjugate to exactly one subgroup of Table 8 and each subgroup in Table 8 is irreducible.*

Table 8: The E_8-irreducible A_1 subgroups of E_8

ID	M	$V_M \downarrow X$	p
1	D_8	$7^{[r]} \otimes 1^{[s]}$ $(rs = 0; r \neq s)$	≥ 11
2		$7^{[r]} \otimes 1^{[s]}$ $(rs = 0; r \neq s)$	≥ 11
3		$3 \otimes 3^{[r]}$ $(r \neq 0)$	≥ 5
4		$3 \otimes 3^{[r]}$ $(r \neq 0)$	≥ 5
5		$5^{[r]} \otimes 1^{[s]} + 1^{[t]} \otimes 1^{[u]}$ $(rstu = 0; r \neq s)$	≥ 7
6		$2^{[r]} \otimes 1^{[s]} \otimes 1^{[t]} + 1^{[u]} \otimes 1^{[v]}$ (r, s, t distinct; $rstu = 0$; $u \leq v$; if $u = v$ then $s < t$)	≥ 3
7		$4^{[r]} \otimes 2^{[s]} + 0$ $(rs = 0; r \neq s)$	≥ 5
8		$1 \otimes 1^{[r]} \otimes 1^{[s]} \otimes 1^{[t]}$ $(0 < r < s < t)$	all

9		$1 \otimes 1^{[r]} \otimes 1^{[s]} \otimes 1^{[t]}$ $(0 < r < s < t)$	≥ 3
10		$14 + 0$	≥ 17
11		$12^{[r]} + 2^{[s]}$ $(rs = 0)$	≥ 13
12		$10^{[r]} + 4^{[s]}$ $(rs = 0)$	≥ 11
13		$10^{[r]} + 1^{[s]} \otimes 1^{[t]} + 0$ $(rs = 0; s < t)$	≥ 11
14		$8^{[r]} + 6^{[s]}$ $(rs = 0)$	≥ 11
15		$8^{[r]} + 2^{[s]} + 1^{[t]} \otimes 1^{[u]}$ ($rst = 0$; $t \leq u$; if $t = u$ then $s < t$)	≥ 11
16		$6^{[r]} + 2^{[s]} \otimes 2^{[t]}$ $(rs = 0;\ s < t)$	≥ 7
17		$6^{[r]} + 4^{[s]} + 1^{[t]} \otimes 1^{[u]}$ $(rst = 0;\ t \leq u)$	≥ 7
18		$6^{[r]} + 1^{[s]} \otimes 1^{[t]} + 1^{[u]} \otimes 1^{[v]} + 0$ $(rs = 0;\ s < t < u < v)$	≥ 7
19		$6^{[r]} + 2^{[s]} + 2^{[t]} + 2^{[u]}$ $(rs = 0;\ s < t < u)$	≥ 7
20		$4 + 4^{[r]} + 4^{[s]} + 0$ $(0 < r < s)$	≥ 5
21		$4^{[r]} + 4^{[s]} + 2^{[t]} + 2^{[u]}$ $(rt = 0;\ r < s;\ t < u)$	≥ 5
22		$4^{[r]} + 2^{[s]} + 3^{[t]} \otimes 1^{[u]}$ ($rsu = 0$; $r \neq s$; $r \leq t$; $t \neq u$; if $r = t$ then $s \leq u$)	≥ 5
23		$4^{[r]} + 2^{[s]} + 1^{[t]} \otimes 1^{[u]} + 1^{[v]} \otimes 1^{[w]}$ ($rstuvw = 0$; see Table 15 for the further conditions on $r, \ldots, w$)	≥ 5
24		$2^{[r]} \otimes 2^{[s]} + 2^{[t]} + 1^{[u]} \otimes 1^{[v]}$ ($rtu = 0$; $r < s$; $u \leq v$; if $u = v$ then $u < t$)	≥ 3
25		$2^{[r]} + 2^{[s]} + 2^{[t]} + 2^{[u]} + 1^{[v]} \otimes 1^{[w]}$ ($rv = 0$; $r < s < t < u$; $v \leq w$; if $v = w$ then $u < v$)	≥ 3
26		$1 \otimes 1^{[r]} + 1^{[s]} \otimes 1^{[t]} + 1^{[u]} \otimes 1^{[v]} + 1^{[w]} \otimes 1^{[x]}$ (see Table 16 for conditions on $r, \ldots, x$)	any
27		$0\vert(2^{[r]} + 2^{[s]} + 2^{[t]})\vert 0 + 1^{[u]} \otimes 1^{[v]} + 1^{[w]} \otimes 1^{[x]}$ ($ruv = 0$; see Table 17 for the further conditions on $r, \ldots, x$)	$= 2$
28		$0\vert(2 + 2^{[r]} + 2^{[s]})\vert 0 + 1^{[t]} \otimes 1^{[u]} \otimes 1^{[v]}$ ($0 < r < s$; $t < u < v$; if $t = 0$ then $r \leq u$; if $t = 0$ and $r = u$ then $s \leq v$)	$= 2$
29		$0\vert(2^{[r]} + 2^{[s]} + 2^{[t]} + 2^{[u]} + 2^{[v]})\vert 0 + 1^{[w]} \otimes 1^{[x]}$ ($rw = 0$; $r < s < t < u < v$; $w < x$)	$= 2$
30		$0\vert(2 + 2^{[r]} + 2^{[s]} + 2^{[t]} + 2^{[u]} + 2^{[v]} + 2^{[w]})\vert 0$ $(0 < r < s < t < u < v < w)$	$= 2$
31	A_1E_7	$(1^{[r]}, E_7(\#15^{\{s,t\}}))$ $(rst = 0; s \neq t)$	≥ 7
32		$(1^{[r]}, E_7(\#16^{\{s,t,u\}}))$ $(rstu = 0; s \neq t; t \neq u)$	≥ 7
33		$(1^{[r]}, E_7(\#17^{\{s,t\}}))$ $(rst = 0; s \neq t)$	≥ 7
34		$(1^{[r]}, E_7(\#18^{\{s,t\}}))$ $(rst = 0)$	≥ 13
35		$(1^{[r]}, E_7(\#19^{\{s,t\}}))$ $(rst = 0;\ s \neq t)$	≥ 5
36		$(1^{[r]}, E_7(\#20)^{[s]})$ $(rs = 0)$	≥ 17
37		$(1^{[r]}, E_7(\#21)^{[s]})$ $(rs = 0)$	≥ 19
38	G_2F_4	$(G_2(\#3)^{[r]}, F_4(\#10)^{[s]})$ $(rs = 0; r \neq s)$	≥ 13
39		$(G_2(\#3)^{[r]}, F_4(\#11^{\{s,t\}}))$ $(rs = 0;\ r < t;\ r \neq s;\ s \neq t)$	≥ 7
40	A_1	1	≥ 23
41	A_1	1	≥ 29
42	A_1	1	≥ 31

The composition factors of $L(E_8)$ restricted to each irreducible subgroup A_1 are given in Table 13.

Proof. We consider each reductive, maximal connected subgroup M of E_8 in turn. By Theorem 3.1, they are D_8, A_8, $\bar{A}_1E_7$, A_2E_6, A_4^2, G_2F_4, B_2 $(p \geq 5)$, A_1A_2 $(p \geq 5)$, A_1 $(p \geq 23)$, A_1 $(p \geq 29)$ and A_1 $(p \geq 31)$. Let X be an M-irreducible subgroup A_1.

Firstly, suppose $M = D_8$. We start by finding the E_8-conjugacy classes of D_8-irreducible subgroups of D_8; we claim that these are $E_8(\#1)$–$E_8(\#30)$ in Table 8 as well as the class of A_1 subgroups acting as $1 \otimes 1^{[r]} \otimes 1^{[s]} \otimes 1^{[t]}$ that are excluded from $E_8(\#9)$ when $p = 2$. This is entirely similar to the case $A_1D_6 < E_7$ and is a mainly routine task of using Lemma 3.4 to find all of the D_8-conjugacy classes of D_8-irreducible subgroups and then considering which classes are fused in E_8. We will just give some details on the D_8-classes which are fused in E_8.

First, we note that the excluded class of A_1 subgroups acting as $1 \otimes 1^{[r]} \otimes 1^{[s]} \otimes 1^{[t]}$ on $V_{D_8}(\lambda_1)$ when $p = 2$ are contained in $B_4(\ddagger)$, with notation from [30, Lemma 7.1]. By [30, Lemma 7.4], this subgroup B_4 is contained in a parabolic subgroup of E_8 and hence so is the class of A_1 subgroups.

The D_8-classes of A_1 subgroups which are fused in E_8 are all contained in the maximal rank subsystem subgroups $A_1^2D_6$ or D_4^2. By [7, Table 11], we have $N_{E_8}(A_1^2D_6) = (A_1^2D_6).2$ where the involution simultaneously acts a graph automorphism of D_6 and swaps the two A_1 factors. Consider a subgroup A_1 acting on $V_{D_8}(\lambda_1)$ as $5^{[r]} \otimes 1^{[s]} + 1^{[t]} \otimes 1^{[u]}$ $(r \neq s; rstu = 0)$ when $p \geq 7$. There are two D_8-classes of such D_8-irreducible A_1 subgroups, since there are two D_6-classes of A_1 subgroups acting as $5^{[r]} \otimes 1^{[s]}$ $(r \neq s)$ on $V_{D_6}(\lambda_1)$. These classes are fused in E_8 by an involution in $N_{E_8}(A_1^2D_6)$, yielding $E_8(\#5)$. Similarly, consider a subgroup A_1 acting on $V_{D_8}(\lambda_1)$ as $2^{[r]} \otimes 1^{[s]} \otimes 1^{[t]} + 1^{[u]} \otimes 1^{[v]}$ (r, s, t distinct; $rstuv = 0$) when $p \neq 2$. There is just one D_8-class of such D_8-irreducible A_1 subgroups, since the graph automorphism of D_6 just swaps s and t. In E_8 we may swap u and v or if $u = v$ then we may swap s and t. Therefore, to have a complete set of representatives without repeats we need $u < v$ or $u = v$ and $s < t$, as in $E_8(\#6)$. Similar arguments apply to yield $E_8(\#13)$, $E_8(\#15)$, $E_8(\#17)$ and $E_8(\#25)$.

We now consider subgroups of D_4^2. We have $N_{E_8}(D_4^2) = (D_4^2).(2 \times S_3)$ by [7, Table 11], where the S_3 acts simultaneously on both D_4 factors and the involution commuting with S_3 swaps the D_4 factors. Firstly, let Y be a subgroup A_1 of D_4^2 acting as $4^{[r]} \otimes 2^{[s]}$ or $3^{[r]} \otimes 1^{[s]}$ (2 classes) on the first D_4 factor and as $4^{[t]} \otimes 2^{[u]}$ or $3^{[t]} \otimes 1^{[u]}$ (2 classes) on the second D_4 factor. By using a triality automorphism, we may assume Y acts as $4^{[r]} + 2^{[s]}$ on the first factor D_4. The projection of Y to the first factor D_4 lies in $\mathrm{SO}_5\mathrm{SO}_3$ and is hence contained in the centraliser of an involution in the S_3. Therefore, we may act by this involution on the second factor D_4 reducing the possibilities, up to E_8-conjugacy, to $4^{[t]} + 2^{[u]}$ or $3^{[t]} \otimes 1^{[u]}$ (1 class). Furthermore, if $r = s$ then the projection of Y lies in A_2, which is the centraliser of a triality automorphism. We may therefore assume that the projection of Y to the second factor D_4 acts as $4^{[t]} + 2^{[u]}$. This analysis leads to the classes $E_8(\#21)$ and $E_8(\#22)$ in Table 8. We note that in $E_8(\#21)$ we may assume $r \leq s$ (and $t \leq u$) since the Weyl group of D_8 contains an involution swapping the stabilisers of the two 5-spaces, $4^{[r]}$ and $4^{[s]}$ (the stabilisers of the two 3-spaces, $2^{[t]}$ and $2^{[u]}$). Also, the involution swapping the two D_4 factors allows us to assume $r \leq t$ in $E_8(\#22)$.

A similar analysis when $p = 2$ and Y acts as either $0|(2^{[r]} + 2^{[s]} + 2^{[t]})|0$ or $1^{[r]} \otimes 1^{[s]} \otimes 1^{[t]}$ on each of the D_4 factors, leads to two collections of conjugacy classes, namely an E_8-reducible one and $E_8(\#28)$.

Next we consider subgroups contained in $A_1^4D_4$. By [7, Table 11], we see that $N_{E_8}(A_1^4D_4) = (A_1^4D_4).S_4$, where the S_4 acts naturally on the four A_1 factors and induces an action of S_3 on the D_4 (with the normal Klein four-subgroup acting trivially). Let Y be a D_8-irreducible subgroup A_1 of $A_1^4D_4$ that is not contained in A_1^8 (we will consider this in the next paragraph). Then the projection of Y to D_4 acts as $6^{[r]} + 0$ $(p \geq 7)$, $4^{[r]} + 2^{[s]}$ $(p \geq 3)$, $3^{[r]} \otimes 1^{[s]}$ $(p \geq 3; r \neq s)$, $0|(2 + 2^{[r]} + 2^{[s]})|0$ $(p = 2; 0 < r < s)$ or

$1 \otimes 1^{[r]} \otimes 1^{[s]}$ ($p = 2; 0 < r < s$) on $V_{D_4}(\lambda_1)$. In the first case the projection of Y is contained in G_2 and hence centralised by the action of S_3. In this case the action of S_4 on A_1^4 allows us to assume $s < t < u < v$, yielding $E_8(\#18)$. In the second and third cases, we may use the triality automorphism to assume Y acts as $4^{[r]} + 2^{[s]}$. If $r = s$ then the projection of Y is centralised by the action of S_3; whereas when $r \neq s$ the projection of Y is only centralised by an involution in S_3. This yields the constraints on the field twists in $E_8(\#23)$. Similarly, the fourth and fifth cases yield $E_8(\#27)$.

Finally, we consider the classes of irreducible A_1 subgroups contained in A_1^8. By [7, Table 11], we have $N_{E_8}(A_1^8) = (A_1^8).\mathrm{AGL}(3,2)$, where $\mathrm{AGL}(3,2) < S_8$ acts on the eight A_1 factors. The subgroup $\mathrm{AGL}(3,2) < S_8$ is generated by $(2,4)(6,8)$, $(2,5,3)(4,6,7)$ and $(1,2)(3,4)(5,6)(7,8)$. To find the E_8-classes of the D_8-irreducible A_1 subgroups, we follow the same method as for $A_1^7 < E_7$, systematically formulating constraints on the field twists, ensuring that each ordered set $0, r, s, t, u, v, w, x$ gives a D_8-irreducible and there are no repeated classes. We note that $\mathrm{AGL}(3,2)$ is 3-transitive, with two orbits on 4-sets, with representatives (in terms of the eight field twists) given by $0, r, s, t$ and $0, r, s, u$. Moreover the stabiliser of a singleton is isomorphic to $\mathrm{PSL}(2,7)$, the stabiliser of a pair is isomorphic to $\mathbb{Z}_2 \times S_4$, the stabiliser of a triple is isomorphic to S_4, as is the stabiliser of either class of quadruples. From this, it is straightforward to prove the ordered sets $0, r, \ldots, x$ satisfying the conditions of Table 16 yield a complete set of E_8-conjugacy classes of D_8-irreducible A_1 subgroups contained in A_1^8, without repeat. This gives $E_8(\#26)$ in Table 8.

For the case $M = D_8$, it remains to prove that $E_8(\#1)$–$E_8(\#30)$ are E_8-irreducible. Firstly, by considering the composition factors from Table 13, Corollary 3.6 shows that $E_8(\#n)$ is E_8-irreducible for the following ID numbers n: 1, 2 ($p \geq 13$), 3 ($p \geq 7$), 4, 5 ($p \geq 11$), 6 ($p \geq 7$), 7, 8 ($p \neq 2$), 9 ($p \geq 5$), 10, 11, 12, 13 ($p \geq 13$), 14, 15 ($p \geq 13$), 16 ($p \geq 11$), 17 ($p \geq 11$), 18 ($p \geq 11$), 19, 20, 21, 22 ($p \geq 7$), 23 ($p \geq 7$), 24 ($p \geq 7$), 25 ($p \geq 5$) and 26 ($p \geq 5$). We now prove the remaining subgroups are E_8-irreducible using Lemma 3.5.

First, let $X = E_8(\#2)$ when $p = 11$. From Table 13, we have $L(E_8) \downarrow X$ has a trivial composition factor only when $r = 0, s = 1$ and Corollary 3.6 applies otherwise. In the case $r = 0, s = 1$, we have $L(E_8) \downarrow X = 40/34/30^2/26^2/22^2/20^2/16^2/14^2/12/10/6^2/4/2/0$. Assume there exists an L-irreducible subgroup Y of a Levi factor L with the same composition factors as X on $L(E_8)$. Using Table 22, we find that the possibilities for L' are A_1E_6, D_7, A_2D_5, A_7, A_3A_4, A_1A_6 and $A_1A_2A_4$. We rule out A_1E_6 since it has a 2-dimensional composition factor on $L(E_8)$. Suppose Y is contained in D_7. Using Table 22, we see that $V_{D_7}(\lambda_1)$ occurs as a multiplicity two composition factor of $L(E_8) \downarrow D_7$. Therefore, Y does not have the same composition factors as X on $L(E_8)$, since there are no combination of composition factors of $L(E_8) \downarrow X$ that form two isomorphic 14-dimensional modules. Now suppose Y is contained in A_2D_5. It follows from the composition factors of X, and by assumption Y, on $L(E_8)$ that $(V_{A_2}(00), V_{D_5}(\lambda_1)) \downarrow Y = 6 + 2^{[1]}$ and thus $(V_{A_2}(00), V_{D_5}(\lambda_4)) \downarrow Y = 6 \otimes 1^{[1]} + 1^{[1]}$. Since $(V_{A_2}(00), V_{D_5}(\lambda_4))$ occurs as a composition factor of A_2D_5 on $L(E_8)$, we see that Y has a 2-dimensional composition factor on $L(E_8)$, a contradiction. Similarly, if Y is contained in A_7 then $V_{A_7}(\lambda_1) \downarrow Y = 14 = 3 \otimes 1^{[1]}$. Therefore, $V_{A_7}(\lambda_3) \downarrow Y$, which occurs as a composition factor of $L(E_8) \downarrow A_7$, has a composition factor of high weight 36, a contradiction. Now suppose Y is contained in A_3A_4. From Table 22, we see that $(V_{A_3}(100), V_{A_4}(0000))$ and $(V_{A_3}(001), V_{A_4}(0000))$ both occur as composition factors of $L(E_8) \downarrow A_3A_4$. But $L(E_8) \downarrow X$ has only one composition factor of dimension four and so the projection of Y to A_3 is not A_3-irreducible, a contradiction. Now suppose Y is contained in A_1A_6. Then we find that $V_{A_6}(\lambda_1) \downarrow Y = 6$ and so both $V_{A_6}(\lambda_3) \downarrow Y$ and $V_{A_6}(\lambda_4) \downarrow Y$ have a composition factor of high weight 12. Therefore, $L(E_8) \downarrow Y$ has at least two composition factors of high weight 12, a contradiction. Finally, suppose Y is contained in $A_1A_2A_4$. The largest dimension of a composition factor of $L(E_8) \downarrow A_1A_2A_4$ is 30 and hence $L(E_8) \downarrow Y$ does not have a composition factor of dimension 32, a contradiction. We have hence shown that no such subgroup Y exists and so Lemma 3.5 shows that X is E_8-irreducible.

Similar arguments show that $E_8(\#3)$ $(p = 5)$, $E_8(\#6)$ $(p = 5)$, $E_8(\#13)$ $(p = 11)$, $E_8(\#16)$ $(p = 7)$, $E_8(\#18)$ $(p = 7)$, $E_8(\#24)$ $(p = 5)$, $E_8(\#25)$ $(p = 3)$ and $E_8(\#26)$ $(p = 3)$ are E_8-irreducible, as they have at most one trivial composition factor on $L(E_8)$.

We next consider the remaining cases when $p \neq 2$. First let $X = E_8(\#5)$ when $p = 7$. Then X has a trivial composition factor on $L(E_8)$ when $r = s - 1 = u$ and X has two trivial composition factors when $r = s - 1 = t = u$. Using Lemma 3.5, we will show that X is E_8-irreducible when $r = s - 1 = t = u$. The case $r = s - 1 = u \neq t$ is similar and in all other cases Corollary 3.6 applies. Since $rstu = 0$, we have $r = t = u = 0$, $s = 1$ and from Table 13, we see that $L(E_8) \downarrow X = 22^2/20/18^3/16/14^3/12^5/10^5/8^2/6/4^2/2^7/0^2$. Suppose there exists an L-irreducible subgroup Y of a Levi factor L with the same composition factors as X on $L(E_8)$. Using Table 22 and considering the number of trivial composition factors of $L(E_8) \downarrow X$, we find that the possibilities for L' are A_1E_6, D_7, A_2D_5, A_2D_4, A_7, A_3A_4, A_1A_6, $A_1A_2A_4$, $A_1^2A_4$, A_3^2 and $A_1^2A_2^2$. We rule out L' being A_1E_6, $A_1^2A_4$ or $A_1^2A_2^2$ since they have 2-dimensional composition factors on $L(E_8)$. We also rule out A_1A_6 since A_6 does not contain an A_6-irreducible subgroup A_1 when $p = 7$, by Lemma 3.4. Now suppose Y is contained in D_7. From Table 22, we see $V_{D_7}(\lambda_1)$ occurs as a multiplicity two composition factor of $L(E_8) \downarrow D_7$. As $L(E_8) \downarrow X$ has only one 7-dimensional composition factor, two 5-dimensional composition factors and two trivial composition factors, it follows that $V_{D_7}(\lambda_1) \downarrow Y = 2^{[a]} + 2^{[b]} + 1^{[c]} \otimes 1^{[d]} + 1^{[e]} \otimes 1^{[f]}$ with $c \neq d$ and $e \neq f$. Therefore, $L(E_8) \downarrow Y$ has at least four 4-dimensional composition factors, a contradiction. Now suppose Y is contained in A_2D_5. Then

$$\begin{aligned} L(E_8) \downarrow A_2D_5 = &(W(11), 0)/(10, \lambda_1)/(10, \lambda_4)/(10, 0)/(01, \lambda_1)/(01, \lambda_5)/(01, 0)/(00, W(\lambda_2))/ \\ &(00, \lambda_4)/(00, \lambda_5)/(00, 0). \end{aligned}$$

The projection of Y to A_2 acts as $2^{[a]}$ on 10 and hence has composition factors $4^{[a]}/2^{[a]}$ on $W(11)$. We also have $V_{D_5}(\lambda_4) = V_{D_5}(\lambda_5)^*$. Therefore, $(10, \lambda_1) \downarrow Y = (01, \lambda_1) \downarrow Y$, $(10, \lambda_4) \downarrow Y = (01, \lambda_5) \downarrow Y$ and $(00, \lambda_4) \downarrow Y = (00, \lambda_5) \downarrow Y$. It follows that $(00, \lambda_2) \downarrow Y$ contains at least one copy each Y-composition factor of $L(E_8)$ occurring with odd multiplicity (except for possibly a composition factor of high weight 2). The sum of the dimensions of such composition factors is 75 which is greater than $45 = \dim(V_{D_5}(\lambda_2))$, a contradiction. The previous argument does not use the D_5-irreducibility of Y and hence also shows that Y is not contained in A_2D_4.

Now suppose Y is contained in A_7. Since $p = 7$ and Y is A_7-irreducible, Y acts as $3^{[a]} \otimes 1^{[b]}$ $(a \neq b)$ or $1^{[a]} \otimes 1^{[b]} \otimes 1^{[c]}$ $(a, b, c$ distinct) on $V_{A_7}(\lambda_1)$. In the latter case, Y is contained in C_4, which has three trivial composition factors on $L(E_8)$ (by [15, Table 8.1]), hence Y has at least three trivial composition factors, a contradiction. So Y acts as $3^{[a]} \otimes 1^{[b]}$ on $V_{A_7}(\lambda_1)$. From Table 22, we have $L(E_8) \downarrow A_7 = (\lambda_1 + \lambda_7)/\lambda_1/\lambda_2/\lambda_3/\lambda_5/\lambda_6/\lambda_7/0$. Since $\lambda_i = \lambda_{8-i}^*$ for $i = 1, 2, 3$ it follows that $(\lambda_1 + \lambda_7) \downarrow Y$ has at least one copy of each odd multiplicity composition factor of $L(E_8) \downarrow Y$. The sum of the dimensions of such composition factors is at least 78, which is greater than 63, a contradiction. Now suppose Y is contained in A_3A_4. Then $L(E_8) \downarrow A_3A_4$ has one trivial composition factor. All of the other composition factors occur in pairs with their duals, except for $(V_{A_3}(101), V_{A_4}(0000))$ and $(V_{A_3}(000), V_{A_4}(1001))$. Since Y has exactly two trivial composition factors, it follows that $V_{A_3}(101)$ restricted to the projection of Y to A_3 or $V_{A_4}(1001)$ restricted to the projection of Y to A_4 has exactly one trivial composition factor (and not both). However, the projection of Y to A_3 and the projection to A_4 are irreducible and so act as $1 \otimes 1^{[a]}$ $(a \neq 0)$ or $3^{[a]}$ and $4^{[a]}$ on the natural module, respectively. Neither action on $V_{A_3}(100)$ yields a trivial composition factor on $V_{A_3}(101)$ and the action on $V_{A_4}(1000)$ does not yield a trivial composition factor on $V_{A_4}(1001)$ either. Hence Y is not contained in A_3A_4. A similar argument also rules out $A_1A_2A_4$. Finally, suppose Y is contained in A_3^2. Then $(V_{A_3}(101), V_{A_3}(000)) \downarrow Y$ and $(V_{A_3}(101), V_{A_3}(000)) \downarrow Y$ have a least one copy of each odd multiplicity composition factor of $L(E_8) \downarrow Y$. As before, the sum of the dimensions of one copy of each odd multiplicity isomorphism class of composition factors is 78, which is greater than 30, a contradiction. We have shown that no such subgroup Y exists and hence X is E_8-irreducible by Lemma 3.5.

Similar arguments show that $E_8(\#15)$ $(p = 11)$, $E_8(\#17)$ $(p = 7)$, $E_8(\#22)$ $(p = 7)$ and $E_8(\#24)$ $(p = 3)$ are E_8-irreducible, as they have at most two trivial composition factors on $L(E_8)$. The remaining cases when $p \neq 2$ are $E_8(\#6)$ $(p = 3)$, $E_8(\#9)$ $(p = 3)$, $E_8(\#22)$ $(p = 5)$ and $E_8(\#23)$ $(p = 3)$. They all have at most four trivial composition factors on $L(E_8)$ (in fact, $E_8(\#12)$ has at most three). We will consider one of the cases in which $E_8(\#6)$ has four trivial composition factors and prove it is E_8-irreducible. The other cases are all similar.

Let $X = E_8(\#6)$ and $p = 3$. When $s = u = v = r - 1 = t - 2$, we see from Table 13 that X has four trivial composition factors on $L(E_8)$. Since $rst = 0$, we have $s = u = v = 0$, $r = 1$, $t = 2$ and $L(E_8) \downarrow X = 30/28/26^2/24/22/18^4/16^5/14^2/12^7/10^2/6/4^2/2^6/0^4$. As usual, we suppose there exists an L-irreducible subgroup Y of a Levi factor L with the same composition factors as X on $L(E_8)$. Since $p = 3$ there are a few Levi subgroups L, that although they have four or fewer trivial composition factors on $L(E_8)$, do not have an L'-irreducible subgroup A_1. Using Table 22, we find that the possibilities for L' are E_7, A_1E_6, D_7, A_2D_5, A_1D_5, A_2D_4, A_7, A_1A_5, A_3^2 and $A_1^2A_2^2$. We immediately rule out L' being A_1E_6, A_1D_5 or A_1A_5 since they have 2-dimensional composition factors on $L(E_8)$. We also rule out L' being A_2D_4, A_3^2 or $A_1^2A_2^2$ since they do not have at least two composition factors of dimension at least 27.

Now suppose that Y is contained in A_7. Then Y acts as $1^{[a]} \otimes 1^{[b]} \otimes 1^{[c]}$ $(a, b, c$ distinct$)$. Therefore, $V_{A_7}(\lambda_1 + \lambda_7) \downarrow Y = 2^{[a]} \otimes 2^{[b]} \otimes 2^{[c]}/2^{[a]} \otimes 2^{[b]}/2^{[a]} \otimes 2^{[c]}/2^{[b]} \otimes 2^{[c]}/2^{[a]}/2^{[b]}/2^{[c]}/0$. In particular, $L(E_8) \downarrow Y$ has at least three 9-dimensional composition factors, a contradiction.

Now suppose Y is contained in E_7. Then Y is conjugate to $E_7(\#3)$, $E_7(\#10)$, $E_7(\#11)$ or $E_7(\#12)$ by Theorem 5. From Table 22, we have $L(E_8) \downarrow E_7 = \lambda_1/\lambda_7^2/0^3$. Since $L(E_8) \downarrow Y$ has exactly two 27-dimensional composition factors, which are isomorphic to each other (both are $26 = 2 \otimes 2^{[1]} \otimes 2^{[2]}$) it follows that $V_{E_7}(\lambda_7) \downarrow Y$ has exactly one 27-dimensional composition factor or $V_{E_7}(\lambda_1) \downarrow Y$ has two isomorphic 27-dimensional composition factors. Using Table 12, we see that this is not true for $E_7(\#10)$, $E_7(\#11)$ and $E_7(\#12)$. Therefore Y is conjugate to $E_7(\#3)$. But then $L(E_8) \downarrow Y$ has a 2-dimensional composition factor coming from $V_{E_7}(\lambda_1) \downarrow Y$, a contradiction.

Suppose Y is contained in D_7. Then since Y is D_7-irreducible and $p = 3$, it acts on $V_{D_7}(\lambda_1)$ as $2^{[a]} + 2^{[b]} + 2^{[c]} + 1^{[d]} \otimes 1^{[e]} + 0$ $(a, b, c$ distinct; $d \neq e)$, $2^{[a]} + 2^{[b]} + 1^{[c]} \otimes 1^{[d]} + 1^{[e]} \otimes 1^{[f]}$ $(a \neq b$; $c \neq d$; $e \neq f$; $\{c, d\} \neq \{e, f\})$ or $2^{[a]} \otimes 2^{[b]} + 1^{[c]} \otimes 1^{[d]} + 0$ $(a \neq b$; $c \neq d)$. The first action is impossible, since $V_{D_7}(\lambda_1)$ occurs as a multiplicity two composition factor in $L(E_8) \downarrow D_7$ and $L(E_8) \downarrow Y$ only has two non-isomorphic 3-dimensional composition factors. Similarly, the latter action is also impossible, since $L(E_8) \downarrow Y$ only has one 9-dimensional composition factor. So Y acts as $2^{[a]} + 2^{[b]} + 1^{[c]} \otimes 1^{[d]} + 1^{[e]} \otimes 1^{[f]}$ $(a \neq b$, $c \neq d$, $e \neq f$ and $\{c, d\} \neq \{e, f\})$ on $V_{D_7}(\lambda_1)$. It follows that

$$\begin{aligned} V_{D_7}(\lambda_2) \downarrow Y = & 2^{[r]} \otimes 2^{[s]}/2^{[r]} \otimes 1^{[t]} \otimes 1^{[u]}/2^{[r]} \otimes 1^{[v]} \otimes 1^{[w]}/2^{[r]}/2^{[s]} \otimes 1^{[t]} \otimes 1^{[u]}/ \\ & 2^{[s]} \otimes 1^{[v]} \otimes 1^{[w]}/2^{[s]}/2^{[t]}/1^{[t]} \otimes 1^{[u]} \otimes 1^{[v]} \otimes 1^{[w]}/2^{[u]}/2^{[v]}/2^{[w]} \end{aligned}$$

and

$$\begin{aligned} V_{D_7}(\lambda_6) \downarrow Y = V_{D_7}(\lambda_7) \downarrow Y = & 1^{[r]} \otimes 1^{[s]} \otimes 1^{[t]} \otimes 1^{[v]}/1^{[r]} \otimes 1^{[s]} \otimes 1^{[t]} \otimes 1^{[w]}/ \\ & 1^{[r]} \otimes 1^{[s]} \otimes 1^{[u]} \otimes 1^{[v]}/1^{[r]} \otimes 1^{[s]} \otimes 1^{[u]} \otimes 1^{[w]}. \end{aligned}$$

Hence $L(E_8) \downarrow Y$ has no 27-dimensional composition factors, a contradiction. Therefore Y does not exist and X is E_8-irreducible by Lemma 3.5.

The final step for $M = D_8$ is to consider the case $p = 2$, where X is one of $E_8(\#n)$ where $n = 8, 26, 27, 28, 29$ or 30. As with the previous cases, we use Lemma 3.5 to prove X is E_8-irreducible. Lemma 3.10 contains all L-irreducible A_1 subgroups of Levi factors L when $p = 2$. For such a subgroup Y, one can write down the composition factors of $L(E_8) \downarrow Y$ and compare them to those of $L(E_8) \downarrow X$ from Table 13. In all cases it is straightforward to show they never match and we leave the details to the reader. This completes the case $M = D_8$.

We next consider the case $M = A_8$. Using Lemma 3.4, we see that X acts on $V_{A_8}(\lambda_1)$ as 8 ($p \geq 11$) or $2 \otimes 2^{[r]}$ ($p \geq 3; r \neq 0$). In both cases X preserves an orthogonal form on $V_{A_8}(\lambda_1)$ and is hence contained in B_4. By [15, Table 8.1], we have that this subgroup B_4 is also contained in D_8 acting irreducibly on the natural module. So in the first case, X is contained in D_8 and is conjugate to $E_8(\#9)$. In the second case, X acts as $W(3) \otimes 1^{[r]} + 1^{[r]} \otimes W(3)$ on $V_{D_8}(\lambda_1)$ (by [15, Prop 2.13]). If $p = 3$ then X is D_8-reducible by Lemma 3.4 and hence E_8-reducible. If $p > 3$, then X is conjugate to $E_8(\#21^{\{0,r,0,r\}})$ and hence E_8-irreducible.

Now let $M = \bar{A}_1 E_7$. The projection of X to E_7 is E_7-irreducible and so by Theorem 5, it is E_7-conjugate to a subgroup in Table 7. Let Y be the projection of X to E_7 so X is a diagonal subgroup of $\bar{A}_1 Y$. We now analyse the different possibilities for Y from Theorem 5. Suppose Y is contained in $\bar{A}_1 D_6$. Then X is contained in $\bar{A}_1^2 D_6$ which is a subgroup of D_8, and has hence already been considered.

Next, suppose Y is $E_7(\#15)$ or $E_7(\#16)$ and so Y is contained in $G_2 C_3$. Consider the first case, $Y = E_7(\#15)$. Then $X \hookrightarrow \bar{A}_1 A_1 A_1$ via $(1^{[r]}, 1^{[s]}, 1^{[t]})$ $(rst = 0; s \neq t)$, where the second factor A_1 is maximal in G_2 and the third factor A_1 is maximal in C_3. Corollary 3.6 shows that X is E_8-irreducible, yielding $E_8(\#31)$. Now let $Y = E_7(\#16)$ and so $X \hookrightarrow \bar{A}_1 A_1 A_1 A_1 < \bar{A}_1 G_2 C_3$ via $(1^{[r]}, 1^{[s]}, 1^{[t]}, 1^{[u]})$ $(rstu = 0; s \neq t; t \neq u)$, where the second factor is maximal in G_2 and $A_1 A_1 < C_3$ acts as $(2, 1)$ on $V_{C_3}(100)$. If $p > 7$ then Corollary 3.6 shows that X is E_8-irreducible. When $p = 7$, Corollary 3.6 applies except when $r = t = 1$, $s = u = 0$. Using the restriction of $L(E_8)$ to $E_8(\#32)$ in Table 13, we calculate that $L(E_8) \downarrow X = 58/44/36/34/30^3/28/26^2/22/14^4/12^2/10^2/2^3/0$. Suppose Z is an L-irreducible subgroup of a Levi factor L having the same composition factors as X on $L(E_8)$. Since $L(E_8) \downarrow X$ has only one trivial composition factor, the possibilities for L' are D_7, A_7, $A_1 E_6$, $A_1 A_6$, $A_2 D_5$, $A_1 E_6$ and $A_3 A_4$. Suppose $L' = D_7$. From Table 22, we see that $V_{D_7}(\lambda_1) \downarrow Z$ occurs as a multiplicity two composition factor of $L(E_8) \downarrow Z$. This is a contradiction, because $L(E_8) \downarrow X$ does not have a set of composition factors that form two isomorphic 14-dimensional modules. Now suppose $L' = A_7$. By considering the even multiplicity 8-dimensional composition factors of X, it follows that Z acts as $10 = 3 \otimes 1^{[1]}$ on $V_{A_7}(\lambda_1)$. Since $V_{A_7}(\lambda_2)$ occurs a composition factor of $L(E_8) \downarrow A_7$, it follows that Z has a composition factor of high weight 18 on $L(E_8)$. This is a contradiction. Suppose $L' = A_1 E_6$. Then $L(E_8) \downarrow A_1 E_6$ has a 2-dimensional composition factor and therefore Z does, a contradiction. Now suppose $L' = A_1 A_6$. Then the projection of Z to A_6 acts as 6 on $V_{A_6}(\lambda_1)$, by Lemma 3.4. From Table 22, we have $L(E_8) \downarrow A_1 A_6$ has a composition factor $(V_{A_1}(0), V_{A_6}(\lambda_1))$ and therefore $L(E_8) \downarrow Z$ has a composition factor of dimension 7, a contradiction. Now let $L' = A_2 D_5$. The only composition factors of $L(E_8) \downarrow A_2 D_5$ with dimension at least 35 are $(V_{A_2}(00), V_{D_5}(\lambda_2))$, $(V_{A_2}(10), V_{D_5}(\lambda_4))$ and $(V_{A_2}(01), V_{D_5}(\lambda_5))$. The composition factor of high weight 34 of $L(E_8) \downarrow X$ has dimension 35 and so one of the three composition factors of dimension at least 35 has 34 as a composition factor when restricted to Z. Since $\lambda_4^* = \lambda_5$ and $10^* = 01$, we have $(V_{A_2}(10), V_{D_5}(\lambda_4)) \downarrow Z = (V_{A_2}(01), V_{D_5}(\lambda_5)) \downarrow Z$ but 34 occurs with multiplicity one. Therefore, $(V_{A_2}(00), V_{D_5}(\lambda_2)) \downarrow Z = 34/M_1/\ldots/M_k$. The sum of the dimensions of $M_1, \ldots, M_k$ is 10 but no set of composition factors of $L(E_8) \downarrow Z$ have dimensions that sum to 10. This is a contradiction, ruling out $A_2 D_5$. Finally, suppose $L' = A_3 A_4$. Then $L(E_8) \downarrow A_3 A_4$ has $(V_{A_3}(000), V_{A_4}(0100))$ as a composition factor. This has dimension 10, but we just noted that $L(E_8) \downarrow Z$ has no set of composition factors whose dimensions sum to 10. This final contradiction shows that Z does not exist and therefore Lemma 3.5 shows X is E_8-irreducible when $p = 7$. This yields $E_8(\#32)$ in Table 8.

Next we consider the case where Y (the projection of X to E_7) is $E_7(\#17)$ and so contained in $A_1 G_2$. Then

$X \hookrightarrow \bar{A}_1 A_1 A_1 < \bar{A}_1 A_1 G_2$ via $(1^{[r]}, 1^{[s]}, 1^{[t]})$ $(p \geq 7; rst = 0; s \neq t)$ where the third factor A_1 is maximal in G_2. We see that X is E_8-irreducible by Corollary 3.6, yielding $E_8(\#33)$. Similarly, when Y is $E_7(\#18)$ we have $X \hookrightarrow \bar{A}_1 A_1 A_1 < \bar{A}_1 A_1 F_4$ via $(1^{[r]}, 1^{[s]}, 1^{[t]})$ $(p \geq 13; rst = 0)$ where the third factor A_1 is maximal in F_4. Again, X is E_8-irreducible by Corollary 3.6, giving $E_8(\#34)$ in Table 8.

Suppose Y is $E_7(\#19)$ and so $X \hookrightarrow \bar{A}_1 A_1 A_1$ via $(1^{[r]}, 1^{[s]}, 1^{[t]})$ $(p \geq 5; st = 0; s \neq t)$ where $A_1 A_1 < E_7$ is maximal. When $p > 5$, Corollary 3.6 applies. When $p = 5$, there is one trivial composition factor on $L(E_8) \downarrow X$ when $(r, s, t) = (0, 1, 0)$ or $(0, 0, 1)$ and none otherwise. We can use Lemma 3.5 in exactly the same way as for $E_8(\#32)$ $(p = 7)$ to show X is E_8-irreducible. This gives $E_8(\#35)$ in Table 8.

Finally, suppose Y is conjugate to $E_7(\#20)$ or $E_7(\#21)$. Then X is E_8-irreducible by Corollary 3.6, yielding $E_8(\#36)$ and $E_8(\#37)$, respectively. This concludes the $M = \bar{A}_1 E_7$ case.

Now let $M = A_2 E_6$. The projection of X to A_2 acts as 2 on $V_{A_2}(10)$ and $p \neq 2$. Let Y be the projection of X to E_6. By Theorem 4, we see that Y is contained in $\bar{A}_1 A_5$, A_2^3, $A_2 G_2$, F_4 or C_4. We claim that in all of the cases X is contained in either D_8 or $\bar{A}_1 E_7$ and has therefore already been considered. If Y is contained in $\bar{A}_1 A_5$ then X is contained in $\bar{A}_1 A_2 A_5$, which is a subgroup of $\bar{A}_1 E_7$. If Y is contained in A_2^3 then it is also contained in C_4 by the proof of Theorem 4. So when Y is either $E_6(\#3)$ or $E_6(\#6)$, X is contained in $A_1 C_4$. The irreducible subgroup A_1 of A_2 is the centraliser of a graph automorphism of A_2 and similarly, C_4 is the centraliser in E_6 of a graph automorphism of E_6. By [7, Table 11], we have $N_{E_8}(A_2 E_6) = (A_2 E_6).2$ where the involution acts as a graph automorphism on both the A_2 and the E_6 factors. Therefore, there exists an involution t in E_8 such that $A_1 C_4 < C_{E_8}(t)^\circ$. By [10, Table 4.3.1], we have $C_{E_8}(t)^\circ$ is either D_8 or $\bar{A}_1 E_7$ and hence X is contained in D_8 or $\bar{A}_1 E_7$. In fact, we have $C_{E_8}(t)^\circ = D_8$.

Next, suppose that Y is contained in $A_2 G_2$. Then the factor G_2 is generated by root subgroups of E_8 and hence X is contained in $G_2 C_{E_8}(G_2)^\circ = G_2 F_4$. In particular, the projection of X to F_4 is contained in $A_2 A_2$. The proof of Theorem 3 shows that the $A_2 A_2$-irreducible A_1 subgroups of $A_2 A_2$ are also contained in $\bar{A}_1 C_3 < F_4$. Therefore, X is contained in $\bar{A}_1 G_2 C_3 < \bar{A}_1 E_7$, as required. Finally, suppose Y is contained in F_4. Then $X < F_4 C_{E_8}(F_4)^\circ = F_4 G_2$. Moreover, the projection of X to G_2 is contained in the maximal subgroup A_2. By Theorem 2, this is also contained in $\bar{A}_1 A_1 < G_2$. Therefore $X < \bar{A}_1 A_1 F_4$ and is hence a subgroup of $\bar{A}_1 E_7$.

Now let $M = A_4^2$. Using Lemma 3.4, the only A_4-irreducible A_1 subgroups act as 4 on $V_{A_4}(\lambda_1)$ $(p \geq 5)$ and are hence contained in $B_2 < A_4$. Therefore X is contained in $B_2^2 < A_4^2$. By [15, p. 63], we have that B_2^2 is also contained in D_8. Hence X has already been considered in the D_8 case and is in fact conjugate to $E_8(\#3)$.

Now let $M = G_2 F_4$. By Theorem 2, the projection of X to G_2 is either contained in $\bar{A}_1 A_1$ or is maximal with $p \geq 7$. In the first case, X is contained in $\bar{A}_1 E_7$ and has already been considered. So suppose the projection of X to G_2 is maximal. Now consider the projection of X to F_4. By Theorem 3, this is contained in B_4, $\bar{A}_1 C_3$, $A_1 G_2$ or is maximal with $p \geq 13$. In the first case X is contained in D_8, since $B_4 C_{E_8}(B_4)^\circ = B_4 B_3 < D_8$ and in the second case X is contained in $\bar{A}_1 E_7$. Now suppose the projection of X to F_4 is $F_4(\#11)$ and hence contained in $A_1 G_2$. The factor G_2 of M and the factor G_2 of $A_1 G_2$ are conjugate in E_8. Furthermore, $N_{E_8}(A_1 G_2^2)$ contains an involution swapping the G_2 factors. Thus, up to E_8-conjugacy, $X \hookrightarrow A_1 A_1^2 < A_1 G_2^2$ via $(1^{[r]}, 1^{[s]}, 1^{[t]})$ $(rs = 0;\ r \neq s;\ r \neq t;\ s \leq t)$. We claim that if $s = t$ then X is contained in $\bar{A}_1 E_7$. Indeed, X is contained in the centraliser of an involution in $N_{E_8}(A_1 G_2^2)$ when $s = t$ and the connected centraliser of that involution is $\bar{A}_1 E_7$. In fact, X is conjugate to $E_8(\#32^{\{r,s,r\}})$. When $s \neq t$, then X is E_8-irreducible by Corollary 3.6, yielding $E_8(\#39)$.

The last case to consider is when the projection of X to F_4 is maximal and hence conjugate to $F_4(\#10)$ $(p \geq 13)$. Then $X \hookrightarrow A_1 A_1 < G_2 F_4$ via $(1^{[r]}, 1^{[s]})$ $(rs = 0)$ with the first factor A_1 maximal in G_2 and the second maximal in F_4. If $r \neq s$, then X is E_8-irreducible by Corollary 3.6, yielding $E_8(\#38)$. If $r = s$ then Theorem 3.9 shows that X is conjugate to $E_8(\#11^{\{0,0\}})$ and has already been considered.

Let $M = B_2$ ($p \geq 5$). There are two cases to consider. Either X is contained in A_1^2 and acts as $1 \otimes 1^{[r]} + 0$ ($r \neq 0$) on $V_{B_2}(10)$ or X is maximal in M and acts as 4 on $V_{B_2}(10)$. In the first case, X is contained in the connected centraliser of an involution in B_2. Hence X is contained in the connected centraliser of an involution in E_8, which is either D_8 or $\bar{A}_1 E_7$ and so X has been considered already.

Now consider the second case, in which X is a maximal subgroup A_1 of M. If $p \geq 11$ then Theorem 3.9 shows that X is conjugate to $E_8(\#10)$. When $p = 7$, we have X is contained in a parabolic subgroup of E_8, as proved in [19, 3.3]. When $p = 5$, we will show that X is contained in an A_7-parabolic subgroup of E_8 and is hence E_8-reducible. To do this, we will use the same method as [30, Lemma 7.9]; we show that $S = A_1(25) < X$ fixes the same subspaces as X on $L(E_8)$ and then show that X fixes an 8-dimensional abelian subalgebra that is ad-nilpotent of exponent 3 i.e. $(\mathrm{ad}\ a)^3 = 0$ for all a.

Firstly, Lemma 3.5 along with Table 22 shows that the only parabolic subgroup X can be contained in is an A_7-parabolic.

To show S and X fix the same subspaces of $L(E_8)$ we use Lemma 3.7. We have $L(E_8) \downarrow X = 18^2/16/14^3/12^4/10^5/8^6/6^8/4/2^3/0^3$ and therefore conditions (i) and (iii) hold. To show condition (ii) holds it suffices to check that the Weyl modules of high weight 18, 16, 12, 10, 8 and 6 are still indecomposable when restricted to S. We then check this in Magma [5]: We construct $S \cong \mathrm{PSL}(2, 25) \cong \mathrm{PSL}(V)$ and the S-modules $S^n(V)$. In each case, we use the inbuilt "Socle" function in Magma to find that the socle of $S^n(V)$ is irreducible and thus $S^n(V)$ is indecomposable for each integer n in the list of high weights. Therefore, X and S fix the same subspaces of $L(E_8)$.

The existence of $M = B_2$ when $p = 5$ is proved in [19, Lemma 5.1.6] using [25, 6.7]. In particular, if α is the long simple root and β is the short simple root in a basis for M then the A_1 generated by $x_{\pm\alpha}(t)$ is contained in the subsystem subgroup $\bar{A}_1 A_5$ and the A_1 generated by $x_{\pm\beta}(t)$ is contained in the subsystem subgroup $A_2 D_5$. Using this, we can write down the generators $x_{\pm\alpha}(t)$, $x_{\pm\beta}(t)$ of M in terms of generators of E_8. From these generators we construct $B_2(25)$ in Magma as a subgroup of the inbuilt finite group of Lie type $E_8(25)$ and then construct S as a maximal subgroup of $B_2(25)$. We now use the inbuilt functionality of Magma to construct the Lie algebra $L(E_8)$ as a module for S and then, again using the inbuilt "Submodules" function, we find all 8-dimensional S-submodules of $L(E_8)$. We find that there is a unique such S-submodule that is an abelian subalgebra, and it is ad-nilpotent of exponent 3.

So S and therefore X fixes an 8-dimensional abelian subalgebra of $L(E_8)$ that is ad-nilpotent of exponent 3. Exponentiating this subalgebra yields an 8-dimensional unipotent subgroup of E_8, normalised by X. Therefore X is contained in a parabolic subgroup of E_8, as required.

Now let $M = A_1 A_2$ ($p \geq 5$). Then the projection of X to A_2 acts as 2 on $V_{A_2}(10)$ and is the centraliser of a graph automorphism of A_2. By [7, Table 11], we have $N_{E_7}(A_1 A_2) = (A_1 A_2).2$ and therefore X is contained in the centraliser of an involution in E_8. Using [10, Table 4.3.1] we find this centraliser is either D_8 or $\bar{A}_1 E_7$. One calculates that it is the latter and X is conjugate to $E_8(\#34^{\{r,s,r\}})$.

Let M be one of the classes of maximal A_1 subgroups. Then they are E_8-irreducible by Theorem 3.1, yielding $E_8(\#40)$, $E_8(\#41)$ and $E_8(\#42)$.

Finally, we check there are no more E_8-conjugacies between any of the irreducible A_1 subgroups by comparing the composition factors in Table 13. This completes the proof of Theorem 6. □

10 Corollaries

In this section we give the proofs of Corollaries 1 to 5. Let G be an exceptional algebraic group over an algebraically closed field of characteristic p. Corollary 1 is an immediate consequence of Theorems 2 to 6, which prove G contains a G-irreducible subgroup A_1 unless $G = E_6$ and $p = 2$.

For Corollary 2, we consider Tables 4 to 8 when $p = 2, 3$. We see that all G_2-irreducible A_1 subgroups of G_2 are contained in $A_1\tilde{A}_1$ from Table 4. When $p = 2$, Table 5 shows that B_4 contains all of the F_4-irreducible A_1 subgroups of F_4 (in fact the same is true for C_4). If $p = 3$, we notice that B_4 and $\bar{A}_1C_3$ both contain F_4-irreducible A_1 subgroups by Theorem 3. Now suppose $G = E_6$. Then if $p = 2$ there are no E_6-irreducible A_1 subgroups. When $p = 3$, the E_6-irreducible A_1 subgroups $E_6(\#2)$ and $E_6(\#3)$ are listed as subgroups of A_1A_5 and A_2^3, respectively. However, $E_6(\#2) < \bar{A}_1C_3 < \bar{A}_1A_5$ and $\bar{A}_1C_3$ is also contained in C_4. Similarly, in the $M = A_2^3$ case of the proof of Theorem 4, $E_6(\#3)$ is proved to be contained in C_4 (acting as $1 \otimes 1^{[r]} \otimes 1^{[s]}$ on $V_{C_4}(\lambda_1)$). Therefore, C_4 contains a conjugacy class representative of each E_6-irreducible subgroup when $p = 3$. For $G = E_7$ and E_8 the result follows immediately from Tables 7 and 8, respectively.

Corollaries 3 and 4 follow from careful consideration of the composition factors listed in Tables 9 to 13.

For the proof of Corollary 5 we first need the following three results.

Lemma 10.1. *Let X be an algebraic group of type A_1 over an algebraically closed field of characteristic $p = 2$. Let W be the X-module $1^{[a]} \otimes 1^{[b]} \otimes 1^{[c]} \otimes 1^{[d]}$ with a, b, c, d distinct. Then the socle of $W \otimes W$ is a 1-dimensional trivial module.*

Proof. It suffices to show that

$$S_n := \dim(\mathrm{Hom}_X(V_{A_1}(n), W \otimes W)) = \begin{cases} 0 \text{ if } & n \neq 0 \\ 1 \text{ if } & n = 0. \end{cases}$$

Firstly, suppose $n = 0$. Then $\mathrm{Hom}_X(0, W \otimes W) \cong \mathrm{Hom}_X(W, W) \cong K$ since W is self-dual and irreducible. Therefore $S_0 = 1$ as required. Now suppose $n \neq 0$ and $S_n \neq 0$. Then $V_{A_1}(n)$ is a composition factor of $W \otimes W = (1^{[a]} \otimes 1^{[a]}) \otimes (1^{[b]} \otimes 1^{[b]}) \otimes (1^{[c]} \otimes 1^{[c]}) \otimes (1^{[d]} \otimes 1^{[d]})$. Since $1 \otimes 1 = T(2) = 0|2|0$ it follows that $V_{A_1}(n)$ is isomorphic to $1^{[a+1]}$, $1^{[a+1]} \otimes 1^{[b+1]}$, $1^{[a+1]} \otimes 1^{[b+1]} \otimes 1^{[c+1]}$ or $1^{[a+1]} \otimes 1^{[b+1]} \otimes 1^{[c+1]} \otimes 1^{[d+1]}$, up to relabelling of a, b, c, d. It remains to check that none of these modules have non-zero homomorphisms to $W \otimes W$. First consider $V_{A_1}(n) \cong 1^{[a+1]}$. Then $\mathrm{Hom}_X(1^{[a+1]}, W \otimes W) \cong \mathrm{Hom}_X(1^{[a+1]} \otimes W, W)$. If $a + 1 \notin \{b, c, d\}$ then $1^{[a+1]} \otimes W$ is irreducible and not isomorphic to W, hence $S_n = 0$. So $a + 1 \in \{b, c, d\}$ and we may assume that $a + 1 = b$. Noting that $1 \otimes 1 \otimes 1 = 3 \oplus 1^2$ (the composition factors are clear and there are no non-trivial extensions between any of them) we have

$$\begin{aligned}
\mathrm{Hom}_X(1^{[a+1]}, W \otimes W) &\cong \mathrm{Hom}_X(1^{[a+1]} \otimes 1^{[a+1]} \otimes 1^{[a+1]}, 1^{[a]} \otimes 1^{[a]} \otimes 1^{[c]} \otimes 1^{[c]} \otimes 1^{[d]} \otimes 1^{[d]}) \\
&\cong \mathrm{Hom}_X((1^{[a+1]} \otimes 1^{[a+2]}) \oplus (1^{[a+1]})^2, 1^{[a]} \otimes 1^{[a]} \otimes 1^{[c]} \otimes 1^{[c]} \otimes 1^{[d]} \otimes 1^{[d]}) \\
&\cong \mathrm{Hom}_X(1^{[a+1]} \otimes 1^{[a+2]}, 1^{[a]} \otimes 1^{[a]} \otimes 1^{[c]} \otimes 1^{[c]} \otimes 1^{[d]} \otimes 1^{[d]}) \oplus \\
&\quad \mathrm{Hom}_X(1^{[a+1]}, 1^{[a]} \otimes 1^{[a]} \otimes 1^{[c]} \otimes 1^{[c]} \otimes 1^{[d]} \otimes 1^{[d]})^2 \\
&\cong \mathrm{Hom}_X(1^{[a]} \otimes 1^{[a+1]} \otimes 1^{[a+2]} \otimes 1^{[c]} \otimes 1^{[d]}, 1^{[a]} \otimes 1^{[c]} \otimes 1^{[d]}) \oplus \\
&\quad \mathrm{Hom}_X(1^{[a]} \otimes 1^{[a+1]} \otimes 1^{[c]} \otimes 1^{[d]}, 1^{[a]} \otimes 1^{[c]} \otimes 1^{[d]})^2 \\
&=: A \oplus B^2.
\end{aligned}$$

We have $1^{[a]} \otimes 1^{[c]} \otimes 1^{[d]}$ and $1^{[a]} \otimes 1^{[a+1]} \otimes 1^{[c]} \otimes 1^{[d]}$ (recall that $a+1=b$ and that a,b,c,d are distinct) are irreducible non-isomorphic modules and so $B=0$. Furthermore, $1^{[a]} \otimes 1^{[a+1]} \otimes 1^{[a+2]} \otimes 1^{[c]} \otimes 1^{[d]}$ is irreducible if and only if $a+2 \notin \{c,d\}$. Therefore $A=0$ unless $a+2 \in \{c,d\}$. We may therefore assume $a+2=c$ and just consider A:

$$\begin{aligned}
A &\cong \mathrm{Hom}_X(1^{[a+2]} \otimes 1^{[a+2]} \otimes 1^{[a+2]}, 1^{[a]} \otimes 1^{[a]} \otimes 1^{[a+1]} \otimes 1^{[d]} \otimes 1^{[d]}) \\
&\cong \mathrm{Hom}_X(1^{[a+2]} \otimes 1^{[a+3]}, 1^{[a]} \otimes 1^{[a]} \otimes 1^{[a+1]} \otimes 1^{[d]} \otimes 1^{[d]}) \oplus \\
&\quad \mathrm{Hom}_X(1^{[a+2]}, 1^{[a]} \otimes 1^{[a]} \otimes 1^{[a+1]} \otimes 1^{[d]} \otimes 1^{[d]})^2 \\
&\cong \mathrm{Hom}_X(1^{[a]} \otimes 1^{[a+1]} \otimes 1^{[a+2]} \otimes 1^{[a+3]} \otimes 1^{[d]}, 1^{[a]} \otimes 1^{[d]}) \oplus \\
&\quad \mathrm{Hom}_X(1^{[a]} \otimes 1^{[a+1]} \otimes 1^{[a+2]} \otimes 1^{[d]}, 1^{[a]} \otimes 1^{[d]})^2 \\
&=: C \oplus D^2.
\end{aligned}$$

As before, $D=0$ and $C=0$ unless $a+3=d$. So finally, we consider C when $a+3=d$.

$$\begin{aligned}
C &\cong \mathrm{Hom}_X(1^{[a+3]} \otimes 1^{[a+3]} \otimes 1^{[a+3]}, 1^{[a]} \otimes 1^{[a]} \otimes 1^{[a+1]} \otimes 1^{[a+2]}) \\
&\cong \mathrm{Hom}_X(1^{[a+3]} \otimes 1^{[a+4]}, 1^{[a]} \otimes 1^{[a]} \otimes 1^{[a+1]} \otimes 1^{[a+2]}) \oplus \\
&\quad \mathrm{Hom}_X(1^{[a+3]}, 1^{[a]} \otimes 1^{[a]} \otimes 1^{[a+1]} \otimes 1^{[a+2]})^2 \\
&\cong \mathrm{Hom}_X(1^{[a]} \otimes 1^{[a+1]} \otimes 1^{[a+2]} \otimes 1^{[a+3]} \otimes 1^{[a+4]}, 1^{[a]}) \oplus \\
&\quad \mathrm{Hom}_X(1^{[a]} \otimes 1^{[a+1]} \otimes 1^{[a+2]} \otimes 1^{[a+3]}, 1^{[a]})^2 \\
&= 0.
\end{aligned}$$

We have proved that $\mathrm{Hom}_X(1^{[a+1]}, W \otimes W) = 0$ and so $S_n = 0$, a contradiction. The three remaining cases are similar and in each one we find that $S_n = 0$, a contradiction. This completes the proof. $\square$

Lemma 10.2. *Let X be a simple algebraic group of type A_1 over an algebraically closed field of characteristic $p=2$. Let W_1 and W_2 be the X-modules $1 \otimes 1^{[r]} \otimes 1^{[s]}$ and $0|(2+2^{[r]}+2^{[s]})|0$, respectively. Then $\bigwedge^2(W_1)$ has a 1-dimensional socle, whereas $\bigwedge^2(W_2)$ has a 7-dimensional socle. In particular, $\bigwedge^2(W_1)$ is not isomorphic to $\bigwedge^2(W_2)$.*

Proof. First, we claim that the socle of $W_1 \otimes W_1$ is 1-dimensional. This follows from a similar calculation to that in the previous lemma. Since $\bigwedge^2(W_1)$ is a submodule of $W_1 \otimes W_1$, it follows that the socle of $\bigwedge^2(W_1)$ is also 1-dimensional. Now we consider $\bigwedge^2(W_2)$. We claim that the socle is isomorphic to $2+2^{[r]}+2^{[s]}+0$. To prove this we consider X as a diagonal subgroup of $Y \cong A_1^3$ via $(1, 1^{[r]}, 1^{[s]})$ and let W_3 be the Y-module $(0,0,0)|((2,0,0)+(0,2,0)+(0,0,2))|(0,0,0)$, so $W_3 \downarrow X = W_2$. The first step is to show that the socle of W_3 is isomorphic to $(2,0,0)+(0,2,0)+(0,0,2)+(0,0,0)$. Define a finite subgroup S of Y, as the direct product of subgroups $\mathrm{SL}(2,4) = \mathrm{SL}(V) < A_1$ for each of the three simple factors of Y. We then construct $W_3 \downarrow S$ in Magma: Given the natural module $(V,0,0)$ for the first factor of S, we have that the tensor product $U_1 = (V,0,0) \otimes (V,0,0) = (0,0,0)|(V^{[1]},0,0)|(0,0,0)$. We then do the same for the second and third factors, forming U_2 and U_3 respectively. It is then straightforward to form the module $W_3 \downarrow S = (0,0,0)|((V^{[1]},0,0)+(0,V^{[1]},0)+(0,0,V^{[1]}))|(0,0,0)$ from the direct sum of U_1, U_2 and U_3. We then use the inbuilt functions for the wedge square and the socle of a module in Magma to conclude that $\mathrm{Soc}(\bigwedge^2(W_3) \downarrow S) \cong (V^{[1]},0,0)+(0,V^{[1]},0)+(0,0,V^{[1]})+(0,0,0)$. Now we check the three conditions of Lemma 3.7, applied to $S < Y$ acting on $\bigwedge^2(W_3)$. The Y-composition factors of $\bigwedge^2(W_3)$ are $(2,2,0)/(2,0,2)/(0,2,2)/(2,0,0)^2/(0,2,0)^2/(0,0,2)^2/(0,0,0)^4$. Therefore conditions (i) and (iii) hold and it remains to show that the restriction map $\mathrm{Ext}^1_Y(M,N) \to \mathrm{Ext}^1_S(M,N)$ is injective for all pairs of Y-composition factors of $\bigwedge^2(W_3)$. Using the Künneth formula [24, 10.85] and Lemma 3.8, we have $\mathrm{Ext}^1_Y(M,N)=0$ unless $M=(2,0,0)$ and $N=(2,2,0)$, $(2,0,2)$ or $(0,0,0)$ (up to swapping M, N and cycling the three A_1 factors), in which case it is 1-dimensional. The map $\mathrm{Ext}^1_Y((2,0,0),(0,0,0)) \to \mathrm{Ext}^1_S((2,0,0),(0,0,0))$ is injective since the non-trivial extension $(2,0,0)|(0,0,0)$ is found in the tensor product $(1,0,0)\otimes(1,0,0)$ for both Y and S. Similarly, the non-trivial extension $(2,0,0)|(2,2,0)$ is found in $(2,1,0)\otimes(0,1,0)$ and $(2,0,0)|(2,0,2)$ is found in $(2,0,1)\otimes(0,0,1)$ for both Y and S. We now apply Lemma 3.7 to conclude that Y and S fix the same subspaces of $\bigwedge^2(W_3)$. Therefore the socle of $\bigwedge^2(W_3)$ as a Y-module is $(2,0,0)+(0,2,0)+(0,0,2)+(0,0,0)$. It follows that the socle of $\bigwedge^2(W_2)$ as an X-module has dimension at least 7. In fact, using Lemma 3.7 again, this time applied to $X<Y$ acting on $\bigwedge^2(W_3)$ implies that the socle of $\bigwedge^2(W_2)$ as an X-module has dimension 7. □

Lemma 10.3. *Let $G=E_8$, $p=5$ and M be a maximal subgroup B_2. Suppose X is a maximal subgroup A_1 of M. Then X is conjugate to $Y<A_8$ acting as $W(8)=8|0$ on $V_{A_8}(\lambda_1)$.*

Proof. In the $M=B_2$ case of the proof of Theorem 6 it is proved that X is contained in an A_7-parabolic subgroup of G. The composition factors of X acting on $L(E_8)$ are $18^2/16/14^3/12^4/10^5/8^6/6^8/4/2^3/0^3$ and it follows that the projection of X to A_7 acts as $3\otimes 1^{[1]}$ on $V_{A_7}(\lambda_1)$. Let $P=QL$ be an A_7-parabolic subgroup of E_8 containing X and let Z be an subgroup A_1 of $L'=A_7$ acting as $3\otimes 1^{[1]}$ on $V_{A_7}(\lambda_1)$, so the projection of X to A_7 is A_7-conjugate to Z. By definition, Y is a subgroup of an A_7-parabolic subgroup of A_8 and hence of E_8, with the projection of Y to A_7 also A_7-conjugate to Z. By using the construction of X in Magma from the $M=B_2$ case of the proof of Theorem 6 and the fact that $L(E_8)\downarrow A_8=(\lambda_1+\lambda_7)\oplus\lambda_3\oplus\lambda_6$ we calculate that X and Y act the same way on $L(E_8)$, specifically as $W(18)+W(18)^*+T(16)+T(12)^2+T(10)^3+14^3+T(6)^3+4$. In particular, neither X nor Y have a trivial submodule on $L(E_8)$ and are thus not contained in an A_7 Levi subgroup. Therefore, both X and Y are non-E_8-cr. To prove that X and Y are conjugate it remains to show that there is only one E_8-conjugacy class of non-E_8-cr A_1 subgroups in QZ. To do this we use the results and methods described in [29] and [22].

We first consider the action of Z on the levels of Q. The action of L' on the levels of Q are as follows

$$\begin{aligned} Q/Q(2)\downarrow A_7 &= \lambda_5,\\ Q(2)/Q(3)\downarrow A_7 &= \lambda_2,\\ Q(3)\downarrow A_7 &= \lambda_7, \end{aligned}$$

and restricting to $Z<L'$ we have

$$\begin{aligned} Q/Q(2)\downarrow Z &= 18+T(12)+T(10),\\ Q(2)/Q(3)\downarrow Z &= 14+10+T(6),\\ Q(3)\downarrow Z &= 8. \end{aligned}$$

In particular, using Lemma 3.8 we see $H^1(Z,Q(i)/Q(i+1))=0$ for $i=1,2$ and $H^1(Z,Q(3))\cong K$. It follows from [29, Proposition 3.2.6, Lemma 3.2.11] that $H^1(Z,Q)\cong K$. Moreover, by [22, Lemma 3.20], there is at most one E_8-conjugacy class of non-E_8-cr A_1 subgroups contained in QZ by considering the action of the 1-dimensional non-trivial torus $Z(L)$. Since X and Y are non-E_8-cr and contained in QZ, there is exactly one class and so X and Y are E_8-conjugate. □

It remains to prove Corollary 5. The strategy for the proof is as follows. For each exceptional algebraic group G we find all M-irreducible A_1 subgroups that are not G-irreducible from the proofs of Theorems 2 to 6. Given such a subgroup X we then check whether it satisfies the hypothesis of Corollary 5. That is to say, we check whether X is contained reducibly in another reductive, maximal connected subgroup

M', or X is contained in a Levi subgroup of G. To do this we use the composition factors of X on the minimal or adjoint module for G, using restriction from M. Of course, since X is G-reducible there must exist some subgroup $Y \cong X$ inside a Levi factor L' having the same composition factors as X. Therefore, we will require the exact module structure of X acting on either the minimal module or adjoint module for G to prove that X is not contained in L'.

Proof of Corollary 5 First consider $G = G_2$. The proof of Theorem 2 shows that the only M-irreducible subgroup A_1 that is G_2-reducible is $X = A_1 \hookrightarrow M = A_1\tilde{A}_1$ via $(1,1)$ when $p = 2$. We need to check whether X satisfies the hypothesis of Corollary 5. The only subgroups of G_2 that have the same composition factors as X are a Levi $\bar{A}_1$ and $A_1 < A_2$ embedded via $W(2)$. However, [28, Theorem 1] shows that X is not conjugate to either of these subgroups. Therefore X is not contained reducibly in another reductive, maximal connected subgroup nor is it contained in a Levi subgroup of G. Hence X satisfies the hypothesis of Corollary 5 and is listed in Table 2.

Now let $G = F_4$. By the proof of Theorem 3, there are no conjugacy classes of M-irreducible A_1 subgroups which are F_4-reducible. Indeed, the B_4-irreducible subgroups acting as $1 \otimes 1^{[r]} \otimes 1^{[s]}$ on $V_{B_4}(\lambda_1)$ when $p = 2$ are shown to be conjugate to B_4-reducible subgroups acting as $0|(2 + 2^{[r]} + 2^{[s]})|0$. Similarly for the C_4-irreducible subgroups acting as $1 \otimes 1^{[r]} \otimes 1^{[s]}$ on $V_{C_4}(\lambda_1)$ when $p = 2$. Therefore there are no A_1 subgroups satisfying the hypothesis of Corollary 5.

Next, suppose $G = E_6$ and consider the M-irreducible A_1 subgroups that are E_6-reducible. These are all found in the proof of Theorem 4. Let X be such a subgroup. If X is contained in a maximal A_2 then $p = 5$ and we claim that X is contained in $\bar{A}_1A_5$ via $(1, W(5))$, hence X does not satisfy the hypothesis of Corollary 5. This is proved in [22, Section 4.1] by showing there is only one conjugacy class of non-E_6-cr subgroups of type A_1 with the same composition factors as X on V_{56}. Now suppose X is contained in C_4 ($p \neq 2$). If X is contained in $\bar{A}_1C_3 < C_4$ then X is also contained in $\bar{A}_1A_5$. By the proof of Theorem 4, every $\bar{A}_1A_5$-irreducible subgroup A_1 is E_6-irreducible. Hence X is contained reducibly in $\bar{A}_1A_5$ and so does not satisfy the hypothesis of the corollary. Similarly, if $X < C_2^2$ then X is contained in a D_5 Levi subgroup and does not satisfy the hypothesis. Finally, suppose X is contained in F_4. Then X is contained in B_4 and hence contained in a D_5 Levi subgroup. Therefore there are no A_1 subgroups satisfying the hypothesis of Corollary 5.

For $G = E_7$ the result is checked in the same way as for E_6. First consider A_7-irreducible A_1 subgroups, all of which are E_7-reducible. If $p > 2$ then such subgroups are contained in C_4, which is contained in an E_6 Levi subgroup by [15, Table 8.2] and so do not satisfy the hypothesis of the corollary . If $p = 2$, then we need to consider a subgroup X acting on $V_{A_7}(\lambda_1)$ as $W_1 = 1 \otimes 1^{[r]} \otimes 1^{[s]}$ $(0 < r < s)$. We claim that X satisfies the hypothesis of Corollary 5. To prove this, we start by considering the action of X on V_{56}. We have $V_{56} \downarrow A_7 = V_{A_7}(\lambda_2) + V_{A_7}(\lambda_6)$ and hence $V_{56} \downarrow X = (\bigwedge^2(W_1))^2$. By Lemma 10.2, we have $\bigwedge^2(W_1)$ is indecomposable and hence X has two direct summands of dimension 28 on V_{56}. From this, it follows that X satisfies the hypothesis of Corollary 5 or X is conjugate to an M-reducible subgroup A_1 of A_7. Suppose the latter is true. Considering the X-composition factors of V_{56}, it follows that X is conjugate to Y, where $V_{A_7}(\lambda_1) \downarrow Y = W_2 = 0|(2 + 2^{[r]} + 2^{[s]})|0$. Now, we have $V_{56} \downarrow Y = (\bigwedge^2(W_2))^2$. Lemma 10.2 shows that X and Y are non-$\mathrm{GL}(56, K)$-conjugate, and thus non-E_7-conjugate. This contradiction proves that X satisfies the hypothesis of the corollary.

Now let X be the irreducible subgroup A_1 contained in the maximal subgroup A_2 when $p = 5, 7$. Then X is E_7-reducible and is in fact non-E_7-cr and conjugate to $Y = A_1 < A_7$ with $V_{A_7}(\lambda_1) \downarrow A_1 = W(7)$ for both $p = 5, 7$. Indeed, by considering its composition factors on V_{56}, we find that the only Levi subgroup that can contain X is E_6. However, in the case $M = A_2$ in the proof of Theorem 5, we calculated that $V_{56} \downarrow X = (0|12|0)^2 + (4|8|4)^2$ when $p = 7$ and therefore X is not contained in E_6 since E_6 has a trivial direct summand on V_{56}. When $p=5$, we have $L(E_7)\downarrow A_2 = 44+11$. We know $V_{A_2}(11)\downarrow X = 4+2$ and need to find $V_{A_2}(44)\downarrow X$. To do this, we first use the fact that 40 and 04 are tilting, so their tensor product is also tilting and thus $40\otimes 04 = 44 + T(33) + T(22)$. Furthermore, $40 = S^4(10)$ and hence $40\downarrow X = S^4(2) = T(8)+4$. Combining this with usual high weight calculations yields that $L(E_7)\downarrow X = T(16)+14+T(12)+T(10)^2+T(8)+4^3+2$. In particular, we find that X has no trivial direct summands on $L(E_7)$ and is therefore not contained in E_6. Thus X is non-E_7-cr. To prove that X is conjugate to $Y < A_7$, we need to prove there is only one E_7-conjugacy class of A_1 subgroups contained in an E_6-parabolic when $p=5,7$. This is done in [22, Sections 5.1, 6.1].

Next, we consider the maximal subgroup A_1A_1 when $p=5$. Then $X\hookrightarrow A_1A_1$ via $(1,1)$ is E_7-reducible. In fact, X is non-E_7-cr and conjugate to $Y = A_1 \hookrightarrow A_1A_1 < A_2A_5$ via $(1,1)$ where the first factor A_1 acts as 2 on $V_{A_2}(10)$ and the second factor A_1 acts as $W(5)$ on $V_{A_5}(\lambda_1)$. We will prove that X is non-E_7-cr and the second statement is proved in [22, Section 5.3]. Looking for a contradiction we suppose X is E_7-cr. Then X is contained irreducibly in some Levi factor L'. By considering the composition factors of X on V_{56}, we have $L' = A_1A_2A_3$. From [19, Table 10.2], we have $V_{56}\downarrow A_1A_1 = ((2,3)|((6,3)+(2,5))|(2,3))+(4,1)$. We can construct such a module for $A_1(25)\times A_1(25)$ in Magma and use it to show that $V_{56}\downarrow X = 9+W(7)+W(7)^*+T(5)^3$. The action of $A_1A_2A_3$ on V_{56} is completely reducible and has a direct summand of dimension 4. All of the direct summands of X on V_{56} have dimension at least 8, a contradiction. It follows that X is non-E_7-cr but is contained reducibly in A_2A_5 and so does not satisfy the hypothesis of Corollary 5.

The last E_7-reducible subgroups to consider are $X_1\hookrightarrow A_1A_1 < A_1G_2$ via $(1,1)$ where the second factor A_1 is maximal in G_2 and $X_2\hookrightarrow A_1A_1 < G_2C_3$ via $(1,1)$ where the first factor A_1 is maximal in G_2 and the second is maximal in C_3, both when $p=7$. From the proof of Theorem 5, we see that X_1 and X_2 are E_7-reducible and by considering their composition factors on V_{56}, they are contained in an $A_1A_2A_3$-parabolic subgroup. Both X_1 and X_2 act on V_{56} as $T(11)+T(9)^2+T(7)$, checked using the restriction of V_{56} to A_1G_2 and G_2C_3 from [19, Table 10.2]. As $V_{56}\downarrow A_1A_2A_3$ has no direct summands of dimension at least 14, it follows that neither X_1 nor X_2 are contained in a $A_1A_2A_3$ Levi subgroup and both satisfy the hypothesis of the corollary. Furthermore, it is shown in [22, Section 6.2] that X_1 is conjugate to X_2, and hence only X_1 appears in Table 2.

Finally, suppose $G=E_8$. First consider $X = A_1 < D_8$ acting as $1\otimes 1^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}$ $(0<r<s<t)$ when $p=2$, where X is the E_8-reducible class of such subgroups contained in $B_4(\ddagger)$, as in [30, Lemma 7.1]. We claim that X satisfies the hypothesis of Corollary 5. To prove this, we will first show that X has a 120-dimensional indecomposable summand on $L(E_8)$. When $p=2$ we have $L(E_8)\downarrow D_8 = (0|\lambda_2|0)+\lambda_7$ and in particular, $\bigwedge^2(\lambda_1) = 0|\lambda_2|0$ is a direct summand. Furthermore, $\bigwedge^2(\lambda_1)$ is a submodule of $\lambda_1\otimes\lambda_1$. Lemma 10.1 shows that the socle of $\lambda_1\otimes\lambda_1\downarrow X$ is a 1-dimensional trivial module and therefore simple. It follows that $\bigwedge^2(\lambda_1)$ has a simple socle and is thus indecomposable. Therefore X has a 120-dimensional indecomposable summand on $L(E_8)$. Now suppose X does not satisfy the hypothesis of the corollary. Then X is a subgroup of a Levi subgroup or another reductive, maximal connected subgroup of E_8. Since X has a 120-dimensional indecomposable summand, it follows that X is an E_7-irreducible subgroup of E_7. As $p=2$, it follows from Corollary 2 that X is contained in $\bar{A}_1D_6$. But the largest dimensional indecomposable summand of $L(E_8)\downarrow \bar{A}_1D_6$ is $(V_{A_1}(1), V_{A_6}(\lambda_6))$, which has dimension 64. This is a contradiction and hence X satisfies the hypothesis of the corollary.

Next, we note that the E_8-reducible subgroup contained A_8-irreducibly in A_8 when $p=3$, is shown to be D_8-reducible in the proof of Theorem 6 and hence does not satisfy the hypothesis of the corollary.

The last subgroup A_1 to consider is a maximal subgroup A_1 of $M=B_2$ when $p=5,7$, let this be X. The proof of [19, Proposition 3.3.3] shows that X is contained in A_8 acting as $W(8)=8|4$ when $p=7$. When

$p = 5$, Lemma 10.3 shows that X is also contained in A_8 acting as $W(8) = 8|0$. Therefore in both cases X is contained reducibly in another reductive maximal connected subgroup of E_8 and so does not satisfy the hypothesis of Corollary 5. □

11 Tables

In this section we give the tables of composition factors for the G-irreducible A_1 subgroups from Theorems 2 to 6 acting on the minimal and adjoint modules for G. The tables use the unique identifier given to G-irreducible A_1 subgroups in Tables 4 to 8 and the composition factors are calculated by restriction from a reductive, maximal connected subgroup M. The notation used in the tables is described in Section 2. The composition factors of M on the minimal and adjoint modules of G are listed in Theorem 3.1.

Table 9: The composition factors of the irreducible A_1 subgroups of G_2.

ID	Comp. factors of $V_7 \downarrow X$	Comp. factors of $L(G_2) \downarrow X$
1	$1^{[r]} \otimes 1^{[s]}/W(2)^{[s]}$	$W(2)^{[r]}/1^{[r]} \otimes W(3)^{[s]}/W(2)^{[s]}$
2	$2^2/0$	$W(4)/2^3$
3	6	$W(10)/2$

Table 10: The composition factors of the irreducible A_1 subgroups of F_4.

ID	Comp. factors of $V_{26} \downarrow X$	Comp. factors of $L(F_4) \downarrow X$
1	$1 \otimes 1^{[r]}/1 \otimes 1^{[s]}/1 \otimes 1^{[t]}/1^{[r]} \otimes 1^{[s]}/$ $1^{[r]} \otimes 1^{[t]}/1^{[s]} \otimes 1^{[t]}/0^2$	$W(2)/1 \otimes 1^{[r]} \otimes 1^{[s]} \otimes 1^{[t]}/1 \otimes 1^{[r]}/1 \otimes 1^{[s]}/1 \otimes 1^{[t]}/W(2)^{[r]}/$ $1^{[r]} \otimes 1^{[s]}/1^{[r]} \otimes 1^{[t]}/W(2)^{[s]}/1^{[s]} \otimes 1^{[t]}/W(2)^{[t]}$
2	$2^{[r]}/1^{[r]} \otimes 1^{[s]} \otimes 1^{[t]}/$ $1^{[r]} \otimes 1^{[s]} \otimes 1^{[u]}/2^{[s]}/1^{[t]} \otimes 1^{[u]}/0^2$	$2^{[r]} \otimes 2^{[s]}/2^{[r]} \otimes 1^{[t]} \otimes 1^{[u]}/2^{[r]}/1^{[r]} \otimes 1^{[s]} \otimes 1^{[t]}/1^{[r]} \otimes 1^{[s]} \otimes 1^{[u]}/$ $2^{[s]} \otimes 1^{[t]} \otimes 1^{[u]}/2^{[s]}/1^{[t]} \otimes 1^{[u]}/2^{[t]}/2^{[u]}/0^4$
3	$2/1 \otimes 1^{[r]} \otimes 1^{[s]} \otimes 1^{[t]}/2^{[r]}/2^{[s]}/2^{[t]}/0^2$	$2 \otimes 2^{[r]}/2 \otimes 2^{[s]}/2 \otimes 2^{[t]}/2/1 \otimes 1^{[r]} \otimes 1^{[s]} \otimes 1^{[t]}/2^{[r]} \otimes 2^{[s]}/$ $2^{[r]} \otimes 2^{[t]}/2^{[r]}/2^{[s]} \otimes 2^{[t]}/2^{[s]}/2^{[t]}/0^4$
4	$2/(1 \otimes 1^{[r]} \otimes 1^{[s]})^2/2^{[r]}/2^{[s]}/0$	$2/2 \otimes 2^{[r]}/2 \otimes 2^{[s]}/(1 \otimes 1^{[r]} \otimes 1^{[s]})^2/2^{[r]} \otimes 2^{[s]}/2^{[r]}/2^{[s]}$
5	$W(3) \otimes 1^{[r]}/2 \otimes 2^{[r]}/1 \otimes W(3)^{[r]}/0$	$W(4) \otimes 2^{[r]}/W(3) \otimes 1^{[r]}/2 \otimes W(4)^{[r]}/2/1 \otimes W(3)^{[r]}/2^{[r]}$
6	$4^{[r]}/3^{[r]} \otimes 1^{[s]}/3^{[r]} \otimes 1^{[t]}/1^{[s]} \otimes 1^{[t]}/0$	$W(6)^{[r]}/4^{[r]} \otimes 1^{[s]} \otimes 1^{[t]}/3^{[r]} \otimes 1^{[s]}/3^{[r]} \otimes 1^{[t]}/2^{[r]}/2^{[s]}/2^{[t]}$
7	$10/8/4/0$	$W(14)/10^2/6/4/2$
8	$1^{[r]} \otimes 5^{[s]}/W(8)^{[s]}/4^{[s]}$	$2^{[r]}/1^{[r]} \otimes W(9)^{[s]}/1^{[r]} \otimes 3^{[s]}/W(10)^{[s]}/6^{[s]}/2^{[s]}$
9	$1^{[r]} \otimes 2^{[s]} \otimes 1^{[t]}/W(4)^{[s]}/2^{[s]} \otimes 2^{[t]}$	$2^{[r]}/1^{[r]} \otimes W(4)^{[s]} \otimes 1^{[t]}/1^{[r]} \otimes W(3)^{[t]}/W(4)^{[s]} \otimes 2^{[t]}/2^{[s]}/2^{[t]}$
10	$W(16)/8$	$W(22)/W(14)/10/2$
11	$4^{[r]}/2^{[r]} \otimes 6^{[s]}$	$4^{[r]} \otimes 6^{[s]}/2^{[r]}/W(10)^{[s]}/2^{[s]}$

Table 11: The composition factors of the irreducible A_1 subgroups of E_6.

ID	Comp. factors of $V_{27} \downarrow X$	Comp. factors of $L(E_6) \downarrow X$
1	$1^{[r]}\otimes 5^{[s]}/W(8)^{[s]}/4^{[s]}/0$	$2^{[r]}/1^{[r]}\otimes W(9)^{[s]}/1^{[r]}\otimes 5^{[s]}/1^{[r]}\otimes 3^{[s]}/W(10)^{[s]}/W(8)^{[s]}/6^{[s]}/4^{[s]}/2^{[s]}$
2	$1^{[r]}\otimes 2^{[s]}\otimes 1^{[t]}/W(4)^{[s]}/2^{[s]}\otimes 2^{[t]}/0$	$2^{[r]}/1^{[r]}\otimes W(4)^{[s]}\otimes 1^{[t]}/1^{[r]}\otimes 2^{[s]}\otimes 1^{[t]}/1^{[r]}\otimes W(3)^{[t]}/W(4)^{[s]}\otimes 2^{[t]}/W(4)^{[s]}/2^{[s]}\otimes 2^{[t]}/2^{[s]}/2^{[t]}$
3	$2\otimes 2^{[r]}/2\otimes 2^{[s]}/2^{[r]}\otimes 2^{[s]}$	$W(4)/(2\otimes 2^{[r]}\otimes 2^{[s]})^2/2/W(4)^{[r]}/2^{[r]}/W(4)^{[s]}/2^{[s]}$
4	$4^{[r]}/2^{[r]}\otimes 6^{[s]}/0$	$4^{[r]}\otimes 6^{[s]}/4^{[r]}/2^{[r]}\otimes 6^{[s]}/2^{[r]}/W(10)^{[s]}/2^{[s]}$
5	$W(16)/8/0$	$W(22)/W(16)/W(14)/10/8/2$
6	$W(12)/8/4$	$W(16)/W(14)/10^2/8/6/4/2$

Table 12: The composition factors of the irreducible A_1 subgroups of E_7.

ID	Comp. factors of $V_{56} \downarrow X$	Comp. factors of $L(E_7) \downarrow X$
1	$1^{[r]}\otimes 5^{[s]}\otimes 1^{[t]}/W(8)^{[s]}\otimes 1^{[t]}/4^{[s]}\otimes 1^{[t]}/3^{[t]}$	$2^{[r]}/1^{[r]}\otimes W(9)^{[s]}/1^{[r]}\otimes 5^{[s]}\otimes 2^{[t]}/1^{[r]}\otimes 3^{[s]}/W(10)^{[s]}/W(8)^{[s]}\otimes 2^{[t]}/6^{[s]}/4^{[s]}\otimes 2^{[t]}/2^{[s]}/2^{[t]}$
2	$1^{[r]}\otimes 5^{[s]}\otimes 1^{[t]}/W(9)^{[s]}/5^{[s]}\otimes 2^{[t]}/3^{[s]}$	$2^{[r]}/1^{[r]}\otimes W(8)^{[s]}\otimes 1^{[t]}/1^{[r]}\otimes 4^{[s]}\otimes 1^{[t]}/1^{[r]}\otimes 3^{[s]}/W(10)^{[s]}/W(8)^{[s]}\otimes 2^{[t]}/6^{[s]}/4^{[s]}\otimes 2^{[t]}/2^{[s]}/2^{[t]}$
3	$1^{[r]}\otimes 2^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}/W(4)^{[s]}\otimes 1^{[u]}/2^{[s]}\otimes 2^{[t]}\otimes 1^{[u]}/W(3)^{[u]}$	$2^{[r]}/1^{[r]}\otimes W(4)^{[s]}\otimes 1^{[u]}/1^{[r]}\otimes 2^{[s]}\otimes 2^{[t]}\otimes 1^{[u]}/1^{[r]}\otimes W(3)^{[u]}/W(4)^{[s]}\otimes 2^{[t]}/W(4)^{[s]}\otimes 2^{[u]}/2^{[s]}\otimes 2^{[t]}\otimes 2^{[u]}/2^{[s]}/2^{[t]}/2^{[u]}$
4	$1^{[r]}\otimes 10^{[s]}/1^{[r]}/W(15)^{[s]}/9^{[s]}/5^{[s]}$	$2^{[r]}/1^{[r]}\otimes W(15)^{[s]}/1^{[r]}\otimes 9^{[s]}/1^{[r]}\otimes 5^{[s]}/W(18)^{[s]}/W(14)^{[s]}/(10^{[s]})^2/6^{[s]}/2^{[s]}$
5	$1^{[r]}\otimes 8^{[s]}/1^{[r]}\otimes 2^{[t]}/10^{[s]}\otimes 1^{[t]}/4^{[s]}\otimes 1^{[t]}$	$2^{[r]}/1^{[r]}\otimes 10^{[s]}\otimes 1^{[t]}/1^{[r]}\otimes 4^{[s]}\otimes 1^{[t]}/W(14)^{[s]}/10^{[s]}/8^{[s]}\otimes 2^{[t]}/6^{[s]}/2^{[s]}/2^{[t]}$
6	$1^{[r]}\otimes 6^{[s]}/1^{[r]}\otimes 4^{[t]}/6^{[s]}\otimes 3^{[t]}/3^{[t]}$	$2^{[r]}/1^{[r]}\otimes 6^{[s]}\otimes 3^{[t]}/1^{[r]}\otimes 3^{[t]}/W(10)^{[s]}/6^{[s]}/6^{[s]}\otimes 4^{[t]}/2^{[s]}/6^{[t]}/2^{[t]}$
7	$1^{[r]}\otimes 6^{[s]}/1^{[r]}\otimes 1^{[t]}\otimes 1^{[u]}/1^{[r]}/6^{[s]}\otimes 1^{[t]}/6^{[s]}\otimes 1^{[u]}/1^{[t]}/1^{[u]}$	$2^{[r]}/1^{[r]}\otimes 6^{[s]}\otimes 1^{[t]}/1^{[r]}\otimes 6^{[s]}\otimes 1^{[u]}/1^{[r]}\otimes 1^{[t]}/1^{[r]}\otimes 1^{[u]}/W(10)^{[s]}/6^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}/(6^{[s]})^2/2^{[s]}/1^{[t]}\otimes 1^{[u]}/2^{[t]}/2^{[u]}/$
8	$1^{[r]}\otimes 4^{[s]}/1^{[r]}\otimes 2^{[t]}/1^{[r]}\otimes 1^{[u]}\otimes 1^{[v]}/3^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}/3^{[s]}\otimes 1^{[t]}\otimes 1^{[v]}$	$W(6)^{[s]}/2^{[r]}/1^{[r]}\otimes 3^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}/1^{[r]}\otimes 3^{[s]}\otimes 1^{[t]}\otimes 1^{[v]}/4^{[s]}\otimes 2^{[t]}/4^{[s]}\otimes 1^{[u]}\otimes 1^{[v]}/2^{[s]}/2^{[t]}\otimes 1^{[u]}\otimes 1^{[v]}/2^{[t]}/2^{[u]}/2^{[v]}$
9	$1^{[r]}\otimes 4^{[s]}/1^{[r]}\otimes 2^{[s]}/1^{[r]}\otimes 1^{[t]}\otimes 1^{[u]}/4^{[s]}\otimes 1^{[t]}/4^{[s]}\otimes 1^{[u]}/2^{[s]}\otimes 1^{[t]}/2^{[s]}\otimes 1^{[u]}$	$2^{[r]}/1^{[r]}\otimes 4^{[s]}\otimes 1^{[t]}/1^{[r]}\otimes 4^{[s]}\otimes 1^{[u]}/1^{[r]}\otimes 2^{[s]}\otimes 1^{[t]}/1^{[r]}\otimes 2^{[s]}\otimes 1^{[u]}/(W(6)^{[s]})^2/4^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}/2^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}/4^{[s]}/(2^{[s]})^3/2^{[t]}/2^{[u]}$
10	$1^{[r]}\otimes 2^{[s]}\otimes 2^{[t]}/1^{[r]}\otimes 2^{[u]}/W(3)^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}/W(3)^{[t]}\otimes 1^{[s]}\otimes 1^{[u]}$	$2^{[r]}/1^{[r]}\otimes W(3)^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}/1^{[r]}\otimes 1^{[s]}\otimes W(3)^{[t]}\otimes 1^{[u]}/W(4)^{[s]}\otimes 2^{[t]}/2^{[s]}\otimes W(4)^{[t]}/2^{[s]}\otimes 2^{[t]}\otimes 2^{[u]}/2^{[s]}/2^{[t]}/2^{[u]}$
11	$1^{[r]}\otimes 2^{[s]}/1^{[r]}\otimes 2^{[t]}/1^{[r]}\otimes 2^{[u]}/1^{[r]}\otimes 2^{[v]}/(1^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}\otimes 1^{[v]})^2$	$2^{[r]}/(1^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}\otimes 1^{[v]})^2/2^{[s]}/2^{[s]}\otimes 2^{[t]}/2^{[s]}\otimes 2^{[u]}/2^{[s]}\otimes 2^{[v]}/2^{[t]}/2^{[t]}\otimes 2^{[u]}/2^{[t]}\otimes 2^{[v]}/2^{[u]}/2^{[u]}\otimes 2^{[v]}/2^{[v]}$
12	$1\otimes 1^{[r]}\otimes 1^{[s]}/1\otimes 1^{[t]}\otimes 1^{[u]}/1\otimes 1^{[v]}\otimes 1^{[w]}/1^{[r]}\otimes 1^{[t]}\otimes 1^{[v]}/1^{[r]}\otimes 1^{[u]}\otimes 1^{[w]}/1^{[s]}\otimes 1^{[t]}\otimes 1^{[w]}/1^{[s]}\otimes 1^{[u]}\otimes 1^{[v]}$	$W(2)/1\otimes 1^{[r]}\otimes 1^{[u]}\otimes 1^{[v]}/1\otimes 1^{[r]}\otimes 1^{[t]}\otimes 1^{[w]}/1\otimes 1^{[s]}\otimes 1^{[t]}\otimes 1^{[v]}/1\otimes 1^{[s]}\otimes 1^{[u]}\otimes 1^{[w]}/W(2)^{[r]}/1^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}/1^{[r]}\otimes 1^{[s]}\otimes 1^{[v]}\otimes 1^{[w]}/W(2)^{[s]}/W(2)^{[t]}/1^{[t]}\otimes 1^{[u]}\otimes 1^{[v]}\otimes 1^{[w]}/W(2)^{[u]}/W(2)^{[v]}/W(2)^{[w]}/$

13	$1^{[r]} \otimes 2^{[s]}/1^{[r]} \otimes 2^{[t]}/1^{[r]} \otimes 2^{[u]}/ 1^{[r]} \otimes 1^{[v]} \otimes 1^{[w]}/(1^{[r]})^2/ 1^{[s]} \otimes 1^{[t]} \otimes 1^{[u]} \otimes 1^{[v]}/ 1^{[s]} \otimes 1^{[t]} \otimes 1^{[u]} \otimes 1^{[w]}$	$2^{[r]}/1^{[r]} \otimes 1^{[s]} \otimes 1^{[t]} \otimes 1^{[u]} \otimes 1^{[v]}/1^{[r]} \otimes 1^{[s]} \otimes 1^{[t]} \otimes 1^{[u]} \otimes 1^{[w]}/ 2^{[s]} \otimes 2^{[t]}/2^{[s]} \otimes 2^{[u]}/2^{[s]} \otimes 1^{[v]} \otimes 1^{[w]}/(2^{[s]})^2/2^{[t]} \otimes 2^{[u]}/ 2^{[t]} \otimes 1^{[v]} \otimes 1^{[w]}/(2^{[t]})^2/2^{[u]} \otimes 1^{[v]} \otimes 1^{[w]}/(2^{[u]})^2/2^{[v]}/ (1^{[v]} \otimes 1^{[w]})^2/2^{[w]}/0^6$
14	$1^{[r]} \otimes 2^{[s]}/1^{[r]} \otimes 2^{[t]}/1^{[r]} \otimes 2^{[u]}/ 1^{[r]} \otimes 2^{[v]}/1^{[r]} \otimes 2^{[w]}/(1^{[r]})^2/ 1^{[s]} \otimes 1^{[t]} \otimes 1^{[u]} \otimes 1^{[v]} \otimes 1^{[w]}$	$2^{[r]}/1^{[r]} \otimes 1^{[s]} \otimes 1^{[t]} \otimes 1^{[u]} \otimes 1^{[v]} \otimes 1^{[w]}/2^{[s]} \otimes 2^{[t]}/2^{[s]} \otimes 2^{[u]}/ 2^{[s]} \otimes 2^{[v]}/2^{[s]} \otimes 2^{[w]}/(2^{[s]})^2/2^{[t]} \otimes 2^{[u]}/2^{[t]} \otimes 2^{[v]}/2^{[t]} \otimes 2^{[w]}/ (2^{[t]})^2/2^{[u]} \otimes 2^{[v]}/2^{[u]} \otimes 2^{[w]}/(2^{[u]})^2/2^{[v]} \otimes 2^{[w]}/(2^{[v]})^2/(2^{[w]})^2/0^6$
15	$6^{[r]} \otimes 5^{[s]}/W(9)^{[s]}/3^{[s]}$	$W(10)^{[r]}/6^{[r]} \otimes W(8)^{[s]}/6^{[r]} \otimes 4^{[s]}/2^{[r]}/W(10)^{[s]}/6^{[s]}/2^{[s]}$
16	$6^{[r]} \otimes 2^{[s]} \otimes 1^{[t]}/4^{[s]} \otimes 1^{[t]}/3^{[t]}$	$W(10)^{[r]}/6^{[r]} \otimes 4^{[s]}/6^{[r]} \otimes 2^{[s]} \otimes 2^{[t]}/2^{[r]}/4^{[s]} \otimes 2^{[t]}/2^{[s]}/2^{[t]}$
17	$3^{[r]} \otimes 6^{[s]}/1^{[r]} \otimes W(10)^{[s]}/1^{[r]} \otimes 2^{[s]}$	$4^{[r]} \otimes 6^{[s]}/2^{[r]} \otimes W(12)^{[s]}/2^{[r]} \otimes 8^{[s]}/2^{[r]} \otimes 4^{[s]}/2^{[r]}/W(10)^{[s]}/2^{[s]}$
18	$3^{[r]}/1^{[r]} \otimes W(16)^{[s]}/1^{[r]} \otimes 8^{[s]}$	$2^{[r]} \otimes W(16)^{[s]}/2^{[r]} \otimes 8^{[s]}/2^{[r]}/W(22)^{[s]}/W(14)^{[s]}/W(10)^{[s]}/2^{[s]}$
19	$W(6)^{[r]} \otimes 3^{[s]}/4^{[r]} \otimes 1^{[s]}/2^{[r]} \otimes 5^{[s]}$	$W(6)^{[r]} \otimes 4^{[s]}/4^{[r]} \otimes W(6)^{[s]}/4^{[r]} \otimes 2^{[s]}/2^{[r]} \otimes W(8)^{[s]}/2^{[r]} \otimes 4^{[s]}/ 2^{[r]}/2^{[s]}$
20	$W(21)/15/11/5$	$W(26)/W(22)/W(18)/16/14/10^2/6/2$
21	$W(27)/17/9$	$W(34)/W(26)/W(22)/18/14/10/2$

Table 13: The composition factors of the irreducible A_1 subgroups of E_8.

ID	Composition Factors of $L(E_8) \downarrow X$
1	$W(16)^{[r]}/W(14)^{[r]}/(W(12)^{[r]} \otimes 2^{[s]})^2/(10^{[r]})^2/(8^{[r]} \otimes 2^{[s]})^2/8^{[r]}/6^{[r]}/(4^{[r]} \otimes 2^{[s]})^2/4^{[r]}/2^{[r]}/4^{[s]}/2^{[s]}$
2	$W(15)^{[r]} \otimes 1^{[s]}/W(14)^{[r]}/W(12)^{[r]} \otimes 2^{[s]}/W(11)^{[r]} \otimes 1^{[s]}/10^{[r]}/9^{[r]} \otimes 1^{[s]}/8^{[r]} \otimes 2^{[s]}/7^{[r]} \otimes 3^{[s]}/6^{[r]}/ 5^{[r]} \otimes 1^{[s]}/4^{[r]} \otimes 2^{[s]}/3^{[r]} \otimes 1^{[s]}/2^{[r]}/2^{[s]}$
3	$W(8)/(W(6) \otimes 4^{[r]})^2/W(6)/(4 \otimes W(6)^{[r]})^2/(4 \otimes 2^{[r]})^2/4/(2 \otimes 4^{[r]})^2/2/W(8)^{[r]}/W(6)^{[r]}/4^{[r]}/2^{[r]}$
4	$W(7) \otimes 3^{[r]}/W(6) \otimes 4^{[r]}/W(6)/W(5) \otimes 3^{[r]}/4 \otimes W(6)^{[r]}/4 \otimes 2^{[r]}/3 \otimes W(7)^{[r]}/3 \otimes W(5)^{[r]}/3 \otimes 1^{[r]}/ 2 \otimes 4^{[r]}/2/1 \otimes 3^{[r]}/W(6)^{[r]}/2^{[r]}$
5	$W(10)^{[r]}/W(9)^{[r]} \otimes 1^{[t]}/W(8)^{[r]} \otimes 2^{[s]}/W(8)^{[r]} \otimes 1^{[s]} \otimes 1^{[u]}/6^{[r]}/5^{[r]} \otimes 2^{[s]} \otimes 1^{[t]}/5^{[r]} \otimes 1^{[s]} \otimes 1^{[t]} \otimes 1^{[u]}/ 4^{[r]} \otimes 2^{[s]}/4^{[r]} \otimes 1^{[s]} \otimes 1^{[u]}/3^{[r]} \otimes 1^{[t]}/2^{[r]}/3^{[s]} \otimes 1^{[u]}/2^{[s]}/2^{[t]}/2^{[u]}$
6	$W(4)^{[r]} \otimes 2^{[s]}/W(4)^{[r]} \otimes 1^{[s]} \otimes 1^{[u]}/W(4)^{[r]} \otimes 2^{[t]}/W(4)^{[r]} \otimes 1^{[t]} \otimes 1^{[v]}/2^{[r]} \otimes 2^{[s]} \otimes 2^{[t]}/ 2^{[r]} \otimes 2^{[s]} \otimes 1^{[t]} \otimes 1^{[v]}/2^{[r]} \otimes 1^{[s]} \otimes 2^{[t]} \otimes 1^{[v]}/2^{[r]} \otimes 1^{[s]} \otimes 1^{[t]} \otimes 1^{[u]} \otimes 1^{[v]}/2^{[r]}/3^{[s]} \otimes 1^{[v]}/2^{[s]}/3^{[t]} \otimes 1^{[u]}/ 2^{[t]}/2^{[u]}/2^{[v]}$
7	$W(9)^{[r]} \otimes 1^{[s]}/W(8)^{[r]} \otimes 2^{[s]}/W(7)^{[r]} \otimes 3^{[s]}/W(6)^{[r]} \otimes 4^{[s]}/W(6)^{[r]}/W(5)^{[r]} \otimes 3^{[s]}/W(5)^{[r]} \otimes 1^{[s]}/ (4^{[r]} \otimes 2^{[s]})^2/3^{[r]} \otimes W(5)^{[s]}/3^{[r]} \otimes 1^{[s]}/2^{[r]} \otimes 4^{[s]}/2^{[r]}/1^{[r]} \otimes 3^{[s]}/2^{[s]}$
8	$W(3) \otimes 1^{[r]} \otimes 1^{[s]} \otimes 1^{[t]}/W(2) \otimes W(2)^{[r]} \otimes W(2)^{[s]}/W(2) \otimes W(2)^{[r]} \otimes W(2)^{[t]}/ W(2) \otimes W(2)^{[s]} \otimes W(2)^{[t]}/W(2)/1 \otimes W(3)^{[r]} \otimes 1^{[s]} \otimes 1^{[t]}/1 \otimes 1^{[r]} \otimes W(3)^{[s]} \otimes 1^{[t]}/ 1 \otimes 1^{[r]} \otimes 1^{[s]} \otimes W(3)^{[t]}/W(2)^{[r]} \otimes W(2)^{[s]} \otimes W(2)^{[t]}/W(2)^{[r]}/W(2)^{[s]}/W(2)^{[t]}$
9	$W(4)/(2 \otimes 2^{[r]} \otimes 2^{[s]})^2/(2 \otimes 2^{[r]} \otimes 2^{[t]})^2/(2 \otimes 2^{[s]} \otimes 2^{[t]})^2/2/W(4)^{[r]}/(2^{[r]} \otimes 2^{[s]} \otimes 2^{[t]})^2/2^{[r]}/W(4)^{[s]}/ 2^{[s]}/W(4)^{[t]}/2^{[t]}$
10	$W(28)/W(26)/W(22)^2/W(18)^2/16/14^3/10^2/8/6/4/2$
11	$W(22)^{[r]}/W(21)^{[r]} \otimes 1^{[s]}/W(18)/W(15)^{[r]} \otimes 1^{[s]}/W(14)^{[r]}/12^{[r]} \otimes 2^{[s]}/11^{[r]} \otimes 1^{[s]}/10^{[r]}/9^{[r]} \otimes 1^{[s]}/6^{[r]}/ 3^{[r]} \otimes 1^{[s]}/2^{[r]}/2^{[s]}$
12	$W(18)^{[r]}/W(15)^{[r]} \otimes 3^{[s]}/W(14)^{[r]}/10^{[r]} \otimes 4^{[s]}/10^{[r]}/9^{[r]} \otimes 3^{[s]}/6^{[r]}/5^{[r]} \otimes 3^{[s]}/2^{[r]}/6^{[s]}/2^{[s]}$

13 $W(18)^{[r]}/W(15)^{[r]}\otimes 1^{[s]}/W(15)^{[r]}\otimes 1^{[t]}/W(14)^{[r]}/10^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}/(10^{[r]})^2/9^{[r]}\otimes 1^{[s]}/9^{[r]}\otimes 1^{[t]}/6^{[r]}/5^{[r]}\otimes 1^{[s]}/5^{[r]}\otimes 1^{[t]}/2^{[r]}/2^{[s]}/1^{[s]}\otimes 1^{[t]}/2^{[t]}$

14 $W(14)^{[r]}/10^{[r]}\otimes 6^{[s]}/(10^{[r]})^2/8^{[r]}\otimes 6^{[s]}/6^{[r]}/4^{[r]}\otimes 6^{[s]}/4^{[r]}/2^{[r]}/10^{[s]}/6^{[s]}/2^{[s]}$

15 $W(14)^{[r]}/10^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}/10^{[r]}\otimes 1^{[s]}\otimes 1^{[u]}/10^{[r]}/8^{[r]}\otimes 2^{[s]}/8^{[r]}\otimes 1^{[t]}\otimes 1^{[u]}/6^{[r]}/4^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}/4^{[r]}\otimes 1^{[s]}\otimes 1^{[u]}/2^{[r]}/2^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}/2^{[s]}/2^{[t]}/2^{[u]}$

16 $W(10)^{[r]}/6^{[r]}\otimes 3^{[s]}\otimes 1^{[t]}/6^{[r]}\otimes 2^{[s]}\otimes 2^{[t]}/6^{[r]}\otimes 1^{[s]}\otimes 3^{[t]}/6^{[r]}/2^{[r]}/4^{[s]}\otimes 2^{[t]}/3^{[s]}\otimes 1^{[t]}/2^{[s]}\otimes 4^{[t]}/2^{[s]}/1^{[s]}\otimes 3^{[t]}/2^{[t]}$

17 $W(10)^{[r]}/6^{[r]}\otimes 4^{[s]}/6^{[r]}\otimes 3^{[s]}\otimes 1^{[t]}/6^{[r]}\otimes 3^{[s]}\otimes 1^{[u]}/6^{[r]}\otimes 1^{[t]}\otimes 1^{[u]}/6^{[r]}/2^{[r]}/6^{[s]}/4^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}/3^{[s]}\otimes 1^{[t]}/3^{[s]}\otimes 1^{[u]}/2^{[s]}/2^{[t]}/2^{[u]}$

18 $W(10)^{[r]}/6^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}/6^{[r]}\otimes 1^{[s]}\otimes 1^{[u]}/6^{[r]}\otimes 1^{[s]}\otimes 1^{[v]}/6^{[r]}\otimes 1^{[t]}\otimes 1^{[u]}/6^{[r]}\otimes 1^{[t]}\otimes 1^{[v]}/6^{[r]}\otimes 1^{[u]}\otimes 1^{[v]}/(6^{[r]})^2/2^{[r]}/2^{[s]}/1^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}\otimes 1^{[v]}/1^{[s]}\otimes 1^{[t]}/1^{[s]}\otimes 1^{[u]}/1^{[s]}\otimes 1^{[v]}/2^{[t]}/1^{[t]}\otimes 1^{[u]}/1^{[t]}\otimes 1^{[v]}/2^{[u]}/1^{[u]}\otimes 1^{[v]}/2^{[v]}$

19 $W(10)^{[r]}/6^{[r]}\otimes 2^{[s]}/(6^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}\otimes 1^{[u]})^2/6^{[r]}\otimes 2^{[t]}/6^{[r]}\otimes 2^{[u]}/6^{[r]}/2^{[r]}/2^{[s]}\otimes 2^{[t]}/2^{[s]}\otimes 2^{[u]}/2^{[s]}/(1^{[s]}\otimes 1^{[t]}\otimes 1^{[u]})^2/2^{[t]}\otimes 2^{[u]}/2^{[t]}/2^{[u]}$

20 $W(6)/4\otimes 4^{[r]}/4\otimes 4^{[s]}/4/(3\otimes 3^{[r]}\otimes 3^{[s]})^2/2/W(6)^{[r]}/4^{[r]}\otimes 4^{[s]}/4^{[r]}/2^{[r]}/W(6)^{[s]}/4^{[s]}/2^{[s]}$

21 $W(6)^{[r]}/4^{[r]}\otimes 4^{[s]}/4^{[r]}\otimes 2^{[t]}/4^{[r]}\otimes 2^{[u]}/(3^{[r]}\otimes 3^{[s]}\otimes 1^{[t]}\otimes 1^{[u]})^2/2^{[r]}/W(6)^{[s]}/4^{[s]}\otimes 2^{[t]}/4^{[s]}\otimes 2^{[u]}/2^{[s]}/2^{[t]}\otimes 2^{[u]}/2^{[t]}/2^{[u]}$

22 $W(6)^{[r]}/4^{[r]}\otimes 2^{[s]}/4^{[r]}\otimes 3^{[t]}\otimes 1^{[u]}/3^{[r]}\otimes 1^{[s]}\otimes 4^{[t]}/3^{[r]}\otimes 1^{[s]}\otimes 3^{[t]}\otimes 1^{[u]}/3^{[r]}\otimes 1^{[s]}\otimes 2^{[u]}/2^{[r]}/2^{[s]}\otimes 3^{[t]}\otimes 1^{[u]}/2^{[s]}/W(6)^{[t]}/4^{[t]}\otimes 2^{[u]}/2^{[t]}/2^{[u]}$

23 $W(6)^{[r]}/4^{[r]}\otimes 2^{[s]}/4^{[r]}\otimes 1^{[t]}\otimes 1^{[u]}/4^{[r]}\otimes 1^{[v]}\otimes 1^{[w]}/3^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}\otimes 1^{[v]}/3^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}\otimes 1^{[w]}/3^{[r]}\otimes 1^{[s]}\otimes 1^{[u]}\otimes 1^{[v]}/3^{[r]}\otimes 1^{[s]}\otimes 1^{[u]}\otimes 1^{[w]}/2^{[r]}/2^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}/2^{[s]}\otimes 1^{[v]}\otimes 1^{[w]}/2^{[s]}/2^{[t]}/1^{[t]}\otimes 1^{[u]}\otimes 1^{[v]}\otimes 1^{[w]}/2^{[u]}/2^{[v]}/2^{[w]}$

24 $W(4)^{[r]}\otimes 2^{[s]}/W(3)^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}/W(3)^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}\otimes 1^{[v]}/2^{[r]}\otimes W(4)^{[s]}/2^{[r]}\otimes 2^{[s]}\otimes 2^{[t]}/2^{[r]}\otimes 2^{[s]}\otimes 1^{[u]}\otimes 1^{[v]}/2^{[r]}/1^{[r]}\otimes W(3)^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}/1^{[r]}\otimes W(3)^{[s]}\otimes 1^{[t]}\otimes 1^{[v]}/2^{[s]}/2^{[t]}\otimes 1^{[u]}\otimes 1^{[v]}/2^{[t]}/2^{[u]}/2^{[v]}$

25 $2^{[r]}\otimes 2^{[s]}/2^{[r]}\otimes 2^{[t]}/2^{[r]}\otimes 2^{[u]}/2^{[r]}\otimes 1^{[v]}\otimes 1^{[w]}/2^{[r]}/(1^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}\otimes 1^{[v]})^2/(1^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}\otimes 1^{[w]})^2/2^{[s]}\otimes 2^{[t]}/2^{[s]}\otimes 2^{[u]}/2^{[s]}\otimes 1^{[v]}\otimes 1^{[w]}/2^{[s]}/2^{[t]}\otimes 2^{[u]}/2^{[t]}\otimes 1^{[v]}\otimes 1^{[w]}/2^{[t]}/2^{[u]}\otimes 1^{[v]}\otimes 1^{[w]}/2^{[u]}/2^{[v]}/2^{[w]}$

26 $W(2)/1\otimes 1^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}/1\otimes 1^{[r]}\otimes 1^{[u]}\otimes 1^{[v]}/1\otimes 1^{[r]}\otimes 1^{[w]}\otimes 1^{[x]}/1\otimes 1^{[s]}\otimes 1^{[u]}\otimes 1^{[w]}/1\otimes 1^{[s]}\otimes 1^{[v]}\otimes 1^{[x]}/1\otimes 1^{[t]}\otimes 1^{[u]}\otimes 1^{[x]}/1\otimes 1^{[t]}\otimes 1^{[v]}\otimes 1^{[w]}/W(2)^{[r]}/1^{[r]}\otimes 1^{[s]}\otimes 1^{[u]}\otimes 1^{[x]}/1^{[r]}\otimes 1^{[s]}\otimes 1^{[v]}\otimes 1^{[w]}/1^{[r]}\otimes 1^{[t]}\otimes 1^{[u]}\otimes 1^{[w]}/1^{[r]}\otimes 1^{[t]}\otimes 1^{[v]}\otimes 1^{[x]}/W(2)^{[s]}/1^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}\otimes 1^{[v]}/1^{[s]}\otimes 1^{[t]}\otimes 1^{[w]}\otimes 1^{[x]}/W(2)^{[t]}/W(2)^{[u]}/1^{[u]}\otimes 1^{[v]}\otimes 1^{[w]}\otimes 1^{[x]}/W(2)^{[v]}/W(2)^{[w]}/W(2)^{[x]}$

27 $2^{[r]}\otimes 2^{[s]}/2^{[r]}\otimes 2^{[t]}/2^{[r]}\otimes 1^{[u]}\otimes 1^{[v]}/2^{[r]}\otimes 1^{[w]}\otimes 1^{[x]}/(2^{[r]})^2/1^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}\otimes 1^{[w]}/1^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}\otimes 1^{[v]}\otimes 1^{[x]}/1^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}\otimes 1^{[x]}/1^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}\otimes 1^{[v]}\otimes 1^{[w]}/2^{[s]}\otimes 2^{[t]}/2^{[s]}\otimes 1^{[u]}\otimes 1^{[v]}/2^{[s]}\otimes 1^{[w]}\otimes 1^{[x]}/(2^{[s]})^2/2^{[t]}\otimes 1^{[u]}\otimes 1^{[v]}/2^{[t]}\otimes 1^{[w]}\otimes 1^{[x]}/(2^{[t]})^2/2^{[u]}/1^{[u]}\otimes 1^{[v]}\otimes 1^{[w]}\otimes 1^{[x]}/(1^{[u]}\otimes 1^{[v]})^2/2^{[v]}/2^{[w]}/(1^{[w]}\otimes 1^{[x]})^2/2^{[x]}/0^8$

28 $2\otimes 2^{[r]}/2\otimes 2^{[s]}/2\otimes 1^{[u]}\otimes 1^{[v]}\otimes 1^{[w]}/2^2/1\otimes 1^{[r]}\otimes 1^{[s]}\otimes 2^{[t]}/1\otimes 1^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}\otimes 1^{[v]}/1\otimes 1^{[r]}\otimes 1^{[s]}\otimes 2^{[u]}/1\otimes 1^{[r]}\otimes 1^{[s]}\otimes 2^{[v]}/(1\otimes 1^{[r]}\otimes 1^{[s]})^2/2^{[r]}\otimes 2^{[s]}/(2^{[r]})^2/2^{[r]}\otimes 1^{[t]}\otimes 1^{[u]}\otimes 1^{[v]}/2^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}\otimes 1^{[v]}/(2^{[s]})^2/2^{[t]}\otimes 2^{[u]}/2^{[t]}\otimes 2^{[v]}/(2^{[t]})^2/(1^{[t]}\otimes 1^{[u]}\otimes 1^{[v]})^2/2^{[u]}\otimes 2^{[v]}/(2^{[u]})^2/(2^{[v]})^2/0^8$

29 $2^{[r]}\otimes 2^{[s]}/2^{[r]}\otimes 2^{[t]}/2^{[r]}\otimes 2^{[u]}/2^{[r]}\otimes 2^{[v]}/2^{[r]}\otimes 1^{[w]}\otimes 1^{[x]}/(2^{[r]})^2/1^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}\otimes 1^{[v]}\otimes 1^{[w]}/1^{[r]}\otimes 1^{[s]}\otimes 1^{[t]}\otimes 1^{[u]}\otimes 1^{[v]}\otimes 1^{[x]}/2^{[s]}\otimes 2^{[t]}/2^{[s]}\otimes 2^{[u]}/2^{[s]}\otimes 2^{[v]}/2^{[s]}\otimes 1^{[w]}\otimes 1^{[x]}/(2^{[s]})^2/2^{[t]}\otimes 2^{[u]}/2^{[t]}\otimes 2^{[v]}/2^{[t]}\otimes 1^{[w]}\otimes 1^{[x]}/(2^{[t]})^2/2^{[u]}\otimes 2^{[v]}/2^{[u]}\otimes 1^{[w]}\otimes 1^{[x]}/(2^{[u]})^2/2^{[v]}\otimes 1^{[w]}\otimes 1^{[x]}/(2^{[v]})^2/2^{[w]}/(1^{[w]}\otimes 1^{[x]})^2/2^{[x]}/0^8$

30	$2 \otimes 2^{[r]}/2 \otimes 2^{[s]}/2 \otimes 2^{[t]}/2 \otimes 2^{[u]}/2 \otimes 2^{[v]}/2 \otimes 2^{[w]}/2^2/1 \otimes 1^{[r]} \otimes 1^{[s]} \otimes 1^{[t]} \otimes 1^{[u]} \otimes 1^{[v]} \otimes 1^{[w]}/2^{[r]} \otimes 2^{[s]}/2^{[r]} \otimes 2^{[t]}/2^{[r]} \otimes 2^{[u]}/2^{[r]} \otimes 2^{[v]}/2^{[r]} \otimes 2^{[w]}/(2^{[r]})^2/2^{[s]} \otimes 2^{[t]}/2^{[s]} \otimes 2^{[u]}/2^{[s]} \otimes 2^{[v]}/2^{[s]} \otimes 2^{[w]}/(2^{[s]})^2/2^{[t]} \otimes 2^{[u]}/2^{[t]} \otimes 2^{[v]}/2^{[t]} \otimes 2^{[w]}/(2^{[t]})^2/2^{[u]} \otimes 2^{[v]}/2^{[u]} \otimes 2^{[w]}/(2^{[u]})^2/2^{[v]} \otimes 2^{[w]}/(2^{[v]})^2/(2^{[w]})^2/0^8$
31	$2^{[r]}/1^{[r]} \otimes 6^{[s]} \otimes 5^{[t]}/1^{[r]} \otimes W(9)^{[t]}/1^{[r]} \otimes 3^{[t]}/W(10)^{[s]}/6^{[s]} \otimes W(8)^{[t]}/6^{[s]} \otimes 4^{[t]}/2^{[s]}/W(10)^{[t]}/6^{[t]}/2^{[t]}$
32	$2^{[r]}/1^{[r]} \otimes 6^{[s]} \otimes 2^{[t]} \otimes 1^{[u]}/1^{[r]} \otimes 4^{[t]} \otimes 1^{[u]}/1^{[r]} \otimes 3^{[u]}/W(10)^{[s]}/6^{[s]} \otimes 4^{[t]}/6^{[s]} \otimes 2^{[t]} \otimes 2^{[u]}/2^{[s]}/4^{[t]} \otimes 2^{[u]}/2^{[t]}/2^{[u]}$
33	$2^{[r]}/1^{[r]} \otimes 3^{[s]} \otimes 6^{[t]}/1^{[r]} \otimes 1^{[s]} \otimes W(10)^{[t]}/1^{[r]} \otimes 1^{[s]} \otimes 2^{[t]}/4^{[s]} \otimes 6^{[t]}/2^{[s]} \otimes W(12)^{[t]}/2^{[s]} \otimes W(8)^{[t]}/2^{[s]} \otimes 4^{[t]}/2^{[s]}/W(10)^{[t]}/2^{[t]}$
34	$2^{[r]}/1^{[r]} \otimes 3^{[s]}/1^{[r]} \otimes 1^{[s]} \otimes W(16)^{[t]}/1^{[r]} \otimes 1^{[s]} \otimes 8^{[t]}/2^{[s]} \otimes W(16)^{[t]}/2^{[s]} \otimes 8^{[t]}/2^{[s]}/W(22)^{[t]}/W(14)^{[t]}/10^{[t]}/2^{[t]}$
35	$2^{[r]}/1^{[r]} \otimes W(6)^{[s]} \otimes 3^{[t]}/1^{[r]} \otimes 4^{[s]} \otimes 1^{[t]}/1^{[r]} \otimes 2^{[s]} \otimes W(5)^{[t]}/W(6)^{[s]} \otimes 4^{[t]}/4^{[s]} \otimes W(6)^{[t]}/4^{[s]} \otimes 2^{[t]}/2^{[s]} \otimes W(8)^{[t]}/2^{[s]} \otimes 4^{[t]}/2^{[s]}/2^{[t]}$
36	$2^{[r]}/1^{[r]} \otimes W(21)^{[s]}/1^{[r]} \otimes 15^{[s]}/1^{[r]} \otimes 11^{[s]}/1^{[r]} \otimes 5^{[s]}/W(26)^{[s]}/W(22)^{[s]}/W(18)^{[s]}/16^{[s]}/14^{[s]}/(10^{[s]})^2/6^{[s]}/2^{[s]}$
37	$2^{[r]}/1^{[r]} \otimes W(27)^{[s]}/1^{[r]} \otimes 17^{[s]}/1^{[r]} \otimes 9^{[s]}/W(34)^{[s]}/W(26)^{[s]}/W(22)^{[s]}/18^{[s]}/14^{[s]}/10^{[s]}/2^{[s]}$
38	$10^{[r]}/6^{[r]} \otimes W(16)^{[s]}/6^{[r]} \otimes 8^{[s]}/2^{[r]}/W(22)^{[s]}/W(14)^{[s]}/10^{[s]}/2^{[s]}$
39	$W(10)^{[r]}/6^{[r]} \otimes 4^{[s]}/6^{[r]} \otimes 2^{[s]} \otimes 6^{[t]}/2^{[r]}/4^{[s]} \otimes 6^{[t]}/2^{[s]}/W(10)^{[t]}/2^{[t]}$
40	$W(38)/W(34)/W(28)/W(26)/22^2/18/16/14/10/6/2$
41	$W(46)/W(38)/W(34)/28/26/22/18/14/10/2$
42	$W(58)/W(46)/W(38)/W(34)/26/22/14/2$

12 Conditions for conjugacy class representatives of G-irreducible A_1 subgroups of $G = E_7$ and E_8

In this section we present the tables referred to within Table 7 and 8. They give the extra restrictions on the field twists in certain embeddings of M-irreducible A_1 subgroups of $M = A_1D_6$ when $G = E_7$ and $M = D_8$ when $G = E_8$, namely $E_7(\#12)$, $E_8(\#23)$, $E_8(\#26)$ and $E_8(\#27)$. These restrictions ensure there is no repetition of conjugacy classes and further, that each conjugacy class is G-irreducible. The restrictions are given in rows of the tables: the first column lists all equalities amongst a subset of the field twists; the second column lists any further requirements. So an ordered set $\{0, r, \ldots\}$ is permitted if it satisfies the conditions in the first and second column of a row of the table (and the set of field twists satisfying each row are mutually exclusive).

We give an example. Consider $X = E_7(\#12)$. Then X is a diagonal subgroup of $A_1^7 < A_1D_6$ via $(1, 1^{[r]}, 1^{[s]}, 1^{[t]}, 1^{[u]}, 1^{[v]}, 1^{[w]})$. Table 14 gives the conditions that an ordered set $0, r, s, t, u, v, w$ needs to satisfy. So $0, r, \ldots, w$ satisfy the conditions of the first row if: $0, r, \ldots, w$ are all distinct; r is the smallest integer of $r, s, \ldots, w$; and t is the smallest integer of t, u, v, w. Similarly, $0, r, \ldots, w$ satisfy the conditions of the ninth row if: $r = 0$; $s = t$; $0, s, u, v, w$ are distinct; and $u < v$.

Table 14: Conditions on field twists for $E_7(\#12)$

All equalities among $0, r, \ldots, w$	Further requirements on $0, r, \ldots, w$
none	$r < \min\{s, t, u, v, w\}$ and $t < \min\{u, v, w\}$
$r = s$	$t < \min\{u, v, w\}$ and $v < w$

$r = t$	$s < v$
$r = s = t$	none
$r = t = w$	$s < u, v$
$r = s = t = v$	$u < w$
$r = t$ and $u = v = w$	none
$r = 0$	$t < \min\{u, v, w\}$ and $v < w$
$r = 0$ and $s = t$	$u < v$
$r = 0$ and $t = u$	$v < w$
$r = 0$ and $s = t = u$	$v < w$
$r = 0$ and $s = t$ and $u = w$	none
$r = s = 0$	$t < u < v < w$
$r = t = 0$	$s < u < v$
$r = t = 0$ and $s = w$	$u < v$
$r = s = t = 0$	$u < v < w$

In the following table, note that the first column lists all equalities occurring among the elements r, s and all equalities among the elements t, u, v, w only. So r is still permitted to be equal to t, for example, in all of the conditions in the first column (this is ruled out by the conditions in the second column for the third and fourth rows).

Table 15: Conditions on field twists for $E_8(\#23)$

All equalities among r, s and all equalities among t, u, v, w	Further requirements on $r, \ldots, w$
none	$t < u$ and $t < v$ and $v < w$
$r = s$	$t < u < v < w$
$r = s$ and $t = u$	$r < t$ and $v < w$
$r = s$ and $t = u = v$	$r \neq t$
$t = u$	$s < t$ and $v < w$
$t = v$	$u < w$
$t = u = v$	none

Table 16: Conditions on field twists for $E_8(\#26)$

All equalities among $0, r, \ldots, w$	Further requirements on $0, r, \ldots, w$
none	$r < s$ and $s < \min\{t, u\}$ and $u < \min\{v, w, x\}$
$r = s$	$u < v, w, x$ and $w < x$
$r = s = t$	$u < v < w < x$
$r = s = u$	$t < v < w$
$r = s = t = u$	$v < w < x$
$r = s$ and $t = u$	$v < w$

$r = s$ and $u = v$	$w < x$
$r = s$ and $t = u = v$	$w < x$
$r = s$ and $t = u$ and $v = w$	none
$r = 0$	$s < \min\{t, u\}$ and $u < \min\{v, w\}$ and $w < x$
$r = 0$ and $s = u$	$t < v$ and $w < x$
$r = 0$ and $s = u = w$	$t < v < x$
$r = 0$ and $s = u$ and $t = w$	none
$r = 0$ and $s = u$ and $t = w$ and $v = x$	none
$r = s = 0$	$u < v < w < x$
$r = s = 0$ and $t = u$	$v < w < x$
$r = s = u = 0$	$t < v < w < x$

Table 17: Conditions on field twists for $E_8(\#27)$

All equalities amongst u, v, w, x	Further requirements on $r, \ldots, x$
none	$r < s < t$ and $u < v$ and $u < w$ and $w < x$
$u = w$	$r < s < t$ and $v < x$

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Appendix A Levi subgroups

Let G be an exceptional algebraic group over an algebraically closed field. This section contains tables of composition factors for Levi subgroups of G on the minimal and adjoint modules for G. If L' is simple then these are found in [15, Tables 8.1–8.7]. If L' is not simple then the composition factors are deduced from those of a maximal subsystem subgroup containing L'.

Table 18: The composition factors for the action of Levi subgroups of G_2 on V_7 and $L(G_2)$.

Levi L'	Comp. factors of $V_7 \downarrow L'$	Comp. factors of $L(G_2) \downarrow L'$
A_1	$1^2/0^3$	$W(2)/1^4/0^3$
$\tilde{A}_1$	$W(2)/1^2$	$W(3)^2/W(2)/0^3$

Table 19: The composition factors for the action of Levi subgroups of F_4 on V_{26} and $L(F_4)$.

Levi L'	Comp. factors of $V_{26} \downarrow L'$	Comp. factors of $L(F_4) \downarrow L'$
B_3	$W(100)/001^2/000^3$	$W(100)^2/W(010)/001^2/000$
B_2	$W(10)/01^4/00^5$	$W(10)^4/W(02)/01^4/00^6$
C_3	$100^2/W(010)$	$W(200)/W(001)^2/000^3$
$A_2\tilde{A}_1$	$(10,1)/(10,0)/(01,1)/(01,0)/(00,W(2))/(00,1)^2/(00,0)$	$(W(11),0)/(10,W(2))/(10,1)/(10,0)/(01,W(2))/(01,1)/(01,0)/(00,W(2))/(00,1)^2/(00,0)$
$\tilde{A}_2A_1$	$(10,1)/(10,0)/(01,1)/(01,0)/(W(11),0)$	$(W(20),1)/(W(20),0)/(W(11),0)/(W(02),1)/(W(02),0)/(00,W(2))/(00,1)^2/(00,0)$
A_2	$10^3/01^3/00^8$	$W(11)/10^6/01^6/00^8$
$\tilde{A}_2$	$10^3/01^3/W(11)$	$W(20)^3/W(11)/W(02)^3/00^8$
$A_1\tilde{A}_1$	$(1,1)^2/(1,0)^2/(0,W(2))/(0,1)^4/(0,0)^3$	$(W(2),0)/(1,W(2))^2/(1,1)^2/(1,0)^4/(0,W(2))^3/(0,1)^4/(0,0)^4$
A_1	$1^6/0^{14}$	$W(2)/1^{14}/0^{21}$
$\tilde{A}_1$	$W(2)/1^8/0^7$	$W(2)^7/1^8/0^{15}$

Table 20: The composition factors for the action of Levi subgroups of E_6 on V_{27} and $L(E_6)$.

Levi L'	Comp. factors of $V_{27} \downarrow L'$	Comp. factors of $L(E_6) \downarrow L'$
D_5	$\lambda_1/\lambda_4/0$	$W(\lambda_2)/\lambda_4/\lambda_5/0$
D_4	$1000/0010/0001/0000^3$	$1000^2/W(0100)/0010^2/0001^2/0000^3$
A_5	λ_1^2/λ_4	$W(\lambda_1+\lambda_5)/\lambda_3^2/0^3$
A_1A_4	$(1,1000)/(1,0000)/(0,0010)/(0,0001)$	$(W(2),0000)/(1,0100)/(1,0010)/(0,W(1001))/(0,1000)/(0,0001)/(0,0000)$
$A_1A_2^2$	$(1,01,00)/(1,00,10)/(0,10,01)/(0,01,00)/(0,00,10)$	$(W(2),00,00)/(1,10,10)/(1,01,01)/(1,00,00)^2/(0,W(11),00)/(0,10,10)/(0,01,01)/(0,00,W(11))/(0,00,00)$
A_4	$1000^2/0010/0001/0000^2$	$W(1001)/1000/0100^2/0010^2/0001/0000^4$
A_1A_3	$(1,100)/(1,000)^2/(0,010)/(0,001)^2/(0,000)$	$(W(2),000)/(1,100)/(1,010)^2/(1,001)/(0,W(101))/(0,100)^2/(0,001)^2/(0,000)^4$
A_2^2	$(10,01)/(01,00)^3/(00,10)^3$	$(W(11),00)/(10,10)^3/(01,01)^3/(00,W(11))/(00,00)^8$
$A_1^2A_2$	$(1,1,00)/(1,0,10)/(1,0,00)/(0,1,01)/(0,1,00)/(0,0,10)/(0,0,01)/(0,0,00)$	$(W(2),0,00)/(1,1,10)/(1,1,01)/(1,0,10)/(1,0,01)/(1,0,00)^2/(0,W(2),00)/(0,1,10)/(0,1,01)/(0,1,00)^2/(0,0,W(11))/(0,0,10)/(0,0,01)/(0,0,00)^2$
A_3	$100^2/010/001^2/000^5$	$W(101)/100^4/010^4/001^4/000^7$
A_1A_2	$(0,10)^3/(1,00)^3/(1,01)/(0,01)/(0,00)^3$	$(W(2),00)/(0,W(11))/(1,10)^3/(1,01)^3/(1,00)^2/(0,10)^3/(0,01)^3/(0,00)^9$

A_1^3	$(1,1,0)/(1,0,1)/(1,0,0)^2/$ $(0,1,1)/(0,1,0)^2/(0,0,1)^2/$ $(0,0,0)^3$	$(W(2),0,0)/(1,1,1)^2/(1,1,0)^2/(1,0,1)^2/(1,0,0)^4/$ $(0,W(2),0)/(0,1,1)^2/(0,1,0)^4/(0,0,W(2))/(0,0,1)^4/(0,0,0)^5$
A_2	$10^3/01^3/00^9$	$W(11)/10^9/01^9/00^{16}$
A_1^2	$(1,1)/(1,0)^4/(0,1)^4/(0,0)^7$	$(W(2),0)/(1,1)^6/(1,0)^8/(0,W(2))/(0,1)^8/(0,0)^{16}$
A_1	$1^6/0^{15}$	$W(2)/1^{20}/0^{35}$

Table 21: The composition factors for the action of Levi subgroups of E_7 on V_{56} and $L(E_7)$.

Levi L'	Comp. factors of $V_{56} \downarrow L'$	Comp. factors of $L(E_7) \downarrow L'$
E_6	$\lambda_1/\lambda_6/0^2$	$\lambda_1/W(\lambda_2)/\lambda_6/0$
D_6	λ_1^2/λ_5	$W(\lambda_2)/\lambda_6^2/0^3$
A_1D_5	$(1,\lambda_1)/(1,0)^2/(0,\lambda_4)/(0,\lambda_5)$	$(W(2),0)/(1,\lambda_4)/(1,\lambda_5)/(0,\lambda_1)^2/(0,W(\lambda_2)/(0,0)$
D_5	$\lambda_1^2/\lambda_4/\lambda_5/0^4$	$\lambda_1^2/W(\lambda_2)/\lambda_4^2/\lambda_5^2/0^4$
A_1D_4	$(1,1000)/(1,0000)^4/(0,0010)^2/$ $(0,0001)^2$	$(W(2),0000)/(1,0010)^2/(1,0001)^2/(0,1000)^4/$ $(0,W(0100))/(0,0000)^6$
D_4	$1000^2/0010^2/0001^2/0000^8$	$1000^4/W(0100)/0010^4/0001^4/0000^9$
A_6	$\lambda_1/\lambda_2/\lambda_5/\lambda_6$	$W(\lambda_1+\lambda_6)/\lambda_1/\lambda_3/\lambda_4/\lambda_6/0$
A_1A_5	$(1,\lambda_1)/(1,\lambda_5)/(0,\lambda_1)/(0,\lambda_3)/(0,\lambda_5)$	$(W(2),0)/(1,\lambda_2)/(1,\lambda_4)/(1,0)^2/(0,W(\lambda_1+\lambda_5))/$ $(0,\lambda_2)/(0,\lambda_4)/(0,0)$
A_2A_4	$(10,1000)/(10,0000)/(01,0001)/$ $(01,0000)/(00,0100)/(00,0010)$	$(W(11),0000)/(10,0010)/(10,0001)/(01,1000)/$ $(01,0100)/ (00,W(1001))/(00,1000)/(00,0001)/$ $(00,0000)$
$A_1A_2A_3$	$(1,10,000)/(1,01,000)/(1,00,010)/$ $(0,10,100)/(0,01,001)/(0,00,100)/$ $(0,00,001)$	$(W(2),00,000)/(1,10,001)/(1,01,100)/(1,00,100)/$ $(1,00,001)/(0,W(11),000)/(0,10,010)/(0,10,000)/$ $(0,01,010)/(0,01,000)/(0,00,W(101))/(0,00,000)$
A_5	$\lambda_1^3/\lambda_3/\lambda_5^3$	$W(\lambda_1+\lambda_5)/\lambda_2^3/\lambda_4^3/0^8$
A_5'	$\lambda_1^2/\lambda_2/\lambda_4/\lambda_5^2/0^2$	$W(\lambda_1+\lambda_5)/\lambda_1^2/\lambda_2/\lambda_3^2/\lambda_4/\lambda_5^2/0^4$
A_1A_4	$(1,1000)/(1,0000)^2/(1,0001)/$ $(0,1000)/(00,0100)/(00,0010)/$ $(0,0001)/(0,0000)^2$	$(W(2),0000)/(1,0000)^2/(1,1000)/(1,0100)/(1,0010)/$ $(1,0001)/(0,W(1001))/(0,1000)^2/(0,0100)/(0,0010)/$ $(0,0001)^2/(0,0000)^2$
A_2A_3	$(10,100)/(10,000)^2/(01,001)/$ $(01,000)^2/(00,100)/(00,010)^2/$ $(00,001)$	$(W(11),000)/(10,010)/(10,001)^2/(10,000)/$ $(01,100)^2/(01,010)/(01,000)/(00,W(101))/$ $(00,100)^2/(00,001)^2/(00,000)^4$
$A_1^2A_3$	$(1,1,000)^2/(1,0,010)/(1,0,000)^2/$ $(0,1,100)/(0,1,001)/(0,0,100)^2/$ $(0,0,001)^2$	$(W(2),0,000)/(1,1,100)/(1,1,001)/(1,0,100)^2/$ $(1,0,001)^2/(0,W(2),000)/(0,1,010)^2/(0,1,000)^4/$ $(0,0,W(101))/(0,0,010)^2/(0,0,000)^4$
A_4	$1000^3/0100/0010/0001^3/0000^6$	$W(1001)/1000^4/0100^3/0010^3/0001^4/0000^9$
A_1A_3	$(1,010)/(1,000)^6/(0,100)^4/(0,001)^4$	$(W(2),000)/(0,W(101))/(1,100)^4/(1,001)^4/(0,010)^6/$ $(0,000)^{15}$
$(A_1A_3)'$	$(1,100)/(1,001)/(1,000)^4/(0,100)^2/$ $(0,010)^2/(0,001)^2/(0,000)^4$	$(W(2),000)/(1,100)^2/(1,010)^2/(1,001)^2/(1,000)^4/$ $(0,W(101))/(0,100)^4/(0,010)^2/(0,001)^4/(0,000)^7$

A_2^2	$(10,10)/(10,00)^3/(01,01)/(01,00)^3/(00,10)^3/(00,01)^3/(00,00)^2$	$(W(11),00)/(10,10)^3/(10,01)/(10,00)^3/(01,10)/(01,01)^3/(01,00)^3/(00,W(11))/(00,10)^3/(00,01)^3/(00,00)^9$
$A_1^2A_2$	$(1,1,00)^2/(1,0,10)/(1,0,01)/(1,0,00)^2/(0,1,10)/(0,1,01)/(0,1,00)^2/(0,0,10)^2/(0,0,01)^2/(0,0,00)^4$	$(W(2),0,00)/(1,1,10)/(1,1,01)/(1,1,00)^2/(1,0,10)^2/(1,0,01)^2/(1,0,00)^4/(0,W(2),00)/(0,1,10)^2/(0,1,01)^2/(0,1,00)^4/(0,0,W(11))/(0,0,10)^3/(0,0,01)^3/(0,0,00)^5$
A_1^4	$(1,1,1,0)/(1,0,0,1)^2/(1,0,0,0)^4/(0,1,0,1)^2/(0,1,0,0)^4/(0,0,1,1)^2/(0,0,1,0)^4$	$(W(2),0,0,0)/(1,1,0,1)^2/(1,1,0,0)^4/(1,0,1,1)^2/(1,0,1,0)^4/(0,W(2),0,0)/(0,1,1,1)^2/(0,1,1,0)^4/(0,0,W(2),0)/(0,0,0,W(2))/(0,0,0,1)^8/(0,0,0,0)^9$
A_3	$100^4/010^2/001^4/000^{12}$	$W(101)/100^8/010^6/001^8/000^{18}$
A_1A_2	$(1,10)/(1,01)/(1,00)^6/(0,10)^4/(0,01)^4/(0,00)^8$	$(W(2),00)/(1,10)^4/(1,01)^4/(1,00)^8/(0,W(11))/(0,10)^7/(0,01)^7/(0,00)^{16}$
A_1^3	$(1,1,0)^2/(1,0,1)^2/(1,0,0)^4/(0,1,1)^2/(0,1,0)^4/(0,0,1)^4/(0,0,0)^8$	$(W(2),0,0)/(1,1,1)^2/(1,1,0)^4/(1,0,1)^4/(1,0,0)^8/(0,W(2),0)/(0,1,1)^4/(0,1,0)^8/(0,0,W(2))/(0,0,1)^8/(0,0,0)^{12}$
$(A_1^3)'$	$(1,1,1)/(1,0,0)^8/(0,1,0)^8/(0,0,1)^8$	$(W(2),0,0)/(1,1,0)^8/(1,0,1)^8/(0,W(2),0)/(0,1,1)^8/(0,0,W(2))/(0,0,0)^{28}$
A_2	$10^6/01^6/00^{20}$	$W(11)/10^{15}/01^{15}/00^{35}$
A_1^2	$(1,1)^2/(1,0)^8/(0,1)^8/(0,0)^{16}$	$(W(2),0)/(1,1)^7/(1,0)^{16}/(0,W(2))/(0,1)^{16}/(0,0)^{31}$
A_1	$1^{12}/0^{32}$	$W(2)/1^{32}/0^{66}$

Table 22: The composition factors for the action of Levi subgroups of E_8 on $L(E_8)$.

Levi L'	Composition factors of $L(E_8) \downarrow L'$
E_7	$W(\lambda_1)/\lambda_7^2/0^3$
A_1E_6	$(W(2),0)/(1,\lambda_1)/(1,\lambda_6)/(1,0)^2/(0,\lambda_1)/(0,W(\lambda_2))/(0,\lambda_6)/(0,0)$
E_6	$\lambda_1^3/W(\lambda_2)/\lambda_6^3/0^8$
D_7	$\lambda_1^2/W(\lambda_2)/\lambda_6/\lambda_7/0$
A_2D_5	$(W(11),0)/(10,\lambda_1)/(10,\lambda_4)/(10,0)/(01,\lambda_1)/(01,\lambda_5)/(01,0)/(00,W(\lambda_2))/(00,\lambda_4)/(00,\lambda_5)/(00,0)$
D_6	$\lambda_1^4/W(\lambda_2)/\lambda_5^2/\lambda_6^2/0^6$
A_1D_5	$(W(2),0)/(1,\lambda_1)^2/(1,\lambda_4)/(1,\lambda_5)/(1,0)^4/(0,\lambda_1)^2/(0,W(\lambda_2))/(0,\lambda_4)^2/(0,\lambda_5)^2/(0,0)^4$
A_2D_4	$(W(11),0000)/(10,1000)/(10,0010)/(10,0001)/(10,0000)^3/(01,1000)/(01,0010)/(01,0001)/(01,0000)^3/(00,1000)^2/(00,W(0100))/(00,0010)^2/(00,0001)^2/(00,0000)^2$
D_5	$\lambda_1^6/W(\lambda_2)/\lambda_4^4/\lambda_5^4/0^{15}$
A_1D_4	$(W(2),0000)/(1,1000)^2/(1,0010)^2/(1,0001)^2/(1,0000)^8/(0,1000)^4/(0,W(0100))/(0,0010)^4/(0,0001)^4/(0,0000)^9$
D_4	$1000^8/W(0100)/0010^8/0001^8/0000^{28}$
A_7	$W(\lambda_1+\lambda_7)/\lambda_1/\lambda_2/\lambda_3/\lambda_5/\lambda_6/\lambda_7/0$

A_3A_4	$(W(101),0000)/(100,1000)/(100,0010)/(100,0000)/(010,1000)/(010,0001)/(001,0100)/(001,0001)/(001,0000)/(000,W(1001))/(000,0100)/(000,0010)/(000,0000)$
A_1A_6	$(W(2),0)/(1,\lambda_1)/(1,\lambda_2)/(1,\lambda_5)/(1,\lambda_6)/(0,W(\lambda_1+\lambda_6))/(0,\lambda_1)/(0,\lambda_3)/(0,\lambda_4)/(0,\lambda_6)/(0,0)$
$A_1A_2A_4$	$(W(2),00,0000)/(1,10,0001)/(1,10,0000)/(1,01,1000)/(1,01,0000)/(1,00,0100)/(1,00,0010)/(0,W(11),0000)/(0,10,1000)/(0,10,0100)/(0,01,0010)/(0,01,0001)/(0,00,W(1001))/(0,00,1000)/(0,00,0001)/(0,00,0000)$
A_6	$W(\lambda_1+\lambda_6)/\lambda_1^3/\lambda_2^2/\lambda_3/\lambda_4/\lambda_5^2/\lambda_6^3/0^4$
A_1A_5	$(W(2),0)/(1,\lambda_1)^2/(1,\lambda_2)/(1,\lambda_4)/(1,\lambda_5)^2/(1,0)^2/(0,W(\lambda_1+\lambda_5))/(0,\lambda_1)^2/(0,\lambda_2)/(0,\lambda_3)^2/(0,\lambda_4)/(0,\lambda_5)^2/(0,0)^4$
A_2A_4	$(W(11),0000)/(10,1000)/(10,0100)/(10,0001)^2/(10,0000)^2/(01,1000)^2/(01,0010)/(01,0001)/(01,0000)^2/(00,W(1001))/(00,1000)/(00,0100)^2/(00,0010)^2/(00,0001)/(00,0000)^4$
$A_1^2A_4$	$(W(2),0,0000)/(1,1,1000)/(1,1,0001)/(1,1,0000)^2/(1,0,1000)/(1,0,0100)/(1,0,0010)/(1,0,0001)/(1,0,0000)^2/(0,W(2),0000)/(0,1,1000)/(0,1,0100)/(0,1,0010)/(0,1,0001)/(0,1,0000)^2/(0,0,W(1001))/(0,0,1000)^2/(0,0,0100)/(0,0,0010)/(0,0,0001)^2/(0,0,0000)^2$
A_3^2	$(W(101),000)/(100,100)/(100,010)/(100,001)/(100,000)^2/(010,100)/(010,001)/(010,000)^2/(001,100)/(001,010)/(001,001)/(001,000)^2/(000,W(101))/(000,100)^2/(000,010)^2/(000,001)^2/(000,000)^2$
$A_1A_2A_3$	$(W(2),00,000)/(1,10,001)/(1,10,000)^2/(1,01,100)/(1,01,000)^2/(1,00,100)/(1,00,010)^2/(1,00,001)/(0,W(11),000)/(0,10,100)^2/(0,10,010)/(0,10,000)/(0,01,010)/(0,01,001)^2/(0,01,000)/(0,00,W(101))/(0,00,100)^2/(0,00,001)^2/(0,00,000)^4$
$A_1^2A_2^2$	$(W(2),0,00,00)/(1,1,10,00)/(1,1,01,00)/(1,1,00,10)/(1,1,00,01)/(1,0,10,01)/(1,0,10,00)/(1,0,01,10)/(1,0,01,00)/(1,0,00,10)/(1,0,00,01)/(1,0,00,00)^2/(0,W(2),00,00)/(0,1,10,10)/(0,1,10,00)/(0,1,01,01)/(0,1,01,00)/(0,1,00,10)/(0,1,00,01)/(0,1,00,00)^2/(0,0,W(11),00)/(0,0,10,10)/(0,0,10,01)/(0,0,10,00)/(0,0,01,10)/(0,0,01,01)/(0,0,01,00)/(0,0,00,W(11))/(0,0,00,10)/(0,0,00,01)/(0,0,00,00)^2$
A_5	$W(\lambda_1+\lambda_5)/\lambda_1^6/\lambda_2^3/\lambda_3^2/\lambda_4^3/\lambda_5^6/0^{11}$
A_1A_4	$(W(2),0000)/(1,1000)^3/(1,0100)/(1,0010)/(1,0001)^3/(1,0000)^6/(0,W(1001))/(0,1000)^4/(0,0100)^3/(0,0010)^3/(0,0001)^4/(0,0000)^9$
A_2A_3	$(W(11),000)/(10,100)^2/(10,010)/(10,001)^2/(10,000)^5/(01,100)^2/(01,010)/(01,001)^2/(01,000)^5/(00,W(101))/(00,100)^4/(00,010)^4/(00,001)^4/(00,000)^7$
$A_1^2A_3$	$(W(2),0,000)/(1,1,100)/(1,1,001)/(1,1,000)^4/(1,0,100)^2/(1,0,010)^2/(1,0,001)^2/(1,0,000)^4/(0,W(2),000)/(0,1,100)^2/(0,1,010)^2/(0,1,001)^2/(0,1,000)^4/(0,0,W(101))/(0,0,100)^4/(0,0,010)^2/(0,0,001)^4/(0,0,000)^7$
$A_1A_2^2$	$(W(2),00,00)/(1,10,01)/(1,10,00)^3/(1,01,10)/(1,01,00)^3/(1,00,10)^3/(1,00,01)^3/(1,00,00)^2/(0,W(11),00)/(0,10,10)^3/(0,10,01)/(0,10,00)^3/(0,01,10)/(0,01,01)^3/(0,01,00)^3/(0,00,W(11))/(0,00,10)^3/(0,00,01)^3/(0,00,00)^9$
$A_1^3A_2$	$(W(2),0,0,00)/(1,1,1,00)^2/(1,1,0,10)/(1,1,0,01)/(1,1,0,00)^2/(1,0,1,10)/(1,0,1,01)/(1,0,1,00)^2/(1,0,0,10)^2/(1,0,0,01)^2/(1,0,0,00)^4/(0,W(2),0,00)/(0,1,1,10)/(0,1,1,01)/(0,1,1,00)^2/(0,1,0,10)^2/(0,1,0,01)^2/(0,1,0,00)^4/(0,0,W(2),00)/(0,0,1,10)^2/(0,0,1,01)^2/(0,0,1,00)^4/(0,0,0,W(11))/(0,0,0,10)^3/(0,0,0,01)^3/(0,0,0,00)^5$
A_4	$W(1001)/1000^{10}/0100^5/0010^5/0001^{10}/0000^{24}$
A_1A_3	$(W(2),000)/(1,100)^4/(1,010)^2/(1,001)^4/(1,000)^{12}/(0,W(101))/(0,100)^8/(0,010)^6/(0,001)^8/(0,000)^{18}$

A_2^2	$(W(11),00)/(10,10)^3/(10,01)^3/(10,00)^9/(01,10)^3/(01,01)^3/(01,00)^9/(00,W(11))/(00,10)^9/(00,01)^9/(00,00)^{16}$
$A_1^2A_2$	$(W(2),0,00)/(1,1,10)/(1,1,01)/(1,1,00)^6/(1,0,10)^4/(1,0,01)^4/(1,0,00)^8/(0,W(2),00)/(0,1,10)^4/(0,1,01)^4/(0,1,00)^8/(0,0,W(11))/(0,0,10)^7/(0,0,01)^7/(0,0,00)^{16}$
A_1^4	$(W(2),0,0,0)/(1,1,1,0)^2/(1,1,0,1)^2/(1,1,0,0)^4/(1,0,1,1)^2/(1,0,1,0)^4/(1,0,0,1)^4/(1,0,0,0)^8/(0,W(2),0,0)/(0,1,1,1)^2/(0,1,1,0)^4/(0,1,0,1)^4/(0,1,0,0)^8/(0,0,W(2),0)/(0,0,1,1)^4/(0,0,1,0)^8/(0,0,0,W(2))/(0,0,0,1)^8/(0,0,0,0)^{12}$
A_3	$W(101)/100^{16}/010^{10}/001^{16}/000^{45}$
A_1A_2	$(W(2),00)/(1,10)^6/(1,01)^6/(1,00)^{20}/(0,W(11))/(0,10)^{15}/(0,01)^{15}/(0,00)^{35}$
A_1^3	$(W(2),0,0)/(1,1,1)^2/(1,1,0)^8/(1,0,1)^8/(1,0,0)^{16}/(0,W(2),0)/(0,1,1)^8/(0,1,0)^{16}/(0,0,W(2))/(0,0,1)^{16}/(0,0,0)^{31}$
A_2	$W(11)/10^{27}/01^{27}/00^{78}$
A_1^2	$(W(2),0)/(1,1)^{12}/(1,0)^{32}/(0,W(2))/(0,1)^{32}/(0,0)^{66}$
A_1	$W(2)/1^{56}/0^{133}$

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