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Borderlines and Probabilities of Borderlines: On the Interconnection Between Vagueness and Uncertainty

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Abstract

We describe an integrated approach to vagueness and uncertainty within a propositional logic setting and based on a combination of three valued logic and probability. Three valued valuations are employed in order to model explicitly borderline cases and in this context we give an axiomatic characterisation of two well known three valued models; supervaluations and Kleene valuations. We then demonstrate the close relationship between Kleene valuations and a sub-class of supervaluations. Belief pairs are lower and upper measures on the sentences of the language generated from a probability distribution defined over a finite set of three valued valuations. We describe links between these measures and other uncertainty theories and we show the close relationship between Kleene belief pairs and a sub-class of supervaluation belief pairs. Finally, a probabilistic approach to conditioning is explored within this framework.

1 Introduction

There is a highly interconnected relationship between vagueness and uncertainty. It is not just that vagueness occurs in conjunction with epistemic uncertainty but also that linguistic uncertainty is integral to vague propositions themselves. The latter refers to uncertainty about the definition or interpretation of concepts in natural language and is a natural result of the empirical manner in which language is learnt. Lawry [13] and Lassiter [12] argue that this form of uncertainty is epistemic in nature and can be modelled probabilistically. In this case, the blurred boundary of a vague category can be modelled by probability defined over possible precise boundaries. There is nonetheless an important distinction between blurred boundaries and the explicit identification of borderline cases. Indeed the latter does not refer to epistemic uncertainty at all but instead results from a non-Boolean truth model. For example, given an exact value for Ethel's height it might be certain that she is borderline short. Vagueness is not only the result of linguistic

uncertainty or of borderline cases but comprises of at least both of these features. Furthermore, vague predicates are everywhere embedded in our statements and beliefs about the world. Consequently, to assess such beliefs we must consider vagueness in conjunction with epistemic uncertainty about the state of the world. This requires an integrated approach capturing both uncertainty about the world and linguistic uncertainty about the conventions of language, together with non-Boolean truth models resulting from more flexible category representation.

In this paper we investigate these ideas in a propositional logic setting by combining probability and three valued valuations i.e. taking truth values true, borderline or false. Initially, we adopt an axiomatic approach and consider what properties should be satisfied by three valued valuations if they are to appropriately represent explicitly borderline cases, and following on from this we then investigate the relationship between two different types of valuations. More specifically, we show that there is a strong relationship between Kleene valuations and a sub-type of supervaluations over a restricted set of formulae of the language. As a means of combining epistemic uncertainty and explicitly borderline cases we will introduce belief pairs in the form of lower and upper measures on the sentences of the language. These are generated from probability distributions defined over three valued valuations. More formally, the lower measure of a sentence will be taken as corresponding to the probability that it is true, and the upper measure as corresponding to the probability that it is not false. We introduce different types of belief pairs based on different underlying three valued truth models, and we investigate some of the relationships between them. We then extend these ideas so as to consider conditional beliefs based on probabilistic conditioning over three valued truth models.

An overview of the paper is as follows: In section 2 we introduce a generic definition of three valued valuation in a propositional logic setting and give Kleene valuations and supervaluations as distinct examples. Section 3 proposes a number of axiomatic properties which we might require a suitable three valued valuation model to satisfy. We discuss the reasonableness of these properties and provide characterisations of both Kleene valuations and supervaluations. The notion of a vagueness ordering of valuations is discussed in section 4 and a candidate partial ordering is proposed. These ideas are then used as the basis of an argument against Łukasiewicz valuations as a model of borderline cases. Belief pairs are introduced in section 6 where we exploit the results in section 5 in order to demonstrate the relationship between Kleene and supervaluation belief pairs. Furthermore, we consider the special case in which uncertainty only concerns the level of vagueness of the language. In section 7 we then outline a model of conditional belief within our proposed framework. Finally, in section 8 we give some discussion and conclusions.

The main contributions of this paper are as follows: Firstly we give axiomatic characterisations of both supervaluations and Kleene valuations as special cases of a very general class of three valued truth functions. This helps to make explicit the assumptions about

the behaviour of borderline cases which is implicit in each case. Secondly, we clarify the relationship between Kleene valuations and a sub-class of supervaluations called complete bounded supervaluations. It is shown that these two types of valuations are equal on the subset of sentences in negated normal form which do not involve both a propositional variable and its negation. Consequently, we have identified a functional class of supervaluations¹, which are similar to Kleene valuations but which preserve classical logic equivalences and tautologies. Thirdly, we extend these results to belief pairs consisting of lower and upper belief measures generated from a probability distribution defined over a finite set of three valued valuations. More specifically, we show that complete bounded supervaluation belief pairs coincide with Kleene belief pairs for the same class of sentences described above. Finally, we investigate conditional belief pairs as generated by conditional probabilities defined over a finite set of three valued valuations. This is rather a novel approach to conditioning for non-classical logic, and is quite distinct from the more usual implication operators defined for many valued logics. We prove a number of results for conditional supervaluation and Kleene belief pairs under different assumptions. In some cases the work presented extends results and employs definitions which have already appeared in the literature including in [14], [15], [16] and [17]. Throughout the paper we will, where appropriate, note the nature and scope of this extension.

2 Three Valued Valuations

In this section we propose a general definition for a three valued valuation of a propositional logic language and give two important examples as well as introducing some useful notation. Let \mathcal{L} be a language of propositional logic with connectives \wedge , \vee and \neg and propositional variables $\mathcal{P} = \{p_1, \dots, p_n\}$. Let $S\mathcal{L}$ denote the sentences of \mathcal{L} as generated recursively from the propositional variables by application of the three connectives. Furthermore, let $L\mathcal{L} = \mathcal{P} \cup \{\neg p_i : p_i \in \mathcal{P}\}$ denote the literals of \mathcal{L} . The general definition of a three valued valuation on \mathcal{L} is then given as follows:

Definition 1. Three Valued Valuation [15]: *A three valued valuation on \mathcal{L} is a function $\mathbf{v} : S\mathcal{L} \rightarrow \{1, \frac{1}{2}, 0\}$ such that $\forall \theta, \varphi \in S\mathcal{L}$ if $\mathbf{v}(\theta) \in \{0, 1\}$ and $\mathbf{v}(\varphi) \in \{0, 1\}$ then $\mathbf{v}(\neg\theta) = 1 - \mathbf{v}(\theta)$, $\mathbf{v}(\theta \wedge \varphi) = \min(\mathbf{v}(\theta), \mathbf{v}(\varphi))$ and $\mathbf{v}(\theta \vee \varphi) = \max(\mathbf{v}(\theta), \mathbf{v}(\varphi))$. Here the truth values denote absolutely true (1), borderline ($\frac{1}{2}$) and absolutely false (0) respectively. The restriction on \mathbf{v} is that it should obey the same rules as Tarski valuations² in the case of Boolean expressions.*

¹Although supervaluations are never truth-function they can be functional in a weaker sense. More details are given later in the paper.

²We use the term Tarski valuations to refer to classical Boolean valuations $v : S\mathcal{L} \rightarrow \{0, 1\}$ defined recursively by the following combination rules for the connectives $\forall \theta, \varphi \in S\mathcal{L}$; $v(\neg\theta) = 1 - v(\theta)$, $v(\theta \wedge \varphi) = \min(v(\theta), v(\varphi))$ and $v(\theta \vee \varphi) = \max(v(\theta), v(\varphi))$.

\neg	1	0
	$\frac{1}{2}$	$\frac{1}{2}$
	0	1

\wedge	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

\vee	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

Table 1: Kleene truth tables

Definition 1 a very broad class of valuations with three truth values in which the only requirement is that they remain consistent with Tarski valuations in the case of sentences with binary truth values. Two well known examples of valuations of this form are supervaluations and Kleene valuations:

Definition 2. Supervaluations [7]: Let \mathbb{T} denote the set of Tarski (classical) valuations defined on \mathcal{L} . A supervaluation is a three valued valuation defined by a set $\Pi \subseteq \mathbb{T}$ of Tarski valuations corresponding to admissible precisifications, such that $\forall \theta \in S\mathcal{L}$;

$$\mathbf{v}(\theta) = \begin{cases} 1 : \min\{v(\theta) : v \in \Pi\} = 1 \\ 0 : \max\{v(\theta) : v \in \Pi\} = 0 \\ \frac{1}{2} : \text{otherwise} \end{cases}$$

Definition 3. Kleene valuations [10]: A Kleene valuation is a three valued valuation defined recursively such that $\forall \theta, \varphi \in S\mathcal{L}$; $\mathbf{v}(\neg\theta) = 1 - \mathbf{v}(\theta)$, $\mathbf{v}(\theta \wedge \varphi) = \min(\mathbf{v}(\theta), \mathbf{v}(\varphi))$ and $\mathbf{v}(\theta \vee \varphi) = \max(\mathbf{v}(\theta), \mathbf{v}(\varphi))$. The truth tables summarizing these combination rules are shown in table 1.

Notice that unlike supervaluations, Kleene valuations are fully truth functional, meaning that not only can the truth value of all sentences in $S\mathcal{L}$ be derived from the truth values of \mathcal{P} , but also that this mapping is based on fixed combination functions for each of the connectives. Supervaluations can never be truth functional although, in section 3, we introduce a sub-type of supervaluations which while not fully truth functional are functional in the weaker sense that all truth values can be derived from those of the propositional variables.

Given a three valued valuation \mathbf{v} , for notational convenience, we also introduce associated lower and upper valuations as follows:

Definition 4. Lower and Upper Valuations: Let \mathbf{v} be a three valued valuation on \mathcal{L} then we define an associated pair of lower and upper valuations on \mathcal{L} as follows: Let $\underline{v}, \bar{v} : S\mathcal{L} \rightarrow \{0, 1\}$ such that $\forall \theta \in S\mathcal{L}$,

$$\underline{v}(\theta) = \begin{cases} 1 : \mathbf{v}(\theta) = 1 \\ 0 : \mathbf{v}(\theta) \neq 1 \end{cases} \quad \text{and} \quad \bar{v}(\theta) = \begin{cases} 1 : \mathbf{v}(\theta) \neq 0 \\ 0 : \mathbf{v}(\theta) = 0 \end{cases}$$

In other words, $\underline{v}(\theta) = 1$ if and only if θ is true, while $\bar{v}(\theta) = 1$ if and only if θ is not false.

Notice that the underlying three valued valuation can be retrieved from the associated lower and upper valuations according to: $\forall \theta \in \mathcal{SL};$

$$\mathbf{v}(\theta) = \frac{\underline{v}(\theta) + \bar{v}(\theta)}{2}$$

Indeed the pair of valuations $\vec{v} = (\underline{v}, \bar{v})$ provides an alternative characterisation of \mathbf{v} as follows: $\forall \theta \in \mathcal{SL};$

$$\begin{aligned} \mathbf{v}(\theta) &= 1 \text{ if and only if } \vec{v}(\theta) = (1, 1) \\ \mathbf{v}(\theta) &= \frac{1}{2} \text{ if and only if } \vec{v}(\theta) = (0, 1) \\ \mathbf{v}(\theta) &= 0 \text{ if and only if } \vec{v}(\theta) = (0, 0) \end{aligned}$$

For supervaluations the lower and upper valuations are given by: $\forall \theta \in \mathcal{SL};$

$$\underline{v}(\theta) = \min\{v(\theta) : v \in \Pi\} \text{ and } \bar{v}(\theta) = \max\{v(\theta) : v \in \Pi\}$$

Notice that lower and upper supervaluations are formally equivalent to Boolean necessity and possibility measures on \mathcal{SL} [4]. Despite this formal identity there is none-the-less a subtle difference in interpretation between the two models according to which possibility theory treats the third truth value as meaning *unknown* rather than *borderline*. See [1] for an in depth discussion of these issues.

For Kleene valuations the lower and upper valuations can be determined recursively according to the following combination rules for the connectives: $\forall \theta, \varphi \in \mathcal{SL};$

- $\underline{v}(\neg\theta) = 1 - \bar{v}(\theta)$ and $\bar{v}(\neg\theta) = 1 - \underline{v}(\theta)$.
- $\underline{v}(\theta \wedge \varphi) = \min(\underline{v}(\theta), \underline{v}(\varphi))$ and $\bar{v}(\theta \wedge \varphi) = \min(\bar{v}(\theta), \bar{v}(\varphi))$.
- $\underline{v}(\theta \vee \varphi) = \max(\underline{v}(\theta), \underline{v}(\varphi))$ and $\bar{v}(\theta \vee \varphi) = \max(\bar{v}(\theta), \bar{v}(\varphi))$.

Throughout the remainder of this paper we will use three valued notation (definition 1) and lower and upper valuations (definition 4) interchangeably according to the relationships identified above.

3 Axioms for Three Valued Valuations

We now introduce a number of axiomatic principles which we might require three valued valuations to satisfy if they are to capture the notion of explicitly borderline cases. As mentioned above, in this paper we do not intend that the middle truth value should represent epistemic uncertainty about the state of the world, but rather that it is due to inherently borderline cases arising as a result of the inherent underlying flexibility of the underlying language [1]. In other words, a truth value of $\frac{1}{2}$ does not represent an

uncertain epistemic state. To illustrate this distinction consider a simple non-Boolean model in which predicates have two distinct boundaries. For example, the predicate *short* could be defined using lower and upper height thresholds $\underline{h} \leq \bar{h}$, according to which a height h is classified as being *absolutely short* if $h \leq \underline{h}$ and *absolutely not short* if $h > \bar{h}$. Intermediate height values where $\underline{h} < h \leq \bar{h}$ are then classified as being *borderline short*. In particular, if we knew that Ethel's height lay in this range then there would be no relevant epistemic uncertainty and we would be *certain* that Ethel was borderline short. The difference between using the middle truth value to represent borderline and using it to represent uncertainty is discussed in detail in [5] and [1]. For this paper we will consider the axioms proposed below in the context in which we interpret the middle truth value as modelling explicitly borderline cases and not epistemic states.

We now adopt this axiomatic approach so as to characterise both supervaluations and Kleene valuations in terms of intuitive properties satisfied by three valued valuations. Initially, we show that supervaluations are characterised by the following three axioms: $\forall \theta, \varphi \in S\mathcal{L}$;

- **P1 Duality:** $\mathbf{v}(\neg\theta) = 1 - \mathbf{v}(\theta)$ ³.
- **P2 Tautology:** If $\models \theta$ then $\mathbf{v}(\theta) = 1$.
- **P3 Equivalence:** If $\theta \equiv \varphi$ then $\mathbf{v}(\theta) = \mathbf{v}(\varphi)$.

where \models and \equiv refer to the classical (Tarski) entailment and equivalence relations respectively. Given definition 1 **P1** simply requires that the negation of a borderline case is also a borderline case. This then seems perhaps the least controversial of all the axioms we consider. **P2** and **P3** require respectively that classical (Tarski) tautologies and equivalences are preserved by three valued valuations.

Theorem 5. *Let \mathbf{v} be a three valued valuation of \mathcal{L} , then \mathbf{v} satisfies **P1**, **P2** and **P3** if and only if \mathbf{v} is a supervaluation.*

Proof. (\Leftarrow) trivial

(\Rightarrow) By the disjunctive normal form theorem of propositional logic and **P3** it follows that there exists functions $\underline{f} : 2^{\mathbb{T}} \rightarrow \{0, 1\}$ and $\bar{f} : 2^{\mathbb{T}} \rightarrow \{0, 1\}$ such that: $\forall \theta \in S\mathcal{L}$,

$$\underline{v}(\theta) = \underline{f}(\{v \in \mathbb{T} : v(\theta) = 1\}) \text{ and } \bar{v}(\theta) = \bar{f}(\{v \in \mathbb{T} : v(\theta) = 1\})$$

By **P1** it follows that $\forall A \subseteq \mathbb{T}$,

$$\bar{f}(A) = 1 - \underline{f}(A^c) \tag{1}$$

³Notice that given **P1** then the lower and upper valuations are dual so that $\forall \theta \in S\mathcal{L}$; $\underline{v}(\neg\theta) = 1 - \bar{v}(\theta)$ and $\bar{v}(\neg\theta) = 1 - \underline{v}(\theta)$.

Now by definition 4 we have that $\underline{v} \leq \bar{v}$. This holds if and only if $\forall A \subseteq \mathbb{T}, \underline{f}(A) \leq \bar{f}(A)$ and by equation 1 this holds if and only if $\forall A \subseteq \mathbb{T}$

$$\underline{f}(A) + \underline{f}(A^c) \leq 1 \quad (2)$$

From equation 2 it follows that $\forall A \subseteq \mathbb{T}, \underline{f}(A) \Rightarrow \underline{f}(A^c) = 0$. Now by **P2** we have that $\underline{f}(\mathbb{T}) = 1$ and hence by equations 1 and 2 it follows that

$$\underline{f}(\emptyset) = 0 \text{ and } \bar{f}(\mathbb{T}) = 1, \bar{f}(\emptyset) = 0 \quad (3)$$

Now by definition 1 three valued valuations must agree with classical valuations for crisp sentences. Hence, by definition 4 we have that $\forall \theta, \varphi \in S\mathcal{L}, \underline{v}(\theta) = 1, \underline{v}(\varphi) = 1 \Rightarrow \underline{v}(\theta \wedge \varphi) = 1$. Therefore,

$$\forall A, B \subseteq \mathbb{T}, \underline{f}(A \cap B) \geq \min(\underline{f}(A), \underline{f}(B)) \quad (4)$$

Also, $\underline{v}(\theta) = 1$ or $\underline{v}(\varphi) = 1 \Rightarrow \underline{v}(\theta \vee \varphi) = 1$ and therefore

$$\forall A, B \subseteq \mathbb{T}, \underline{f}(A \cup B) \geq \max(\underline{f}(A), \underline{f}(B)) \quad (5)$$

We can now show that \underline{f} is an increasing function in the sense that if $A \subseteq B \subseteq \mathbb{T}$ then $\underline{f}(A) \leq \underline{f}(B)$: Suppose $A \subseteq B$ and $\underline{f}(A) = 1$ then since $A \cup B = B$ it follows by equation 5 that $\underline{f}(B) \geq 1 \Rightarrow \underline{f}(B) = 1$. It also follows that \bar{f} is increasing in same sense. Suppose again that $A \subseteq B$ and that $\bar{f}(A) = 1$ then by equation 1 $1 - \underline{f}(A^c) = 1 \Rightarrow \underline{f}(A^c) = 0$ since \underline{f} is increasing⁴. Therefore $1 - \underline{f}(B^c) = 1 \Rightarrow \bar{f}(B) = 1$.

We can now use the fact that \underline{f} is increasing together with equations 3 and 4 to show that \underline{f} is a Boolean possibility measure [4]. This is a well known result but we include the details for completeness. Suppose $\exists A, B \subseteq \mathbb{T}$ such that $\underline{f}(A \cap B) > \min(\underline{f}(A), \underline{f}(B))$. This would imply that $\underline{f}(A \cap B) = 1$ and $\min(\underline{f}(A), \underline{f}(B)) = 0$. However, $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Hence, since \underline{f} is an increasing function, then $\underline{f}(A \cap B) = 1 \Rightarrow \underline{f}(A) = 1$ and $\underline{f}(B) = 1$ which is a contradiction. Therefore,

$$\forall A, B \subseteq \mathbb{T}, \underline{f}(A \cap B) \leq \min(\underline{f}(A), \underline{f}(B))$$

Hence, by equation 4 it holds that

$$\forall A, B \subseteq \mathbb{T}, \underline{f}(A \cap B) = \min(\underline{f}(A), \underline{f}(B)) \quad (6)$$

Furthermore, by equation 1 it follows that:

$$\begin{aligned} \bar{f}(A \cup B) &= 1 - \underline{f}(A^c \cap B^c) = 1 - \min(\underline{f}(A^c), \underline{f}(B^c)) \text{ by equation 6} \\ &= 1 - \min(1 - \bar{f}(A), 1 - \bar{f}(B)) = \max(\bar{f}(A), \bar{f}(B)) \text{ by equation 1} \end{aligned}$$

⁴Suppose $\underline{f}(B^c) = 1 \Rightarrow \underline{f}(A^c) = 1$ (since $B^c \subseteq A^c$) which is a contradiction.

Therefore, \underline{f} and \overline{f} are Boolean necessity and possibility measures on $2^{\mathbb{T}}$ respectively. From this it follows that [4], $\forall \theta \in S\mathcal{L}$;

$$\underline{v}(\theta) = \min\{v(\theta) : v \in \Pi\} \text{ and } \overline{v}(\theta) = \min\{v(\theta) : v \in \Pi\} \text{ where } \Pi = \{v \in \mathbb{T} : \overline{f}(\{v\}) = 1\}$$

as required. \square

There is a slight variant of this characterisation based on the following property:

- **P4 Non-Vacuous:** $\exists \theta \in S\mathcal{L}$ such that $\mathbf{v}(\theta) \neq \frac{1}{2}$.

Corollary 6. *A three valued valuation \mathbf{v} on \mathcal{L} satisfies **P1**, **P3** and **P4** if and only if \mathbf{v} is a supervaluation.*

Proof. (\Leftarrow) Trivial. (\Rightarrow) We show that if \mathbf{v} satisfies **P1**, **P3** and **P4** then \mathbf{v} also satisfies **P2**. In the proof of theorem 5 we showed using properties **P1** and **P3** only together with definition 1 that, $\forall \theta \in S\mathcal{L}$, $\underline{v}(\theta) = \underline{f}(\{v \in \mathbb{T} : v(\theta) = 1\})$ and $\overline{v}(\theta) = \overline{f}(\{v \in \mathbb{T} : v(\theta) = 1\})$ where both \underline{f} and \overline{f} are increasing functions. By **P4** we assume w.l.o.g. that $\exists \theta \in S\mathcal{L}$ such that $\mathbf{v}(\theta) = 1$. Let $A = \{v \in \mathbb{T} : v(\theta) = 1\}$ then $\underline{f}(A) = \overline{f}(A) = 1$. Therefore, since $A \subseteq \mathbb{T}$ and since both \underline{f} and \overline{f} are increasing it follows that $\underline{f}(\mathbb{T}) = \overline{f}(\mathbb{T}) = 1$. Hence, **P2** holds. The result then follows trivially from theorem 5. \square

These results improve on a characterisation result for supervaluations given in [15] which required the additional axiom that $\forall \theta, \varphi \in S\mathcal{L}$, $\overline{v}(\theta \vee \varphi) = \max(\overline{v}(\theta), \overline{v}(\varphi))$.

Theorem 5 would initially seem to provide the basis of a strong case for adopting supervaluations as a three valued truth model, at least in a simple propositional logic setting. This would certainly be true if we were to interpret the middle truth value as resulting from uncertainty about an underlying Tarski truth model. Given such an interpretation then there would be a very strong case for preserving classical equivalences and tautologies. For instance, suppose that $\mathbf{v}(\theta) = \frac{1}{2}$ were simply to mean that the Boolean truth value of θ is unknown. In this case θ could not be a classical tautology since it would be known to be true in all Tarski valuations, hence removing all uncertainty. Also, since any two classically equivalent sentences have the same truth value for all Tarski valuations then there could only be uncertainty about one if there was also the same uncertainty about the other. Indeed as mentioned earlier, Boolean possibility theory provides an alternative epistemic interpretation of supervaluations consistent with exactly such a view. However, since here we are using the third truth value to represent explicitly borderline cases then the situation is much less clear cut. From this perspective the three truth values are primitives resulting from an inherently non-Boolean interpretation of the language, and consequently whether or not classical tautologies and equivalences are preserved is an open question. On the other hand, it is not clear why simply allowing for borderline cases in the

language should, in itself, result in **P2** or **P3** being violated. Hence, we might then argue that, in the absence of good reasons to the contrary, we should preserve Tarski tautologies and equivalences as we move from a two to a three valued setting.

We now introduce four additional axioms which when taken together with **P1** provide a characterisation of Kleene valuations: $\forall \theta, \varphi, \psi \in S\mathcal{L}$;

- **P5 Commutativity:** $\mathbf{v}(\theta \wedge \varphi) = \mathbf{v}(\varphi \wedge \theta)$ and $\mathbf{v}(\theta \vee \varphi) = \mathbf{v}(\varphi \vee \theta)$.
- **P6 Bounds:** If $\mathbf{v}(\theta) \neq 1$ or $\mathbf{v}(\varphi) \neq 1$ then $\mathbf{v}(\theta \wedge \varphi) \neq 1$, and if $\mathbf{v}(\theta) \neq 0$ or $\mathbf{v}(\varphi) \neq 0$ then $\mathbf{v}(\theta \vee \varphi) \neq 0$.⁵
- **P7 Monotonicity:** If $\mathbf{v}(\psi) < \mathbf{v}(\varphi)$ then $\mathbf{v}(\theta \wedge \psi) \leq \mathbf{v}(\theta \wedge \varphi)$ and $\mathbf{v}(\theta \vee \psi) \leq \mathbf{v}(\theta \vee \varphi)$.
- **P8 Borderline:** If $\mathbf{v}(\theta) = \mathbf{v}(\varphi) = \frac{1}{2}$ then $\mathbf{v}(\theta \wedge \varphi) = \mathbf{v}(\theta \vee \varphi) = \frac{1}{2}$.

Lemma 7. *Let \mathbf{v} be a three valued valuation on \mathcal{L} satisfying **P5**, **P6** and **P7** then $\forall \theta, \varphi \in S\mathcal{L}$;*

$$\underline{\mathbf{v}}(\theta \wedge \varphi) = \min(\underline{\mathbf{v}}(\theta), \underline{\mathbf{v}}(\varphi)) \text{ and } \overline{\mathbf{v}}(\theta \wedge \varphi) \leq \min(\overline{\mathbf{v}}(\theta), \overline{\mathbf{v}}(\varphi))$$

and

$$\underline{\mathbf{v}}(\theta \vee \varphi) \geq \max(\underline{\mathbf{v}}(\theta), \underline{\mathbf{v}}(\varphi)) \text{ and } \overline{\mathbf{v}}(\theta \vee \varphi) = \max(\overline{\mathbf{v}}(\theta), \overline{\mathbf{v}}(\varphi))$$

Proof. By definition 1 it follows that if $\mathbf{v}(\theta) = \mathbf{v}(\varphi) = 1$ then $\mathbf{v}(\theta \wedge \varphi) = 1$. In other words, if $\underline{\mathbf{v}}(\theta) = \underline{\mathbf{v}}(\varphi) = 1$ then $\underline{\mathbf{v}}(\theta \wedge \varphi) = 1 = \min(\underline{\mathbf{v}}(\theta), \underline{\mathbf{v}}(\varphi))$. In all other cases, we have that either $\mathbf{v}(\theta) \neq 1$ or $\mathbf{v}(\varphi) \neq 1$ and hence by **P6** $\mathbf{v}(\theta \wedge \varphi) \neq 1$. Therefore, $\underline{\mathbf{v}}(\theta \wedge \varphi) = 0 = \min(\underline{\mathbf{v}}(\theta), \underline{\mathbf{v}}(\varphi))$ as required.

Suppose that $\overline{\mathbf{v}}(\theta \wedge \varphi) > \min(\overline{\mathbf{v}}(\theta), \overline{\mathbf{v}}(\varphi))$. This implies that $\overline{\mathbf{v}}(\theta \wedge \varphi) = 1$ and $\min(\overline{\mathbf{v}}(\theta), \overline{\mathbf{v}}(\varphi)) = 0$ i.e. that either $\mathbf{v}(\theta) = 0$ or $\mathbf{v}(\varphi) = 0$. Now w.l.o.g. by **P5** we can assume that $\mathbf{v}(\theta) = 0$. If $\mathbf{v}(\varphi) = 1$ then by definition 1 $\mathbf{v}(\theta \wedge \varphi) = 0$. Otherwise, $\mathbf{v}(\varphi) \in \{0, \frac{1}{2}\}$. In this case can assume that there exists $\psi \in S\mathcal{L}$ such that $\mathbf{v}(\psi) = 1$ ⁶ so that by definition 1 $\mathbf{v}(\theta \wedge \psi) = 0$, and hence by **P7** $\mathbf{v}(\theta \wedge \varphi) = 0$. Hence, in all cases $\underline{\mathbf{v}}(\theta \wedge \varphi) = 0$. This is a contradiction and therefore $\overline{\mathbf{v}}(\theta \wedge \varphi) \leq \min(\overline{\mathbf{v}}(\theta), \overline{\mathbf{v}}(\varphi))$ as required.

Also, by definition 1 it follows that if $\mathbf{v}(\theta) = 0$ and $\mathbf{v}(\varphi) = 0$ then $\mathbf{v}(\theta \vee \varphi) = 0$. In other words, if $\overline{\mathbf{v}}(\theta) = \overline{\mathbf{v}}(\varphi) = 0$ then $\overline{\mathbf{v}}(\theta \vee \varphi) = 0 = \max(\overline{\mathbf{v}}(\theta), \overline{\mathbf{v}}(\varphi))$. In all other cases, we have that either $\mathbf{v}(\theta) \neq 0$ or $\mathbf{v}(\varphi) \neq 0$ and hence by **P6** $\mathbf{v}(\theta \vee \varphi) \neq 0$. Therefore $\overline{\mathbf{v}}(\theta \vee \varphi) = 1 = \max(\overline{\mathbf{v}}(\theta), \overline{\mathbf{v}}(\varphi))$ as required.

⁵The name *bounds* for this property is motivated by the fact that when translated into lower and upper valuation notation it requires that $\forall \theta, \varphi \in S\mathcal{L}$, $\underline{\mathbf{v}}(\theta \wedge \varphi) \leq \min(\underline{\mathbf{v}}(\theta), \underline{\mathbf{v}}(\varphi))$ and $\overline{\mathbf{v}}(\theta \vee \varphi) \geq \max(\overline{\mathbf{v}}(\theta), \overline{\mathbf{v}}(\varphi))$

⁶Otherwise $\forall \theta \in S\mathcal{L}$, $\mathbf{v}(\theta) = \frac{1}{2}$ which trivially implies that \mathbf{v} is a Kleene valuation and immediately satisfies the required inequality.

Suppose that $\underline{v}(\theta \vee \varphi) < \max(\underline{v}(\theta), \underline{v}(\varphi))$. This implies that $\underline{v}(\theta \vee \varphi) = 0$ and $\max(\underline{v}(\theta), \underline{v}(\varphi)) = 1$ i.e. that either $\mathbf{v}(\theta) = 1$ or $\mathbf{v}(\varphi) = 1$. Now w.l.o.g. by **P5** we can assume that $\mathbf{v}(\theta) = 1$. If $\mathbf{v}(\varphi) = 0$ then by definition 1 $\mathbf{v}(\theta \vee \varphi) = 1$. Otherwise, $\mathbf{v}(\varphi) \in \{\frac{1}{2}, 1\}$. In this case we can assume that there exists $\psi \in S\mathcal{L}$ such that $\mathbf{v}(\psi) = 0$ so that by definition 1 $\mathbf{v}(\theta \vee \psi) = 1$, and hence by **P7** $\mathbf{v}(\theta \vee \varphi) = 1$. Hence, in all cases $\underline{v}(\theta \vee \varphi) = 1$. This is a contradiction and therefore $\underline{v}(\theta \vee \varphi) \geq \max(\underline{v}(\theta), \underline{v}(\varphi))$ as required. \square

Theorem 8. *Let \mathbf{v} be a three valued valuation on \mathcal{L} , then \mathbf{v} satisfies **P1**, **P5**, **P6**, **P7** and **P8** if and only if \mathbf{v} is a Kleene valuation.*

Proof. (\Leftarrow) It is trivial to show that Kleene valuations satisfy all of **P1**, **P5**, **P6**, **P7** and **P8**.

(\Rightarrow) Given **P1** and by lemma 7 it is only required to show that $\forall \theta, \varphi \in S\mathcal{L}$, $\bar{v}(\theta \wedge \varphi) = \min(\bar{v}(\theta), \bar{v}(\varphi))$ and $\underline{v}(\theta \vee \varphi) = \max(\underline{v}(\theta), \underline{v}(\varphi))$.

Suppose $\bar{v}(\theta \wedge \varphi) < \min(\bar{v}(\theta), \bar{v}(\varphi))$. In this case $\bar{v}(\theta \wedge \varphi) = 0$ and $\min(\bar{v}(\theta), \bar{v}(\varphi)) = 1$ i.e. $\mathbf{v}(\theta) \in \{\frac{1}{2}, 1\}$ and $\mathbf{v}(\varphi) \in \{\frac{1}{2}, 1\}$. Now if $\mathbf{v}(\theta) = \frac{1}{2}$ and $\mathbf{v}(\varphi) = \frac{1}{2}$ then by **P8** $\mathbf{v}(\theta \wedge \varphi) = \frac{1}{2}$ in which case $\bar{v}(\theta \wedge \varphi) = 1$ which is a contradiction. Furthermore, if $\mathbf{v}(\theta) = 1$ and $\mathbf{v}(\varphi) = 1$ then by definition 1 $\mathbf{v}(\theta \wedge \varphi) = 1$ in which case $\bar{v}(\theta \wedge \varphi) = 1$ which is a contradiction. Hence, by **P5** we need now only consider the case in which $\mathbf{v}(\theta) = \frac{1}{2}$ and $\mathbf{v}(\varphi) = 1$. Now by **P8** $\mathbf{v}(\theta \wedge \varphi) = \frac{1}{2}$ and hence by **P7** $\mathbf{v}(\theta \wedge \varphi) \in \{\frac{1}{2}, 1\}$. This implies that $\bar{v}(\theta \wedge \varphi) = 1$ which is a contradiction. Hence, $\bar{v}(\theta \wedge \varphi) \not< \min(\bar{v}(\theta), \bar{v}(\varphi))$ and by lemma 7 $\bar{v}(\theta \wedge \varphi) = \min(\bar{v}(\theta), \bar{v}(\varphi))$.

Suppose $\underline{v}(\theta \vee \varphi) > \max(\underline{v}(\theta), \underline{v}(\varphi))$. In this case $\underline{v}(\theta \vee \varphi) = 1$ and $\max(\underline{v}(\theta), \underline{v}(\varphi)) = 0$ i.e. $\mathbf{v}(\theta) \in \{0, \frac{1}{2}\}$ and $\mathbf{v}(\varphi) \in \{0, \frac{1}{2}\}$. Now if $\mathbf{v}(\theta) = \frac{1}{2}$ and $\mathbf{v}(\varphi) = \frac{1}{2}$ then by **P8** $\mathbf{v}(\theta \vee \varphi) = \frac{1}{2}$ in which case $\underline{v}(\theta \vee \varphi) = 0$ which is a contradiction. Furthermore, if $\mathbf{v}(\theta) = 0$ and $\mathbf{v}(\varphi) = 0$ then by definition 1 $\mathbf{v}(\theta \vee \varphi) = 0$ in which case $\underline{v}(\theta \vee \varphi) = 0$ which is a contradiction. Hence, by **P5** we need now only consider the case in which $\mathbf{v}(\theta) = \frac{1}{2}$ and $\mathbf{v}(\varphi) = 0$. Now by **P8** $\mathbf{v}(\theta \vee \varphi) = \frac{1}{2}$ and hence by **P7** $\mathbf{v}(\theta \vee \varphi) \in \{0, \frac{1}{2}\}$. This implies that $\underline{v}(\theta \vee \varphi) = 0$ which is a contradiction. Hence, $\underline{v}(\theta \vee \varphi) \not> \max(\underline{v}(\theta), \underline{v}(\varphi))$ and by lemma 7 $\underline{v}(\theta \vee \varphi) = \max(\underline{v}(\theta), \underline{v}(\varphi))$. \square

Theorem 9. *Supervaluations satisfy **P5**, **P6** and **P7**.*

Proof. **P5** follows trivially from the commutativity of Tarski valuations. Now given **P5** let \mathbf{v} be a supervaluation then suppose that $\mathbf{v}(\theta) \neq 1$ then $\exists v \in \Pi$ such that $v(\theta) = 0 \Rightarrow v(\theta \wedge \varphi) = 0$ and hence $\mathbf{v}(\theta \wedge \varphi) \neq 1$. Also, suppose $\mathbf{v}(\theta) \neq 0$ then $\exists v \in \Pi$ such that $v(\theta) = 1 \Rightarrow v(\theta \vee \varphi) = 1$ and hence $\mathbf{v}(\theta \vee \varphi) \neq 0$. Hence, **P6** holds.

For **P7** we must consider only the following cases (assuming **P5**)

- $\mathbf{v}(\theta) = 1, \mathbf{v}(\varphi) = 1$: In this case $\mathbf{v}(\theta \wedge \varphi) = 1 \geq \mathbf{v}(\theta \wedge \psi)$ for any ψ .
- $\mathbf{v}(\theta) = 1, \mathbf{v}(\varphi) = \frac{1}{2}$: In this case $\mathbf{v}(\theta \wedge \varphi) = \frac{1}{2}$. If $\mathbf{v}(\psi) = 0$ then $\mathbf{v}(\theta \wedge \psi) = 0 < \mathbf{v}(\theta \wedge \varphi)$.
- $\mathbf{v}(\theta) = \frac{1}{2}, \mathbf{v}(\varphi) = 1$: In this case $\mathbf{v}(\theta \wedge \varphi) = \frac{1}{2} \geq \mathbf{v}(\theta \wedge \psi)$ for any ψ .
- $\mathbf{v}(\theta) = \frac{1}{2}, \mathbf{v}(\varphi) = \frac{1}{2}$: In this case $\mathbf{v}(\theta \wedge \varphi) \in \{0, \frac{1}{2}\}$. If $\mathbf{v}(\psi) = 0$ then $\mathbf{v}(\theta \wedge \psi) = 0 \leq \mathbf{v}(\theta \wedge \varphi)$.
- $\mathbf{v}(\theta) = 0, \mathbf{v}(\varphi) = 1$: In this case $\mathbf{v}(\theta \wedge \varphi) = 0 = \mathbf{v}(\theta \wedge \psi)$ for any ψ .
- $\mathbf{v}(\theta) = 0, \mathbf{v}(\varphi) = \frac{1}{2}$: In this case $\mathbf{v}(\theta \wedge \varphi) = 0 = \mathbf{v}(\theta \wedge \psi)$ for any ψ .

The result for disjunction follows similarly. □

Theorems 8 and 9 suggest perhaps that **P8** is the most controversial of these additional properties. The intuition behind it is that the conjunction or disjunction of two borderline sentences should not take any truth value other than borderline. In general, **P8** is clearly inconsistent with **P2** as we can see by considering excluded middle tautologies $\theta \vee \neg\theta$ when θ , and consequently by **P1** also $\neg\theta$, is a borderline case⁷. Of course, **P8** also means that contradictions $\theta \wedge \neg\theta$ have a non-zero truth value if $\mathbf{v}(\theta) = \frac{1}{2}$. In the seminal paper [7] Fine argues that a theory of vagueness should be able to account for penumbral connections, these being logical relations between borderline sentences. So, for example, Fine argues that even if θ has a borderline truth value then $\theta \vee \neg\theta$ and $\theta \wedge \neg\theta$ are true and false respectively. However, in the case of vague sentences this seem rather a subjective judgement. One way of gaining insight into this issue would be to undertake experimental studies into how people actually deal with penumbral connections in natural language. There have been only a few examples of such studies reported in the literature and broadly speaking the results are mixed (see [28] for an overview). However, in one such study Ripley [24] finds that there is evidence that people are willing to accept contradictions in borderline cases. We will return to the issue of penumbral connection in the discussion in section 8. In the sequel we show that it is possible to identify a subclass of supervaluations which behave as Kleene valuations on a restricted set of sentences and hence which satisfy **P2** generally and **P8** in this fragment of the language. Initially, in the following section we introduce a natural vagueness ordering on three valued valuations and argue that this provides a strong case against another well known three valued valuation as a model of explicit borderlines.

⁷Note that it is not the case that Kleene valuations can be characterised simply by adding **P8** to **P1** and **P3** since Kleene valuations do not satisfy **P3**. To see this consider a Kleene valuation for which $\mathbf{v}(p_1) = 1$ and $\mathbf{v}(p_2) = \frac{1}{2}$. Now $p_1 \equiv (p_1 \wedge p_2) \vee (p_1 \wedge \neg p_2)$ but for Kleene valuations $\mathbf{v}((p_1 \wedge p_2) \vee (p_1 \wedge \neg p_2)) = \frac{1}{2}$.

\neg	1	0
	$\frac{1}{2}$	$\frac{1}{2}$
	0	1

\wedge	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	0	0
0	0	0	0

\vee	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

Table 2: Łukasiewicz truth tables

4 A Vagueness Ordering

Semantic precision [15] is a natural partial ordering on three valued valuations and concerns the situation in which one valuation admits more borderline cases than another but where otherwise their truth values agree. More formally, valuation \mathbf{v}_1 is less semantically precise than \mathbf{v}_2 , denoted $\mathbf{v}_1 \preceq \mathbf{v}_2$, if they disagree only for some set of sentences of \mathcal{L} , which being identified as either absolutely true or absolutely false by \mathbf{v}_2 , are classified as being borderline cases by \mathbf{v}_1 . In other words, \mathbf{v}_1 is less semantically precise than \mathbf{v}_2 if all the 1 and 0 valuations of \mathbf{v}_1 are preserved by \mathbf{v}_2 . Hence, we might think of semantic precision as ordering three valued valuations according to their relative vagueness. Shapiro [26] proposed essentially the same ordering of interpretations which he refers to as *sharpening* i.e. $\mathbf{v}_1 \preceq \mathbf{v}_2$ means that \mathbf{v}_2 extends or sharpens \mathbf{v}_1 .

Definition 10. Semantic Precision [15]: *For three valued valuations \mathbf{v}_1 and \mathbf{v}_2 , $\mathbf{v}_1 \preceq \mathbf{v}_2$ if and only if $\forall \theta \in S\mathcal{L}$, $\underline{v}_1(\theta) \leq \underline{v}_2(\theta)$ and $\bar{v}_1(\theta) \geq \bar{v}_2(\theta)$. Furthermore, $\mathbf{v}_1 \prec \mathbf{v}_2$ if $\mathbf{v}_1 \preceq \mathbf{v}_2$ and $\mathbf{v}_1 \neq \mathbf{v}_2$. Note that if **P1** holds then $\mathbf{v}_1 \preceq \mathbf{v}_2$ if and only if $\forall \theta \in S\mathcal{L}$, $\mathbf{v}_1(\theta) = 1$ implies that $\mathbf{v}_2(\theta) = 1$.*

Theorem 11. Semantic Precision for Kleene and Supervaluations [14], [15]:

- *If \mathbf{v} is a Kleene valuation on \mathcal{L} then let $P = \{p_i \in \mathcal{P} : \mathbf{v}(p_i) = 1\}$ and $N = \{p_i \in \mathcal{P} : \mathbf{v}(p_i) = 0\}$. Then for Kleene valuations \mathbf{v}_1 and \mathbf{v}_2 , $\mathbf{v}_1 \preceq \mathbf{v}_2$ if and only if $P_1 \subseteq P_2$ and $N_1 \subseteq N_2$.*
- *Let \mathbf{v}_1 and \mathbf{v}_2 be supervaluations on \mathcal{L} with sets of admissible valuations Π_1 and Π_2 respectively. Then $\mathbf{v}_1 \preceq \mathbf{v}_2$ if and only if $\Pi_1 \supseteq \Pi_2$.*

At this point it is interesting to consider Łukasiewicz three valued valuations as a possible model of explicit borderlines. As for Kleene valuations, these are truth functional and are defined recursively as follows:

Definition 12. Łukasiewicz Valuations [20]: *A Łukasiewicz valuation is a three valued valuation defined recursively such that $\forall \theta, \varphi \in S\mathcal{L}$; $\mathbf{v}(\neg\theta) = 1 - \mathbf{v}(\theta)$, $\mathbf{v}(\theta \wedge \varphi) = \max(0, \mathbf{v}(\theta) + \mathbf{v}(\varphi) - 1)$ and $\mathbf{v}(\theta \vee \varphi) = \min(1, \mathbf{v}(\theta) + \mathbf{v}(\varphi))$. Truth tables summarizing these combination rules are given in table 2.*

Lukasiewicz valuations satisfy a number of the principles introduced in section 3, including **P1**, **P4**, **P5**, **P6** and **P7**. Although, of course, unlike Kleene valuations they do not satisfy the borderline principle **P8**. The following result, however, suggests that Lukasiewicz valuations may not be a good model for borderline vagueness since such valuations cannot represent differing levels of vagueness in terms of the semantic precision ordering.

Theorem 13. *Let \mathbf{v}_1 and \mathbf{v}_2 be Lukasiewicz valuations then $\mathbf{v}_1 \not\prec \mathbf{v}_2$.*

Proof. Let \mathbf{v}_1 and \mathbf{v}_2 be Lukasiewicz valuations such that $\mathbf{v}_1 \neq \mathbf{v}_2$, and let $B_1 = \{l \in LL : \mathbf{v}_1(l) = \frac{1}{2}\}$ and $B_2 = \{l \in LL : \mathbf{v}_2(l) = \frac{1}{2}\}$. Now using proof by contradiction we assume that $\mathbf{v}_1 \prec \mathbf{v}_2$. In this case by definition 10 it follows that $B_2 \subset B_1$. Let $l \in B_1 - B_2$. Then $\mathbf{v}_1(l) = \frac{1}{2}$ and by definition 12 $\mathbf{v}_1(l \wedge l) = 0$. Also, $\mathbf{v}_2(l) \neq \frac{1}{2}$ and w.l.o.g we can assume $\mathbf{v}_2(l) = 1$ (otherwise consider $\neg l$). Hence, $\mathbf{v}_2(l \wedge l) = 1$ and therefore by definition 12 $\mathbf{v}_1 \not\prec \mathbf{v}_2$ since both $\underline{v}_1(l \wedge l) = 0 < \underline{v}_2(l \wedge l) = 1$ and $\overline{v}_1(l \wedge l) = 0 < \overline{v}_2(l \wedge l) = 1$ which is a contradiction. \square

The negative result in theorem 13 is due to Lukasiewicz valuations satisfying a property which is dual to **P8**, namely that the conjunction (or disjunction) of two borderline sentences is *never* borderline. But this would seem to be completely counter to our intuitive understanding of vagueness in certain cases. For example, consider the two statements ‘Ethel is short’ and ‘Ethel is rich’ where *short* and *rich* are adjectives defined on completely independent scales and where the two propositions are logically independent in the sense that a priori, fixing the truth value of one does not constrain the truth value of the other. However, in this case all Lukasiewicz valuations for which both propositions are borderline, give the truth value false to ‘Ethel is short and rich’. In other words in *all* states of the world in which Ethel is both borderline rich and borderline short we are forced to accept a strong penumbral connection according to which Ethel being short and rich is completely rule out. It is hard to envisage an intuitive interpretation of these propositions which would justify such an assumption.

Notice that there do exist distinct Lukasiewicz valuations for which $\mathbf{v}_1 \prec \mathbf{v}_2$ if we adopt a weaker semantic precision ordering according to which $\mathbf{v}_1 \preceq \mathbf{v}_2$ provided that the constraints $\underline{v}_1(l) \leq \underline{v}_2(l)$ and $\overline{v}_1(l) \geq \overline{v}_2(l)$ are satisfied for all literals l . However, this weaker definition would seem to give unwarranted preference to literals rather than comparing the vagueness of two valuations across the whole language.

5 Relating Kleene and Supervaluations

We now introduce a particular class of supervaluations which we refer to as *complete bounded supervaluations*. These are shown to be strongly related to Kleene valuations,

agreeing with the latter on a fragment of \mathcal{L} , but otherwise more semantically precise. A complete bounded supervaluation on \mathcal{L} , is uniquely determined by its truth values on the propositional variables, but this is via a non-truth functional mapping.

Definition 14. Complete Bounded Supervaluations: *Let \sqsubseteq be the partial ordering on \mathbb{T} according to which $v_1 \sqsubseteq v_2$ if and only if $\forall p_i \in \mathcal{P}$, $v_1(p_i) \leq v_2(p_i)$. Then a complete bounded supervaluation is a supervaluation with the set of admissible precisifications of the form $\Pi = \{v \in \mathbb{T} : v_* \sqsubseteq v \sqsubseteq v^*\}$ where $\forall p_i \in \mathcal{P}$, $v_*(p_i) = \min\{v(p_i) : v \in \Pi\}$ and $v^*(p_i) = \max\{v(p_i) : v \in \Pi\}$.*

The following is the definition of a particular subset of the sentences of \mathcal{L} , on which we will later show that complete bounded supervaluations and Kleene valuations coincide.

Definition 15. A Restricted Set of Sentences: *Let $\mathbb{A} = \{A \subseteq L\mathcal{L} : \forall p_i \in \mathcal{P}, \{p_i, \neg p_i\} \not\subseteq A, \{p_i, \neg p_i\} \cap A \neq \emptyset\}$ where $L\mathcal{L}$ denotes the literals of \mathcal{L} ⁸. For $A \in \mathbb{A}$, let $S\mathcal{L}_A \subseteq S\mathcal{L}$ denote the set of sentences of \mathcal{L} generated recursively from A using only the connectives \wedge and \vee . Then we define $S\mathcal{L}^* = \bigcup_{A \in \mathbb{A}} S\mathcal{L}_A$. Notice that $S\mathcal{L}^*$ is the subset of the sentences of \mathcal{L} in negated normal form, for which it is not the case that both a propositional variable and its negation appear.*

We now define a family of partial orderings on the set of Tarski valuations of which the ordering used in definition 14 is a particular example. This is mainly a technical device that is useful in several of the proofs in this section.

Definition 16. A Family of Partial Orderings on \mathbb{T} : *For $A \in \mathbb{A}$ we define the ordering \sqsubseteq_A on \mathbb{T} such that $\forall v_1, v_2 \in \mathbb{T}$, $v_1 \sqsubseteq_A v_2$ if and only if $\forall l \in A$, $v_1(l) \leq v_2(l)$. Notice that the partial ordering \sqsubseteq in definition 14 corresponds to $\sqsubseteq_{\mathcal{P}}$.*

Definition 17. Minimal and Maximal Valuations: *Let \mathbf{v} be a supervaluation with admissible valuations $\Pi \subseteq \mathbb{T}$, then the maximal and minimal Tarski valuations of \mathbf{v} , relative to the ordering \sqsubseteq_A for $A \in \mathbb{A}$, are defined as follows: $v_A^*, v_{A^*} \in \mathbb{T}$ such that $\forall l \in A$,*

$$\begin{aligned} v_A^*(l) &= \max\{v(l) : v \in \Pi\} = \bar{v}(l) \text{ and} \\ v_{A^*}(l) &= \min\{v(l) : v \in \Pi\} = \underline{v}(l) \end{aligned}$$

Notice that v_ and v^* in definition 14 correspond to $v_{\mathcal{P}^*}$ and $v_{\mathcal{P}}$ respectively.*

Lemma 18. *Let \mathbf{v} be a supervaluation then $\forall A \in \mathbb{A}$, $v_* \sqsubseteq v_A^* \sqsubseteq v^*$ and $v_* \sqsubseteq v_{A^*} \sqsubseteq v^*$.*

Proof. Notice that trivially $v_A^* \sqsubseteq v^*$ and $v_* \sqsubseteq v_{A^*}$, since $\forall p_i \in \mathcal{P}$, $v^*(p_i) = \bar{v}(p_i)$ and $v_*(p_i) = \underline{v}(p_i)$ and by definitions 4 and 17 $v_A^*(p_i) \leq \bar{v}(p_i)$ and $v_{A^*}(p_i) \geq \underline{v}(p_i)$. We now show that $v_* \sqsubseteq v_{A^*}$ by considering the following two cases:

⁸Notice that each $A \in \mathbb{A}$ defines a unique Tarski valuation such that $v(p) = 1$ if $p \in A$ and $v(p) = 0$ if $\neg p \in A$.

- 1) Suppose $p_i \in A$ then $v_A^*(p_i) = 0 \Rightarrow$ (by definitions 4 and 17) $\bar{v}(p_i) = 0 \Rightarrow \underline{v}(p_i) = 0 \Rightarrow v_*(p_i) = 0$.
- 2) Suppose $\neg p_i \in A$ then $v_A^*(p_i) = 0 \Rightarrow$ (by definitions 4 and 17) $v_A^*(\neg p_i) = \bar{v}(\neg p_i) = 1 \Rightarrow \underline{v}(p_i) = 0 \Rightarrow v_*(p_i) = 0$

Similarly, we show that $v_{A*} \leq v^*$ by considering the following cases:

- 1) Suppose $p_i \in A$ then $v_{A*}(p_i) = 1 \Rightarrow$ (by definitions 4 and 17) $\underline{v}(p_i) = 1 \Rightarrow \bar{v}(p_i) = 1 \Rightarrow v^*(p_i) = 1$.
- 2) Suppose $\neg p_i \in A$ then $v_{A*}(p_i) = 1 \Rightarrow$ (by definitions 4 and 17) $v_{A*}(\neg p_i) = \underline{v}(\neg p_i) = 0 \Rightarrow \bar{v}(p_i) = 1 \Rightarrow v^*(p_i) = 1$

as required. \square

Lemma 19. *Let $A \in \mathbb{A}$ then for $v_1, v_2 \in \mathbb{T}$, if $v_1 \leq_A v_2$ then it holds that $\forall \theta \in S\mathcal{L}_A$, $v_1(\theta) = 1 \Rightarrow v_2(\theta) = 1$*

Proof. We proceed by induction on $S\mathcal{L}_A$. Let $S\mathcal{L}_A^0 = A$ and $S\mathcal{L}_A^k = S\mathcal{L}_A^{k-1} \cup \{\theta \vee \varphi, \theta \wedge \varphi : \theta, \varphi \in S\mathcal{L}_A^{k-1}\}$. Now for $l \in A$ the result follows trivially by definition of 17. Now if $\psi \in S\mathcal{L}_A^k$ then either $\psi \in S\mathcal{L}_A^{k-1}$ and the result follows trivially by the inductive hypothesis or one of the follow holds: For $\theta, \varphi \in S\mathcal{L}_A^{k-1}$,

- $\psi = \theta \wedge \varphi$: In this case $v_1(\psi) = 1 \rightarrow v_1(\theta \wedge \varphi) = 1 \Rightarrow v_1(\theta) = 1, v_1(\varphi) = 1 \Rightarrow v_2(\theta) = 1, v_2(\varphi) = 1$ by induction $\Rightarrow v_2(\theta \wedge \varphi) = 1 \Rightarrow v_2(\psi) = 1$ as required.
- $\psi = \theta \vee \varphi$: In this case $v_1(\psi) = 1 \rightarrow v_1(\theta \vee \varphi) = 1 \Rightarrow v_1(\theta) = 1$ or $v_1(\varphi) = 1 \Rightarrow v_2(\theta) = 1$ or $v_2(\varphi) = 1$ by induction $\Rightarrow v_2(\theta \vee \varphi) = 1 \Rightarrow v_2(\psi) = 1$ as required.

\square

Theorem 20. *Let \mathbf{v} be a complete bounded supervaluation and, for some $A \in \mathbb{A}$, let v_{A*} and v_A^* be minimal and maximal valuations generated from \mathbf{v} according to definition 17. Let \mathbf{v}' be the supervaluation with admissible valuations $\Pi' = \{v_{A*}, v_A^*\}$ then $\forall \theta \in S\mathcal{L}_A$, $\mathbf{v}(\theta) = \mathbf{v}'(\theta)$.*

Proof. We show that $\forall \theta \in S\mathcal{L}_A$, $\underline{v}(\theta) = \underline{v}'(\theta)$ and $\bar{v}(\theta) = \bar{v}'(\theta)$.

For $\theta \in S\mathcal{L}_A$ if $\bar{v}(\theta) = 1$ then $\exists v \in \Pi$ such that $v(\theta) = 1 \Rightarrow v_A^*(\theta) = 1$ since $v \leq_A v_A^*$ and by lemma 19. Also, if $\bar{v}'(\theta) = 1$ then by lemma 19 $v_A^*(\theta) = 1$. Now by lemma 18 $v_A^* \in \Pi$ since \mathbf{v} is a complete bounded supervaluation pair. Hence $\bar{v}(\theta) = 1$.

Furthermore, if $\underline{v}(\theta) = 1$ then $\forall v \in \Pi, v(\theta) = 1$. Now by lemma 18 $v_{A*}, v_A^* \in \Pi$ since \mathbf{v} is a complete bounded supervaluation pair. Hence $\underline{v}'(\theta) = 1$. Also, if $\underline{v}'(\theta) = 1 \Rightarrow v_{A*}(\theta) = 1$. Hence since $\forall v \in \Pi v_{A*} \leq_A v$, then by lemma 19 it holds that $\underline{v}(\theta) = 1$. \square

Corollary 21. *Let \mathbf{v} be a complete bounded supervaluation and let $A \in \mathbb{A}$. Then $\forall \theta, \varphi \in S\mathcal{L}_A$, $\bar{\mathbf{v}}(\theta \wedge \varphi) = \min(\bar{\mathbf{v}}(\theta), \bar{\mathbf{v}}(\varphi))$ and $\underline{\mathbf{v}}(\theta \vee \varphi) = \max(\underline{\mathbf{v}}(\theta), \underline{\mathbf{v}}(\varphi))$.*

Proof. $\forall \theta, \varphi \in S\mathcal{L}_A$, $\bar{\mathbf{v}}(\theta \wedge \varphi) = v_A^*(\theta \wedge \varphi)$ by theorem 20 $= \min(v_A^*(\theta), v_A^*(\varphi)) = \min(\bar{\mathbf{v}}(\theta), \bar{\mathbf{v}}(\varphi))$ by theorem 20 as required. Also, by theorem 20 $\underline{\mathbf{v}}(\theta \vee \varphi) = v_{A*}(\theta \vee \varphi) = \max(v_{A*}(\theta), v_{A*}(\varphi)) = \max(\underline{\mathbf{v}}(\theta), \underline{\mathbf{v}}(\varphi))$ as required. \square

As an immediate consequence of corollary 21 we have that complete bounded supervaluation pairs obey all the Kleene combination rules for \wedge and \vee when restricted to $S\mathcal{L}_A$ for any $A \in \mathbb{A}$. In particular, complete bounded supervaluations satisfy **P8** when restricted to $S\mathcal{L}_A$ for some $A \in \mathbb{A}$. The following result summarizes the strong relationship between Kleene valuations and complete bounded supervaluations.

Theorem 22. *Let \mathbf{v}_{cbs} be a complete bounded supervaluation, then there exists a unique Kleene valuation \mathbf{v}_k such that $\mathbf{v}_k \preceq \mathbf{v}_{cbs}$ and $\forall \theta \in S\mathcal{L}^*$, $\mathbf{v}_k(\theta) = \mathbf{v}_{cbs}(\theta)$.*

Proof. We define \mathbf{v}_k such that $\forall p_i \in \mathcal{P}$, $\mathbf{v}_k(p_i) = \mathbf{v}_{cbs}(p_i)$. Notice that trivially we have that $\forall l \in L\mathcal{L}$, $\mathbf{v}_k(l) = \mathbf{v}_{cbs}(l)$. We now proceed by induction to show that for any $A \in \mathbb{A}$ and $\forall \theta \in S\mathcal{L}_A$, $\mathbf{v}_k(\theta) = \mathbf{v}_{cbs}(\theta)$. For $l \in A$, the result holds trivially as above. If $\psi \in S\mathcal{L}_A^k$ then either $\psi \in S\mathcal{L}_A^{k-1}$ in which case the result follows trivially by the inductive hypothesis or one of the follow hold: For $\theta, \varphi \in S\mathcal{L}_A^{k-1}$,

- $\psi = \theta \wedge \varphi$: In this case $\underline{\mathbf{v}}(\theta \wedge \varphi) = \min(\underline{\mathbf{v}}_k(\theta), \underline{\mathbf{v}}_k(\varphi))$ by induction $= \min(\underline{\mathbf{v}}_{cbs}(\theta), \underline{\mathbf{v}}_{cbs}(\varphi)) = \underline{\mathbf{v}}_{cbs}(\theta \wedge \varphi)$ by definition 2. Also, $\bar{\mathbf{v}}_k(\theta \wedge \varphi) = \min(\bar{\mathbf{v}}_k(\theta), \bar{\mathbf{v}}_k(\varphi))$ by induction $= \min(\bar{\mathbf{v}}_{cbs}(\theta), \bar{\mathbf{v}}_{cbs}(\varphi)) = \bar{\mathbf{v}}_{cbs}(\theta \wedge \varphi)$ by corollary 21.
- $\psi = \theta \vee \varphi$: In this case $\underline{\mathbf{v}}_k(\theta \vee \varphi) = \max(\underline{\mathbf{v}}_k(\theta), \underline{\mathbf{v}}_k(\varphi))$ by induction $= \max(\underline{\mathbf{v}}_{cbs}(\theta), \underline{\mathbf{v}}_{cbs}(\varphi)) = \underline{\mathbf{v}}_{cbs}(\theta \vee \varphi)$ by corollary 21. Also, $\bar{\mathbf{v}}_k(\theta \vee \varphi) = \max(\bar{\mathbf{v}}_k(\theta), \bar{\mathbf{v}}_k(\varphi))$ by induction $= \max(\bar{\mathbf{v}}_{cbs}(\theta), \bar{\mathbf{v}}_{cbs}(\varphi)) = \bar{\mathbf{v}}_{cbs}(\theta \vee \varphi)$ by definition 2.

Clearly, \mathbf{v}_k is the only Kleene valuation which agrees with \mathbf{v}_{cbs} on $S\mathcal{L}^*$ since for any other Kleene valuation there must exist a propositional variable where it disagrees with \mathbf{v}_{cbs} .

We now show by induction that $\mathbf{v}_k \preceq \mathbf{v}_{cbs}$. Let $S\mathcal{L}^0 = \mathcal{P}$ and $S\mathcal{L}^k = S\mathcal{L}^{k-1} \cup \{\theta \wedge \varphi, \theta \vee \varphi, \neg \theta : \theta, \varphi \in S\mathcal{L}^{k-1}\}$. Clearly by the definition of \mathbf{v}_k , it holds that $\forall p_i \in \mathcal{P}$, $\mathbf{v}_k(p_i) = \mathbf{v}_{cbs}(p_i)$ and hence the result holds for $S\mathcal{L}^0$. Now suppose that $\psi \in S\mathcal{L}^k$ then either $\psi \in S\mathcal{L}^{k-1}$, in which case the result holds trivially, or $\exists \theta, \varphi \in S\mathcal{L}^{k-1}$ such that one of the following cases holds:

- $\psi = \theta \wedge \varphi$: In this case, if $\underline{\mathbf{v}}_k(\theta \wedge \varphi) = 1$ then by definition 3 $\min(\underline{\mathbf{v}}_k(\theta), \underline{\mathbf{v}}_k(\varphi)) = 1$ which implies that $\underline{\mathbf{v}}_k(\theta) = 1$ and $\underline{\mathbf{v}}_k(\varphi) = 1$. Hence, by induction $\underline{\mathbf{v}}_{cbs}(\theta) = 1$ and $\underline{\mathbf{v}}_{cbs}(\varphi) = 1$ and by definition 2 $\underline{\mathbf{v}}_{cbs}(\theta \wedge \varphi) = 1$ as required. Also, if $\bar{\mathbf{v}}_{cbs}(\theta \wedge \varphi) = 1$

then by definition 2 $\bar{v}_{cbs}(\theta) = 1$ and $\bar{v}_{cbs}(\varphi) = 1$ which implies by induction that $\bar{v}_k(\theta) = 1$ and $\bar{v}_k(\varphi) = 1$. Hence, $\min(\bar{v}_k(\theta), \bar{v}_k(\varphi)) = \bar{v}_k(\theta \wedge \varphi) = 1$ as required.

- $\psi = \theta \vee \varphi$: In this case, if $\underline{v}_k(\theta \vee \varphi) = 1$ then by definition 3 $\max(\underline{v}_k(\theta), \underline{v}_k(\varphi)) = 1$ which implies that $\underline{v}_k(\theta) = 1$ or $\underline{v}_k(\varphi) = 1$. Hence, by induction $\underline{v}_{cbs}(\theta) = 1$ or $\underline{v}_{cbs}(\varphi) = 1$ and by definition 2 $\underline{v}_{cbs}(\theta \vee \varphi) = 1$ as required. Also, if $\bar{v}_{cbs}(\theta \vee \varphi) = 1$ then by definition 2 $\bar{v}_{cbs}(\theta) = 1$ or $\bar{v}_{cbs}(\varphi) = 1$ which implies by induction that $\bar{v}_k(\theta) = 1$ or $\bar{v}_k(\varphi) = 1$. Hence, $\max(\bar{v}_k(\theta), \bar{v}_k(\varphi)) = \bar{v}_k(\theta \vee \varphi) = 1$ as required.
- $\psi = \neg\theta$: In this case, if $\underline{v}_k(\neg\theta) = 1$ then $\bar{v}_k(\theta) = 0$ which implies by induction that $\bar{v}_{cbs}(\theta) = 0$. Hence, $\underline{v}_{cbs}(\neg\theta) = 1$ as required. Also, if $\bar{v}_{cbs}(\neg\theta) = 1$ then $\underline{v}_{cbs}(\theta) = 0$ which implies by induction that $\underline{v}_k(\theta) = 0$. Hence, $\bar{v}_k(\neg\theta) = 1$ as required.

□

Theorem 22 is related to an existing result in [15] which, while similar, only holds for sentences in $S\mathcal{L}_A$ where either $A = \mathcal{P}$ or $A = \{\neg p_i : p_i \in \mathcal{P}\}$ although it does hold for a slightly broader class of supervaluations.

6 Lower and Upper Belief Measures

As repeatedly emphasised above we are not using the third truth value in order to stand for uncertain or unknown but rather to represent an explicitly borderline case. Instead we propose that in our current setting, epistemic uncertainty should be quantified by defining probabilities over three valued valuations. This is merely an extension of the usual possible worlds approach to defining measures of belief on the sentences of \mathcal{L} , but extended to three valued truth models. As we will see below such an approach naturally yields a pair of lower and upper measures on $S\mathcal{L}$.

It is important to note here that we *do not* intend for there to be a crisp division between epistemic uncertainty and vagueness, with the latter referring only to borderline cases. As noted by Keefe and Smith [9], vagueness is a multifaceted phenomenon and vague predicates exhibit blurred boundaries as well as borderline cases. We have consistently argued that the former can be understood as resulting from a type of epistemic uncertainty about what is the correct definition of predicates in language [13], [15], [17]. This semantic or linguistic uncertainty [12] naturally results from the distributed manner in which language is learned through repeated interactions between individuals [23], [13]. On the other hand, there is also often epistemic uncertainty about the state of the world occurring in conjunction with both blurred boundaries and explicit borderlines.

To illustrate these ideas recall the earlier example of the predicate *short* defined by lower and upper threshold values $\underline{h} \leq \bar{h}$. In this case linguistic uncertainty manifests itself

in terms of uncertainty about the exact values of the thresholds \underline{h} and \overline{h} . Furthermore, if we are interested in the truth value of the proposition ‘Ethel is short’ then we also need to take account of Ethel’s height h about which we might also be uncertain i.e. this being uncertainty about the state of the world. Hence, by treating both types of uncertainty as being epistemic in nature, and defining a joint distribution over $\underline{h}, \overline{h}$ and h together with similar variables relevant to the other propositions in the language, would then naturally result in a probability distribution over the valuations of \mathcal{L} . Given such a distribution we can naturally define lower and upper belief measures on $S\mathcal{L}$ as follows:

Definition 23. Belief Pairs [14], [15]: *Let \mathbb{V} be a finite set of three valued valuations and w be a probability distribution on \mathbb{V} then we define a belief pair as a pair of lower and upper measures $\vec{\mu} = (\underline{\mu}, \overline{\mu})$ where $\underline{\mu}, \overline{\mu} : S\mathcal{L} \rightarrow [0, 1]$ such that $\forall \theta \in S\mathcal{L};$
 $\underline{\mu}(\theta) = w(\{\mathbf{v} \in \mathbb{V} : \mathbf{v}(\theta) = 1\})$ and $\overline{\mu}(\theta) = w(\{\mathbf{v} \in \mathbb{V} : \mathbf{v}(\theta) \neq 0\})$.⁹*

It can also be interesting to consider mid-point belief degrees generated from a belief pair by taking the average of the lower and upper measures as follows:

Definition 24. Mid-Point Belief Degrees [17]: *Let \mathbb{V} be a finite set of three valued valuations and w be a probability distribution on \mathbb{V} and let $\vec{\mu} = (\underline{\mu}, \overline{\mu})$ be the corresponding belief pair as given in definition 23 then the corresponding mid-point belief degree (belief degree for short) $\beta : S\mathcal{L} \rightarrow [0, 1]$ is defined as follows: $\forall \theta \in S\mathcal{L};$*

$$\beta(\theta) = \frac{\underline{\mu}(\theta) + \overline{\mu}(\theta)}{2} = w(\{\mathbf{v} \in \mathbb{V} : \mathbf{v}(\theta) = 1\}) + \frac{w(\{\mathbf{v} \in \mathbb{V} : \mathbf{v}(\theta) = \frac{1}{2}\})}{2}$$

Hence, we can think of $\beta(\theta)$ as being determined by reallocating the probability associated with truth value $\frac{1}{2}$ evenly between the probabilities associated with truth values 0 and 1. Furthermore, $\beta(\theta)$ is in fact the expected truth value of θ given distribution w , i.e. $\beta(\theta) = \mathbb{E}(\mathbf{v}(\theta))$ [29]. The use of the term belief degree in this content is therefore consistent with Smith’s proposal [27] that the degree of belief of a sentence should be generally defined as its expected truth value.

In earlier work [17] we have used the term *truth degree* instead of *belief degree* for mid-point measures of the above form. This is perhaps not ideal since here we are referring to a measure of subjective belief rather than to a truth value in infinite valued logic, in which context the term *truth degree* it is more typically applied. On the other hand, later in this section we will describe how belief degrees can provide a characterisation of a certain type of infinite valued truth (see theorem 25), thus providing a strong link between these two concepts.

⁹We are assuming here that there is sufficient information to allow agents to quantify their uncertainty using a precise probability distribution on \mathbb{V} . It would also be interesting to consider the case in which uncertainty was quantified by a set of probability distributions (a credal set) over three valued valuations but this is beyond the scope of the current paper. Also notice that we are abusing notation slightly here and using the same symbol for the probability distribution w and the measure that it generates.

In the cases that \mathbb{V} is restricted only to supervaluations or to Kleene valuations we refer to $\vec{\mu}$ as a *supervaluation belief pair* or a *Kleene belief pair* respectively. Furthermore, $\vec{\mu}$ is a *complete bounded supervaluation belief pair* if \mathbb{V} is restricted to complete bounded supervaluations¹⁰. It is well known that *supervaluation belief pairs* correspond to Dempster-Shafer belief and plausibility measures on $S\mathcal{L}$ [8], [6]. However, the specific properties of complete bounded supervaluation belief pairs have only recently been studied [15] and we recall some of them later in this section. There is also a clear link between belief degrees determined from supervaluations belief pairs and *credibility measures* as initially proposed by Dubois and Prade [3] and later developed at some length by liu and liu [19]. In fact, credibility measures are defined from necessity and possibility measures and hence, in the current context, relate to cases in which there is only uncertainty about the correct level of vagueness at which the language should be interpreted, as represented by the semantic precision ordering. We will consider exactly this case in the sequel.

Kleene belief pairs have been proposed independently in [14] and [29]. We now recap on some of their properties including a surprising characterisation of min-max fuzzy logic as shown in [17]. Furthermore, we exploit theorem 22 in order to extend a result in [15] and hence to clarify the relationship between Kleene belief pairs and complete bounded supervaluation belief pairs.

Notice that if \mathbb{V} is restricted to valuations which satisfy **P1** then trivially by definition 23 we have duality between the lower and upper measures so that $\forall \theta \in S\mathcal{L}$;

$$\underline{\mu}(\neg\theta) = 1 - \overline{\mu}(\theta) \text{ and } \overline{\mu}(\neg\theta) = 1 - \underline{\mu}(\theta)$$

Furthermore, by theorem 5 it follows that supervaluation belief pairs satisfy the following

- If $\models \theta$ then $\vec{\mu}(\theta) = (1, 1)$.
- If $\theta \equiv \varphi$ then $\vec{\mu}(\theta) = \vec{\mu}(\varphi)$.

By definition 23 it follows immediately that $\forall \theta \in S\mathcal{L}$;

$$w(\{\mathbf{v} \in \mathbb{V} : \mathbf{v}(\theta) = \frac{1}{2}\}) = \overline{\mu}(\theta) - \underline{\mu}(\theta)$$

In addition, for Kleene belief pairs we have that $\forall \theta \in S\mathcal{L}$;

$$\overline{\mu}(\theta \wedge \neg\theta) = 2\beta(\theta \wedge \neg\theta) = w(\{\mathbf{v} \in \mathbb{V} : \mathbf{v}(\theta) = \frac{1}{2}\})$$

In contrast, for supervaluation belief pairs the fact that supervaluations satisfy **P2** ensures that:

$$\overline{\mu}(\theta \wedge \neg\theta) = \beta(\theta \wedge \neg\theta) = 0$$

¹⁰In light of the discussion in section 4 we will not consider probabilities defined over Lukasiewicz valuations in this paper. Instead we refer the reader to the work of Mundici [21], [22] which investigates probability measures defined over MV algebras with Lukasiewicz operators.

In general, Kleene belief pairs are additive so that $\forall \theta, \varphi \in S\mathcal{L}$;

$$\underline{\mu}(\theta \vee \varphi) = \underline{\mu}(\theta) + \underline{\mu}(\varphi) - \underline{\mu}(\theta \wedge \varphi) \text{ and } \overline{\mu}(\theta \vee \varphi) = \overline{\mu}(\theta) + \overline{\mu}(\varphi) - \overline{\mu}(\theta \wedge \varphi)$$

In contrast, for supervaluation belief pairs $\underline{\mu}$ is super-additive and $\overline{\mu}$ is sub-additive so that $\forall \theta, \varphi \in S\mathcal{L}$;

$$\underline{\mu}(\theta \vee \varphi) \geq \underline{\mu}(\theta) + \underline{\mu}(\varphi) - \underline{\mu}(\theta \wedge \varphi) \text{ and } \overline{\mu}(\theta \vee \varphi) \leq \overline{\mu}(\theta) + \overline{\mu}(\varphi) - \overline{\mu}(\theta \wedge \varphi)$$

We now consider a special case of belief pairs in which an agent's uncertainty relates *only* to the level of vagueness at which \mathcal{L} should be interpreted. More formally this means that w is non-zero only on a set of valuations which can be totally ordered according to semantic precision (definition 10). The following results taken from [14] and [17] summarize the properties of Kleene belief pairs under this assumption and in particular that the resulting mid-point belief degrees provide a complete characterisation of min-max fuzzy logic [30].

Theorem 25. [17] *Let $\zeta : S\mathcal{L} \rightarrow [0, 1]$ then ζ satisfies $\forall \theta, \varphi \in S\mathcal{L}$, $\zeta(\neg\theta) = 1 - \zeta(\theta)$, $\zeta(\theta \wedge \varphi) = \min(\zeta(\theta), \zeta(\varphi))$ and $\zeta(\theta \vee \varphi) = \max(\zeta(\theta), \zeta(\varphi))$ if and only if $\forall \theta \in S\mathcal{L}$, $\zeta(\theta) = \frac{\underline{\mu}(\theta) + \overline{\mu}(\theta)}{2}$ where $\vec{\mu} = (\underline{\mu}, \overline{\mu})$ is a Kleene Belief pair generated by a probability distribution w over Kleene valuations such that $\{\mathbf{v} : w(\mathbf{v}) > 0\} = \{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ and $\mathbf{v}_1 \preceq \dots \preceq \mathbf{v}_r$.*

Theorem 26. [17] *Let $\beta : S\mathcal{L} \rightarrow [0, 1]$ be a belief degree generated from a Kleene belief pair as in theorem 25. Then $\forall \theta \in S\mathcal{L}$;*

$$\underline{\mu}(\theta) = \max(0, 2\beta(\theta) - 1) \text{ and } \overline{\mu}(\theta) = \min(1, 2\beta(\theta))$$

Corollary 27. [14] [17] *Let $\vec{\mu}$ be a Kleene belief pair generated from a probability distribution w for which $\{\mathbf{v} : w(\mathbf{v}) > 0\} = \{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ where $\mathbf{v}_1 \preceq \dots \preceq \mathbf{v}_r$. Then $\forall \theta, \varphi \in S\mathcal{L}$;*

$$\begin{aligned} \underline{\mu}(\theta \wedge \varphi) &= \min(\underline{\mu}(\theta), \underline{\mu}(\varphi)), \quad \overline{\mu}(\theta \wedge \varphi) = \min(\overline{\mu}(\theta), \overline{\mu}(\varphi)) \text{ and} \\ \underline{\mu}(\theta \vee \varphi) &= \max(\underline{\mu}(\theta), \underline{\mu}(\varphi)), \quad \overline{\mu}(\theta \vee \varphi) = \max(\overline{\mu}(\theta), \overline{\mu}(\varphi)) \end{aligned}$$

We now consider the relationship between Kleene belief pairs and complete bounded supervaluation pairs. The following results extend those given in [15] to a wider class of propositional formulae. Theorem 28 exploits corollary 21 and theorem 22 to show the properties of complete bounded supervaluation belief pairs under the above assumption that all uncertainty concerns semantic precision, while theorem 30 demonstrates the equivalence between Kleene belief pairs and complete bounded supervaluation belief pairs for all sentences in $S\mathcal{L}^*$.

Theorem 28. *Let $\vec{\mu}$ be a complete bounded supervaluation belief pair generated from a probability distribution w for which $\{\mathbf{v} : w(\mathbf{v}) > 0\} = \{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ where $\mathbf{v}_1 \preceq \dots \preceq \mathbf{v}_r$. Then for any $A \in \mathbb{A}$, it holds that $\forall \theta, \varphi \in S\mathcal{L}_A$;*

$$\underline{\mu}(\theta \vee \varphi) = \max(\underline{\mu}(\theta), \underline{\mu}(\varphi)) \text{ and } \overline{\mu}(\theta \wedge \varphi) = \min(\overline{\mu}(\theta), \overline{\mu}(\varphi))$$

Proof. By the conditions of the theorem it follows that for any $\theta \in S\mathcal{L}$, if $\underline{v}_i(\theta) = 1$ then $\underline{v}_j(\theta) = 1$ for $j = i + 1, \dots, r$. Hence, $\exists t \leq r$ such that $\{\mathbf{v}_i : \underline{v}_i(\theta) = 1\} = \{\mathbf{v}_t, \dots, \mathbf{v}_r\}$. Similarly for $\varphi \in S\mathcal{L}$, $\exists t' \leq r$ such that $\{\mathbf{v}_i : \underline{v}_i(\varphi) = 1\} = \{\mathbf{v}_{t'}, \dots, \mathbf{v}_r\}$. Now if $\theta, \varphi \in S\mathcal{L}_A$ then by corollary 21 we have that:

$$\begin{aligned} \underline{\mu}(\theta \vee \varphi) &= w(\{\mathbf{v} : \underline{v}(\theta \vee \varphi) = 1\}) = w(\{\mathbf{v} : \max(\underline{v}(\theta), \underline{v}(\varphi)) = 1\}) \\ &= \sum_{j=\min(t,t')}^r w(\mathbf{v}_j) = \max(\underline{\mu}(\theta), \underline{\mu}(\varphi)) \end{aligned}$$

The result for $\bar{\mu}(\theta \wedge \varphi)$ also follows similarly from corollary 21. \square

Corollary 29. *Let $\bar{\mu}$ be a complete bounded supervaluation belief pair generated from a probability distribution w for which $\{\mathbf{v} : w(\mathbf{v}) > 0\} = \{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ where $\mathbf{v}_1 \preceq \dots \preceq \mathbf{v}_r$. Then for any $A \in \mathbb{A}$, it holds that $\forall \theta, \varphi \in S\mathcal{L}_A$;*

$$\underline{\mu}(\theta \vee \varphi) = \max(\underline{\mu}(\theta), \underline{\mu}(\varphi)) \text{ and } \bar{\mu}(\theta \wedge \varphi) = \min(\bar{\mu}(\theta), \bar{\mu}(\varphi))$$

and $\forall \theta, \varphi \in S\mathcal{L}$;

$$\underline{\mu}(\theta \wedge \varphi) = \min(\underline{\mu}(\theta), \underline{\mu}(\varphi)) \text{ and } \bar{\mu}(\theta \vee \varphi) = \max(\bar{\mu}(\theta), \bar{\mu}(\varphi))$$

Proof. The restriction that w is non-zero only on $\mathbf{v}_1 \preceq \dots \preceq \mathbf{v}_r$ ensures that $\underline{\mu}$ and $\bar{\mu}$ are necessity and possibility measures on $S\mathcal{L}$ respectively. Hence, $\forall \theta, \varphi \in S\mathcal{L}$, $\underline{\mu}(\theta \wedge \varphi) = \min(\underline{\mu}(\theta), \underline{\mu}(\varphi))$ and $\bar{\mu}(\theta \vee \varphi) = \max(\bar{\mu}(\theta), \bar{\mu}(\varphi))$. The result then follows trivially from theorem 28. \square

Theorem 30. *Let $\bar{\mu}_1$ be a complete bounded supervaluation belief pair on $S\mathcal{L}$, then there is a Kleene belief pair $\bar{\mu}_2$ on $S\mathcal{L}$ such that $\forall \theta \in S\mathcal{L}^*$, $\bar{\mu}_1(\theta) = \bar{\mu}_2(\theta)$ and $\forall \theta \in S\mathcal{L}$, $\underline{\mu}_1(\theta) \geq \underline{\mu}_2(\theta)$ and $\bar{\mu}_1(\theta) \leq \bar{\mu}_2(\theta)$.*

Proof. Let \mathbb{V}_k and \mathbb{V}_{cbs} denote the sets of Kleene valuations and complete bounded supervaluations on \mathcal{L} respectively. For any complete bounded supervaluation \mathbf{v}_{cbs} , let \mathbf{v}_k be the unique Kleene valuation determined by $\forall p_i \in \mathcal{P}$, $\mathbf{v}_k(p_i) = \mathbf{v}_{cbs}(p_i)$ as in the proof of theorem 22. Furthermore, let $f : \mathbb{V}_{cbs} \rightarrow \mathbb{V}_k$ denote the bijective functional mapping according to which $f(\mathbf{v}_{cbs}) = \mathbf{v}_k$. Now let w_1 be a probability distribution on \mathbb{V}_{cbs} then we define a corresponding distribution w_2 on \mathbb{V}_k , such that $\forall \mathbf{v} \in \mathbb{V}_{cbs}$, $w_2(f(\mathbf{v})) = w_1(\mathbf{v})$. Then by theorem 22 we have that: $\forall \theta \in S\mathcal{L}^*$

$$\begin{aligned} \underline{\mu}_1(\theta) &= w_1(\{\mathbf{v} \in \mathbb{V}_{cbs} : \mathbf{v}(\theta) = 1\}) = w_1(\{\mathbf{v} \in \mathbb{V}_{cbs} : f(\mathbf{v})(\theta) = 1\}) \\ &= w_2(\{\mathbf{v} \in \mathbb{V}_k : \mathbf{v}(\theta) = 1\}) = \underline{\mu}_2(\theta) \end{aligned}$$

It then follows similarly that $\bar{\mu}_1(\theta) = \bar{\mu}_2(\theta)$.

Furthermore, from theorem 22 we have that $f(\mathbf{v}) \preceq \mathbf{v}$. Hence, $\forall \theta \in S\mathcal{L}$;

$$\begin{aligned} \underline{\mu}_2(\theta) &= w_2(\{v \in \mathbb{V}_k : \mathbf{v}(\theta) = 1\}) = w_1(\{\mathbf{v} \in \mathbb{V}_{cbs} : f(\mathbf{v})(\theta) = 1\}) \\ &\leq w_1(\{\mathbf{v} \in \mathbb{V}_{cbs} : \mathbf{v}(\theta) = 1\}) = \underline{\mu}_1(\theta) \end{aligned}$$

The proof that $\bar{\mu}_2(\theta) \geq \bar{\mu}_1(\theta)$ follows similarly. \square

7 Conditional Belief Pairs

In this section we propose a conditioning model according to which belief pairs can be updated on the basis of new information about the truth value of sentences of \mathcal{L} . In view of the inherently probabilistic nature of belief pairs we will adopt an approach based on conditional probability. For this approach we assume that new knowledge takes the form of constraints on the three valued truth values of sentences of \mathcal{L} . Given the interconnection between vagueness and uncertain discussed in the previous section then we can think of such constraints as providing new information both about the state of the world and about the underlying interpretation of \mathcal{L} .

Definition 31. Conditional Belief Pairs [16]: *Suppose an agent obtains new knowledge regarding sentences in $S\mathcal{L}$ in the form of a set of constraints K on three valued valuations of the following form:*

$$K = \{\mathbf{v}(\varphi_i) \in Z_i : i = 1, \dots, t\} \text{ where } Z_i \subseteq \{0, \frac{1}{2}, 1\} \text{ and } \varphi_i \in S\mathcal{L} \text{ for } i = 1, \dots, t$$

Given a prior probability distribution on \mathbb{V} then we define lower and upper belief pairs conditional on K as follows: $\forall \theta \in S\mathcal{L}$;

$$\underline{\mu}(\theta|K) = \frac{w(\{\mathbf{v} \in \mathbb{V}(K) : \underline{v}(\theta) = 1\})}{w(\mathbb{V}(K))} \text{ and } \bar{\mu}(\theta|K) = \frac{w(\{\mathbf{v} \in \mathbb{V}(K) : \bar{v}(\theta) = 1\})}{w(\mathbb{V}(K))}$$

where $\mathbb{V}(K) \subseteq \mathbb{V}$ is the set of three valued valuations in \mathbb{V} which satisfy K . The corresponding conditional belief degree given K is then defined by:

$$\beta(\theta|K) = \frac{\underline{\mu}(\theta|K) + \bar{\mu}(\theta|K)}{2}$$

Theorem 32. [16] *If w is defined on a subset of the Kleene valuations on \mathcal{L} , then $\forall \theta, \varphi \in S\mathcal{L}$ such that $\underline{\mu}(\varphi) > 0$;*

$$\begin{aligned} \underline{\mu}(\theta|\mathbf{v}(\varphi) = 1) &= \frac{\underline{\mu}(\theta \wedge \varphi)}{\underline{\mu}(\varphi)}, \quad \bar{\mu}(\theta|\mathbf{v}(\varphi) = 1) = \frac{\bar{\mu}(\theta \vee \neg\varphi) - \bar{\mu}(\neg\varphi)}{1 - \bar{\mu}(\neg\varphi)} \text{ and} \\ \underline{\mu}(\theta|\mathbf{v}(\varphi) \neq 0) &= \frac{\underline{\mu}(\theta \vee \neg\varphi) - \underline{\mu}(\neg\varphi)}{1 - \underline{\mu}(\neg\varphi)}, \quad \bar{\mu}(\theta|\mathbf{v}(\varphi) \neq 0) = \frac{\bar{\mu}(\theta \wedge \varphi)}{\bar{\mu}(\varphi)} \end{aligned}$$

Theorem 33. *If w is defined on a subset of the supervaluations on \mathcal{L} , then $\forall \theta, \varphi \in S\mathcal{L}$ such that $\underline{\mu}(\varphi) > 0$;*

$$\begin{aligned}\underline{\mu}(\theta|\mathbf{v}(\varphi) = 1) &= \frac{\underline{\mu}(\theta \wedge \varphi)}{\underline{\mu}(\varphi)}, \quad \overline{\mu}(\theta|\mathbf{v}(\varphi) = 1) = \frac{\overline{\mu}(\theta \vee \neg\varphi) - \overline{\mu}(\neg\varphi)}{1 - \overline{\mu}(\neg\varphi)} \text{ and} \\ \underline{\mu}(\theta|\mathbf{v}(\varphi) \neq 0) &\leq \frac{\underline{\mu}(\theta \vee \neg\varphi) - \underline{\mu}(\neg\varphi)}{1 - \underline{\mu}(\neg\varphi)}, \quad \overline{\mu}(\theta|\mathbf{v}(\varphi) \neq 0) \geq \frac{\overline{\mu}(\theta \wedge \varphi)}{\overline{\mu}(\varphi)}\end{aligned}$$

Proof. We assume that \mathbb{V} is a subset of the supervaluations on \mathcal{L} . Then

$$\begin{aligned}\underline{\mu}(\theta|\mathbf{v}(\varphi) = 1) &= \frac{w(\{\mathbf{v} \in \mathbb{V} : \mathbf{v}(\theta) = 1, \mathbf{v}(\varphi) = 1\})}{w(\{\mathbf{v} \in \mathbb{V} : \mathbf{v}(\varphi) = 1\})} \\ &= \frac{w(\{\mathbf{v} \in \mathbb{V} : \mathbf{v}(\theta \wedge \varphi) = 1\})}{w(\{\mathbf{v} \in \mathbb{V} : \mathbf{v}(\varphi) = 1\})} = \frac{\underline{\mu}(\theta \wedge \varphi)}{\underline{\mu}(\varphi)}\end{aligned}$$

Furthermore,

$$\begin{aligned}\overline{\mu}(\theta|\mathbf{v}(\varphi) \neq 0) &= \frac{w(\{\mathbf{v} \in \mathbb{V} : \mathbf{v}(\theta) \neq 0, \mathbf{v}(\varphi) \neq 0\})}{w(\{\mathbf{v} \in \mathbb{V} : \mathbf{v}(\varphi) \neq 0\})} \\ &\geq \frac{w(\{\mathbf{v} \in \mathbb{V} : \mathbf{v}(\theta \wedge \varphi) \neq 0\})}{w(\{\mathbf{v} \in \mathbb{V} : \mathbf{v}(\varphi) \neq 0\})} = \frac{\overline{\mu}(\theta \wedge \varphi)}{\overline{\mu}(\varphi)}\end{aligned}$$

This follows since $\mathbf{v}(\theta \wedge \varphi) \neq 0 \Rightarrow \mathbf{v}(\theta) \neq 0$ and $\mathbf{v}(\varphi) \neq 0$ whilst the converse does not hold. In particular, it is possible that both the sets $\{v \in \Pi : v(\theta) = 1\}$ and $\{v \in \Pi : v(\varphi) = 1\}$ are non-empty but that their intersection is empty. The remaining results then follow by duality by considering the following relationships:

$$\overline{\mu}(\theta|\mathbf{v}(\varphi) = 1) = 1 - \underline{\mu}(\neg\theta|\mathbf{v}(\varphi) = 1) \text{ and } \underline{\mu}(\theta|\mathbf{v}(\varphi) \neq 0) = 1 - \overline{\mu}(\neg\theta|\mathbf{v}(\varphi) \neq 0)$$

□

Corollary 34. *If w is defined on a subset of the complete bounded supervaluations on \mathcal{L} , then for $A \in \mathbb{A}$ it holds that $\forall \theta, \varphi \in S\mathcal{L}_A$;*

$$\begin{aligned}\underline{\mu}(\theta|\mathbf{v}(\varphi) = 1) &= \frac{\underline{\mu}(\theta \wedge \varphi)}{\underline{\mu}(\varphi)}, \quad \overline{\mu}(\theta|\mathbf{v}(\varphi) = 1) = \frac{\overline{\mu}(\theta \vee \neg\varphi) - \overline{\mu}(\neg\varphi)}{1 - \overline{\mu}(\neg\varphi)} \text{ and} \\ \underline{\mu}(\theta|\mathbf{v}(\varphi) \neq 0) &= \frac{\underline{\mu}(\theta \vee \neg\varphi) - \underline{\mu}(\neg\varphi)}{1 - \underline{\mu}(\neg\varphi)}, \quad \overline{\mu}(\theta|\mathbf{v}(\varphi) \neq 0) = \frac{\overline{\mu}(\theta \wedge \varphi)}{\overline{\mu}(\varphi)}\end{aligned}$$

Proof. By corollary 21 we have that $\overline{v}(\theta \wedge \varphi) = \min(\overline{v}(\theta), \overline{v}(\varphi))$. Hence,

$$\begin{aligned}w(\{\mathbf{v} \in \mathbb{V} : \mathbf{v}(\theta) \neq 0, \mathbf{v}(\varphi) \neq 0\}) &= w(\{\mathbf{v} \in \mathbb{V} : \overline{v}(\theta) = 1, \overline{v}(\varphi) = 1\}) \\ &= w(\{\mathbf{v} \in \mathbb{V} : \min(\overline{v}(\theta), \overline{v}(\varphi)) = 1\}) = w(\{\mathbf{v} \in \mathbb{V} : \overline{v}(\theta \wedge \varphi) = 1\})\end{aligned}$$

Therefore,

$$\overline{\mu}(\theta|\mathbf{v}(\varphi) \neq 0) = \frac{\overline{\mu}(\theta \wedge \varphi)}{\overline{\mu}(\varphi)}$$

Consequently by duality we have that,

$$\underline{\mu}(\theta|\mathbf{v}(\varphi) \neq 0) = \frac{\underline{\mu}(\theta \vee \neg\varphi) - \underline{\mu}(\neg\varphi)}{1 - \underline{\mu}(\neg\varphi)}$$

The result then follows by theorem 33. \square

Given that supervaluation belief pairs correspond to Dempster-Shafer belief and plausibility measure on $S\mathcal{L}$ [8], [6], then from theorem 33 we note that by taking $K = \{\mathbf{v}(\varphi) = 1\}$ we obtain the standard model of Dempster-Shafer conditioning as originally proposed [25].

In view of our assumption about the non-epistemic nature of the borderline truth value, it then makes sense for an agent to condition their beliefs given the information that a particular sentence is borderline. For example, if we learn that Ethel is borderline *short* then this provides us with new information about her height. In contrast, simply being told that it is unknown whether or not Ethel is short provides us with no additional information about her height. For the case of Kleene belief pairs the resulting lower and upper measures have the following simple form [16]:

$$\underline{\mu}(\theta|\mathbf{v}(\varphi) = \frac{1}{2}) = \frac{\underline{\mu}(\theta \vee \varphi \vee \neg\varphi) - \underline{\mu}(\varphi \vee \neg\varphi)}{1 - \underline{\mu}(\varphi \vee \neg\varphi)} \text{ and } \overline{\mu}(\theta|\mathbf{v}(\varphi) = \frac{1}{2}) = \frac{\overline{\mu}(\theta \wedge \varphi \wedge \neg\varphi)}{\overline{\mu}(\varphi \wedge \neg\varphi)}$$

It is also interesting to consider conditioning in those situations in which the only uncertainty relates to the level of vagueness of \mathcal{L} , as ordered according to semantic precision (definition 10). In the light of theorem 25 we focus on belief degree conditioning in this context.

Lemma 35. *Let $\mathbf{v}_1 \preceq \dots \preceq \mathbf{v}_r$ be an totally ordered set of Kleene valuations on \mathcal{L} . Let $b_i : S\mathcal{L} \rightarrow \{0, 1\}$ for $i = 1, \dots, 2r$ such that $\forall \theta \in S\mathcal{L}$;*

$$b_i(\theta) = \begin{cases} \underline{v}_i(\theta) & : i \leq r \\ \overline{v}_{2r+1-i} & : i > r \end{cases}$$

Furthermore, for $\theta \in S\mathcal{L}$ let $i_\theta = \min\{i : b_i(\theta) = 1\}$. Then the following hold: $\forall \theta, \varphi \in S\mathcal{L}$;

(1) $i_{\theta \wedge \varphi} = \max(i_\theta, i_\varphi)$ and $i_{\theta \vee \varphi} = \min(i_\theta, i_\varphi)$.

(2) $i_{\neg\theta} = 2r + 2 - i_\theta$.

Proof. The proof of part (1) is given in [17].

(2) For $i \leq r$ we have that $b_i(\neg\theta) = 1$ if and only if $\underline{v}_i(\neg\theta) = 1$ if and only if $\overline{v}_i(\theta) = 0$ if and only if $b_{2r+1-i}(\theta) = 0$.

Also, for $i > r$ we have that $b_i(\neg\theta) = 1$ if and only if $\overline{v}_{2r+1-i}(\neg\theta) = 1$ if and only if $\underline{v}_{2r+1-i}(\theta) = 0$ if and only if $b_{2r+1-i}(\theta) = 0$. Hence, $\forall i$, $b_i(\neg\theta) = 1$ if and only if $b_{2r+1-i}(\theta) = 0$ if and only if $2r + i - 1 \leq i_\theta - 1$ if and only if $i \geq 2r + 2 - i_\theta$ as required. \square

Theorem 36. Let $\vec{\mu}$ be a Kleene valuation pair generated from a probability distribution w for which $\{\mathbf{v} : w(\mathbf{v}) > 0\} = \{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ where $\mathbf{v}_1 \preceq \dots \preceq \mathbf{v}_r$ and let $\beta : S\mathcal{L} \rightarrow [0, 1]$ be the associated belief degree. Then if $\underline{\mu}(\varphi) > 0$ it holds that $\forall \theta \in S\mathcal{L}$;

$$\beta(\theta | \mathbf{v}(\varphi) = 1) = \begin{cases} \min(1, \frac{\beta(\theta) + \beta(\varphi) - 1}{2\beta(\varphi) - 1}) : \beta(\theta) > \beta(\neg\varphi) \\ 0 : \text{otherwise} \end{cases}$$

Proof. Let $i_K = \min\{i : v_i(\varphi) = 1\}$ so that $\{\mathbf{v} : \underline{v}(\varphi) = 1\} = \{\mathbf{v}_{i_K}, \dots, \mathbf{v}_r\}$. Hence, the condition $K = \{\mathbf{v}(\varphi) = 1\}$ restricts us to a subset of valuations which can then be used to generate the following sequence of binary mappings:

$$\underline{v}_{i_K} \leq \dots \leq \underline{v}_r \leq \bar{v}_r \leq \dots \leq \bar{v}_{i_K}$$

which can also be written as:

$$b_{i_K} \leq \dots \leq b_r \leq b_{r+1} \leq \dots \leq b_{2r+1-i_K}$$

Notice that since $\underline{\mu}(\varphi) > 0$ then $i_K \leq r$. From this it follows that $b_i(\varphi) = 1$ if and only if $i \geq i_K$ if and only if $i_K = i_\varphi$. Furthermore, by theorem 26

$$\underline{\mu}(\varphi) = \max(0, 2\beta(\varphi) - 1)$$

Hence, $\underline{\mu}(\varphi) > 0$ if and only if $2\beta(\varphi) - 1 > 0$ if and only if $\beta(\varphi) > \frac{1}{2}$. Therefore,

$$w(\mathbf{v}_i | \mathbf{v}(\varphi) = 1) = \begin{cases} \frac{w(\mathbf{v}_i)}{2\beta(\varphi) - 1} : i \geq i_K \\ 0 : i < i_K \end{cases}$$

Hence, we can define:

$$w'(b_i | \mathbf{v}(\varphi) = 1) = \begin{cases} \frac{w(\mathbf{v}_i | \mathbf{v}(\varphi) = 1)}{2} : i \leq r \\ \frac{w(\mathbf{v}_{2r+1-i} | \mathbf{v}(\varphi) = 1)}{2} : i > r \end{cases}$$

Therefore,

$$w'(b_i | \mathbf{v}(\varphi) = 1) = \begin{cases} \frac{w'(b_i)}{2\beta(\varphi) - 1} : i = i_K, \dots, 2r + 1 - i_K \\ 0 : \text{otherwise} \end{cases}$$

Now from above it follows that:

$$\beta(\theta | \mathbf{v}(\varphi) = 1) = \sum_{i=i_\theta}^{2r} w'(b_i | \mathbf{v}(\varphi) = 1)$$

Now $w'(\bar{v}_i | \mathbf{v}(\varphi) = 1) > 0$ if and only if $i_\varphi = i_K \leq i \leq 2r + 1 - i_K = 2r + 1 - i_\varphi = i_{\neg\varphi} - 1$ by lemma 35. Hence, if $i_\theta \geq i_{\neg\varphi}$ if and only if $\beta(\theta) \leq \beta(\neg\varphi)$ then $\beta(\theta | \mathbf{v}(\varphi) = 1) = 0$. Otherwise:

$$\beta(\theta | \mathbf{v}(\varphi) = 1) = \frac{\sum_{i=\max(i_K, i_\theta)}^{2r+1-i_K} w'(b_i)}{2\beta(\varphi) - 1} = \frac{\sum_{i=\max(i_\varphi, i_\theta)}^{i_{\neg\varphi}-1} w'(b_i)}{2\beta(\varphi) - 1}$$

Also by lemma 35,

$$\begin{aligned} \sum_{i=\max(i_\varphi, i_\theta)}^{i_{\neg\varphi}-1} w'(b_i) &= \sum_{i=i_\theta \wedge \varphi}^{i_{\neg\varphi}-1} w'(b_i) = \sum_{i=i_\theta \wedge \varphi}^{2r} w'(b_i) - \sum_{i=i_{\neg\varphi}}^{2r} w'(b_i) = \beta(\theta \wedge \varphi) - \beta(\neg\varphi) \\ &= \beta(\theta \wedge \varphi) - (1 - \beta(\varphi)) = \min(\beta(\theta), \beta(\varphi)) + \beta(\varphi) - 1 \end{aligned}$$

Hence,

$$\beta(\theta | \mathbf{v}(\varphi) = 1) = \frac{\min(\beta(\theta), \beta(\varphi)) + \beta(\varphi) - 1}{2\beta(\varphi) - 1} = \min\left(1, \frac{\beta(\theta) + \beta(\varphi) - 1}{2\beta(\varphi) - 1}\right)$$

as required. \square

Theorem 37. Let $\bar{\mu}$ be a Kleene valuation pair generated from a probability distribution w for which $\{\mathbf{v} : w(\mathbf{v}) > 0\} = \{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ where $\mathbf{v}_1 \preceq \dots \preceq \mathbf{v}_r$ and let $\beta : S\mathcal{L} \rightarrow [0, 1]$ be the associated belief degree. Then if $\bar{\mu}(\varphi) > 0$ it holds that $\forall \theta \in S\mathcal{L}$;

$$\beta(\theta | \mathbf{v}(\varphi) \neq 0) = \begin{cases} \frac{2\beta(\varphi) + \beta(\theta) - 1}{2\beta(\varphi)} : \beta(\theta) > \beta(\neg\varphi) \\ \frac{1}{2} : \beta(\varphi) \leq \beta(\theta) \leq \beta(\neg\varphi) \\ \frac{\beta(\theta)}{2\beta(\varphi)} : \beta(\theta) < \beta(\varphi) \end{cases}$$

Proof. Let $i_K = \max\{i : \bar{v}_i(\varphi) = 1\}$ so that $\{\mathbf{v}_i : \bar{v}_i(\varphi) = 1\} = \{\mathbf{v}_1, \dots, \mathbf{v}_{i_K}\}$. Hence, the condition $K = \{\mathbf{v}(\varphi) \neq 0\}$ restricts us to a subset of valuations which can then be used to generate the following sequence of binary mappings:

$$\underline{v}_1 \leq \dots \leq \underline{v}_{i_K} \leq \bar{v}_{i_K} \leq \dots \leq \bar{v}_1$$

which can also be written as:

$$b_1 \leq \dots \leq b_{i_K} \leq b_{2r+1-i_K} \leq \dots \leq b_{2r}$$

Now there are two possibilities:

- 1) $i_\varphi \leq r$ if and only if $\forall i, \bar{v}_i(\varphi) = 1$ if and only if $i_K = r$
- 2) $i_\varphi > r$ if and only if $\bar{v}_i(\varphi) = 1$ for $i = 1, \dots, 2r+1-i_\varphi$ if and only if $i_K = 2r+1-i_\varphi = i_{\neg\varphi} - 1$ by lemma 35.

Now for case 1) we have that $w(\{\mathbf{v}(\varphi) \neq 0\}) = 1$ and hence $\beta(\theta | \mathbf{v}(\varphi) \neq 0) = \beta(\theta)$. Furthermore, $w(\{\mathbf{v}(\varphi) \neq 0\}) = \bar{\mu}(\varphi) = 1$ if and only if $\beta(\varphi) \geq \frac{1}{2}$ by theorem 26. Now consider case 2) where $\bar{\mu}(\varphi) < 1$ if and only if $\beta(\varphi) < \frac{1}{2}$. In this case by theorem 26 we have that:

$$\bar{\mu}(\varphi) = \min(1, 2\beta(\varphi)) = 2\beta(\varphi)$$

Hence,

$$w(\mathbf{v}_i | \mathbf{v}(\varphi) \neq 0) = \begin{cases} \frac{w(\mathbf{v}_i)}{2\beta(\varphi)} : i = 1, \dots, i_K \\ 0 : \text{otherwise} \end{cases}$$

Taking,

$$w'(b_i | \mathbf{v}(\varphi) \neq 0) = \begin{cases} \frac{w(\mathbf{v}_i | \mathbf{v}(\varphi) \neq 0)}{2} : i \leq r \\ \frac{w(\mathbf{v}_{2r+1-i})}{2} : i > r \end{cases}$$

then we have that:

$$w'(b_i | \mathbf{v}(\varphi) \neq 0) = \begin{cases} \frac{w'(b_i)}{2\beta(\varphi)} : i \leq i_K \text{ or } i \geq 2r + 1 - i_K \\ 0 : \text{otherwise} \end{cases}$$

For $\theta \in S\mathcal{L}$ we now consider the following three cases:

2 a) $i_\theta \leq i_K = i_{\neg\varphi} - 1$ if and only if $\beta(\theta) > \beta(\neg\varphi)$: In this case:

$$\begin{aligned} \beta(\theta | \mathbf{v}(\varphi) \neq 0) &= \sum_{i=i_\theta}^{2r} w'(b_i | \mathbf{v}(\varphi) \neq 0) = \frac{\sum_{i=i_\theta}^{i_K} w'(b_i) + \sum_{i=2r+1-i_K}^{2r} w'(b_i)}{2\beta(\varphi)} \\ &= \frac{\sum_{i=i_\theta}^{i_K} w'(b_i) + \sum_{i=i_\varphi}^{2r} w'(b_i)}{2\beta(\varphi)} = \frac{\sum_{i=i_\theta}^{i_K} w'(b_i) + \beta(\varphi)}{2\beta(\varphi)} \end{aligned}$$

Now

$$\begin{aligned} \sum_{i=i_\theta}^{i_K} w'(b_i) &= \sum_{i=i_\theta}^{i_{\neg\varphi}-1} w'(b_i) = \sum_{i=i_\theta}^{2r} w'(b_i) - \sum_{i=i_{\neg\varphi}}^{2r} w'(b_i) \\ &= \beta(\theta) - \beta(\neg\varphi) = \beta(\theta) + \beta(\varphi) - 1 \end{aligned}$$

Hence,

$$\beta(\theta | \mathbf{v}(\varphi) \neq 0) = \frac{\beta(\theta) + \beta(\varphi) - 1 + \beta(\varphi)}{2\beta(\varphi)} = \frac{2\beta(\varphi) + \beta(\theta) - 1}{2\beta(\varphi)}$$

as required.

2 b) $i_{\neg\varphi} - 1 = i_K < i_\theta \leq 2r + 1 - i_K = i_\varphi$ if and only if $\beta(\varphi) \leq \beta(\theta) \leq \beta(\neg\varphi)$: In this case,

$$\begin{aligned} \beta(\theta | \mathbf{v}(\varphi) \neq 0) &= \sum_{i=i_\theta}^{2r} w'(b_i | \mathbf{v}(\varphi) \neq 0) = \sum_{i=2r+1-i_K}^{2r} w'(b_i | \mathbf{v}(\varphi) \neq 0) = \frac{\sum_{i=2r+1-i_K}^{2r} w'(b_i)}{2\beta(\varphi)} \\ &= \frac{\sum_{i=i_\varphi}^{2r} w'(b_i)}{2\beta(\varphi)} = \frac{\beta(\varphi)}{2\beta(\varphi)} = \frac{1}{2} \end{aligned}$$

as required.

2 c) $i_\theta > 2r + 1 - i_K = i_\varphi$ if and only if $\beta(\theta) < \beta(\varphi)$: In this case,

$$\beta(\theta|\mathbf{v}(\varphi) \neq 0) = \sum_{i=i_\theta}^{2r} w'(b_i|\mathbf{v}(\varphi) \neq 0) = \frac{\sum_{i=i_\theta}^{2r} w'(b_i)}{2\beta(\varphi)} = \frac{\beta(\theta)}{2\beta(\varphi)}$$

as required. □

Note that despite the characterisation of min-max fuzzy logic given by theorem 25, there does not appear to be a close link between the conditional belief degrees given in theorems 36 and 37 and any of the fuzzy logic implication operators proposed in the literature (see [11] for an overview). This is perhaps not surprising given our use of probabilistic conditioning, this being in contrast to fuzzy implication operators which are usually envisaged as being a many valued generalisation of the implication connective in classical logic.

The following example serves to illustrate the main ideas concerning conditional belief in a three valued setting as introduced in this section.

Example 38. *Consider an experiment in which a fair die is tossed twice such that the first and second throws are independent. Let X and Y be the random variables corresponding to the sum of the two scores and the maximum of the two scores respectively. An agent aims to evaluate their beliefs in the propositions $p_i : i = 1, \dots, 12$ and $q_j : j = 1, \dots, 6$ where $p_i = 'X \text{ is about } i'$ and $q_j = 'Y \text{ is about } j'$. These propositions are interpreted according to the following three valued valuation:*

$$\mathbf{v}(p_i) = \begin{cases} 1 : X = i \\ \frac{1}{2} : X \in \{i-1, i+1\} \\ 0 : \text{otherwise} \end{cases} \quad \text{and} \quad \mathbf{v}(q_j) = \begin{cases} 1 : Y = j \\ \frac{1}{2} : Y \in \{j-1, j+1\} \\ 0 : \text{otherwise} \end{cases}$$

Now suppose that the agent learns that the outcome of the experiment is such that $\mathbf{v}(q_3) = \frac{1}{2}$ then they can then infer that $Y \in \{2, 4\}$ and consequently that the outcome of the experiment is one of the following pairs of scores; (1, 2), (1, 4), (2, 1), (2, 2), (2, 4), (3, 4), (4, 1), (4, 2), (4, 3) or (4, 4). Given that each of these outcomes is equally likely then the agent infers the following conditional beliefs for p_5 :

$$\begin{aligned} \underline{\mu}(p_5|\mathbf{v}(q_3) = \frac{1}{2}) &= P(X = 5|Y \in \{2, 4\}) = \frac{1}{5} \\ \overline{\mu}(p_5|\mathbf{v}(q_3) = \frac{1}{2}) &= P(X \in \{4, 5, 6\}|Y \in \{2, 4\}) = \frac{1}{2} \\ \beta(p_5|\mathbf{v}(q_3) = \frac{1}{2}) &= \frac{\frac{1}{5} + \frac{1}{2}}{2} = \frac{7}{20} \end{aligned}$$

Furthermore, suppose that in addition to uncertainty about the outcome of the experiment, the agent also has semantic uncertainty about how p_i and q_j should be interpreted. More

specifically, the agent considers that there are two types of interpretation of these propositions, one strict and one more relaxed. Here we assume that the strict interpretation is as above, while the relaxed interpretation is defined by the following three valued valuation:

$$\mathbf{v}(p_i) = \begin{cases} 1 : X = i \\ \frac{1}{2} : X \in \{i - 2, i - 1, i + 1, i + 2\} \\ 0 : \text{otherwise} \end{cases} \quad \text{and}$$

$$\mathbf{v}(q_j) = \begin{cases} 1 : Y = j \\ \frac{1}{2} : Y \in \{j - 2, j - 1, j + 1, j + 2\} \\ 0 : \text{otherwise} \end{cases}$$

We assume that the agent believes that either the strict interpretation should apply to all propositions with probability $\frac{1}{2}$ or that the relaxed interpretation should apply to all propositions also with probability $\frac{1}{2}$. Furthermore, we assume that the choice between strict and relaxed interpretations is independent of the outcome of the experiment. Now if the agent learns that $\mathbf{v}(q_3) = \frac{1}{2}$ then under the strict interpretation this implies that $Y \in \{2, 4\}$ and under the relaxed interpretation that $Y \in \{1, 2, 4, 5\}$. The former identifies the outcomes described above while the latter identifies the following outcomes of the experiment; (1, 1), (1, 2), (1, 4), (1, 5), (2, 1), (2, 2), (2, 4), (2, 5), (3, 4), (3, 5), (4, 1), (4, 2), (4, 4), (4, 5), (5, 1), (5, 2), (5, 4), (5, 5) and (5, 5). Hence, the agent now evaluates the following conditional beliefs for p_5 :

$$\begin{aligned} \underline{\mu}(p_5 | \mathbf{v}(q_3) = \frac{1}{2}) &= \\ \frac{1}{2}P(X = 5 | Y \in \{2, 4\}) + \frac{1}{2}P(X = 5 | Y \in \{1, 2, 4, 5\}) &= \frac{1}{2} \left(\frac{1}{5} \right) + \frac{1}{2} \left(\frac{1}{10} \right) = \frac{3}{20} \\ \overline{\mu}(p_5 | \mathbf{v}(q_3) = \frac{1}{2}) &= \\ \frac{1}{2}P(X \in \{4, 5, 6\} | Y \in \{2, 4\}) + \frac{1}{2}P(X \in \{3, 4, 5, 6, 7\} | Y \in \{1, 2, 4, 5\}) &= \\ = \frac{1}{2} \left(\frac{1}{2} \right) + \frac{1}{2} \left(\frac{3}{5} \right) &= \frac{11}{20} \\ \beta(p_5 | \mathbf{v}(q_3) = \frac{1}{2}) &= \frac{\frac{3}{20} + \frac{11}{20}}{2} = \frac{7}{20} \end{aligned}$$

8 Conclusion and Discussion

In this paper we have explored the interconnection between vagueness and uncertainty in a propositional logic setting by considering explicitly borderline cases in conjunction with epistemic uncertainty. The former are not a result of uncertainty at all, but arise as the result of an inherently non-Boolean underlying truth model. Furthermore, the latter includes both uncertainty about the state of the world as well as linguistic (semantic) uncertainty about the interpretation of the language. Indeed we have argued that the

blurred boundaries which are typical of vague predicates result from linguistic uncertainty, this reinforcing the claim that there is no strict division between epistemic uncertainty and vagueness. Instead explicit borderlines and blurred boundaries are both part of the complex phenomenon of vagueness, some aspects of which are probabilistic and some of which are non-probabilistic in nature. Also, vagueness usually occurs together with uncertainty about the state of the world which is often probabilistic.

We have proposed an integrated model which combines three value logic, representing borderline cases, and probability, quantifying uncertainty. A summary of the main contributions of this paper is as follows. We have provided axiomatic characterisations of two well known types of three valued valuations, supervaluations and Kleene valuations, and we have explored the sometimes close relationship between them. To a certain extent this clarifies the assumptions made in both cases, so as to help us judge if they are reasonable given our interpretation of the third truth value as meaning borderline. Furthermore, we have described belief pairs of lower and upper measures, as naturally generated from a probability distribution defined over a finite set of three valued valuations. By defining probabilities over supervaluations we obtain Dempster-Shafer belief and plausibility measures over the sentences of the language. By exploiting the results relating Kleene valuations and supervaluations, it is then shown that there is a close relationship between a special case of these measures and Kleene belief pairs generated from a probability distribution defined over Kleene valuations. The latter also provide a complete characterisation of min-max fuzzy logic in the case when the uncertainty concerns only the level of vagueness at which the language should be interpreted. Finally, in keeping with the probabilistic underpinnings of this approach we have defined conditional belief pairs based on conditional probabilities over three valued valuations and have given some new results for both Kleene and supervaluation belief pairs in this context.

The results presented in section 5 of this paper show that complete bounded supervaluations are equivalent to Kleene valuations on a significant subset of the sentences of \mathcal{L} , whilst still preserving classical equivalences as required by theorem 5. Kleene valuations are completely truth functional so that the truth value of any compound sentence can be determined from the truth values of its components by means of recursive application of a set of truth functions, one for each of the connectives in the language. Complete bounded supervaluations are also functional but in a weaker sense. For this class of valuations, while it is not the case that there is a fixed set of truth functions associated with the connectives which can be applied recursively in order to determine the truth value of any sentence, it is nonetheless the case that the truth values of the propositional variables completely determine the truth values of all the sentences of \mathcal{L} . To see this notice that by definition 14, for a complete bounded supervaluation the set of admissible valuations and hence the entire valuation is determined by v_* and v^* . Furthermore, since for all propositional variables $v_*(p_i) = \underline{v}(p_i)$ and $v^*(p_i) = \overline{v}(p_i)$ it follows that Π can be completely

determined from $\mathbf{v}(p_i) : i = 1, \dots, n$. Now for Kleene valuations truth functionality underlies their inability to represent penumbral connections, including those in the form of classical tautologies. Complete bounded supervaluations do capture the latter but they are still severely limited in the type of penumbral connections that they can encode. For example, consider the propositions $p_1 = \text{'Ethel is middle class'}$ and $p_2 = \text{'Ethel is rich'}$ and suppose that $\mathbf{v}(p_1) = \mathbf{v}(p_2) = \frac{1}{2}$. In this case we might expect that there would be a penumbral connection between p_1 and p_2 according to which being middle class would rule out being rich and vice versa. This would suggest that Π should not contain any Tarski valuations for which $v(p_1 \wedge p_2) = 1$. In fact it might well be appropriate to assume an even stronger relationship according to which, if both p_1 and p_2 have borderline truth values then p_1 is equivalent to $\neg p_2$ ¹¹. In this case Π should also not contain valuations for which $v(\neg p_1 \wedge \neg p_2) = 1$. However, by definition 14 any complete bounded supervaluation for which $\mathbf{v}(p_1) = \mathbf{v}(p_2) = \frac{1}{2}$ must have a set of admissible valuations containing at least one valuation for which $v(p_1 \wedge p_2) = 1$ and at least one for which $v(\neg p_1 \wedge \neg p_2) = 1$. Consequently, we see that complete bounded supervaluations will tend to be inappropriate when there are important penumbral connections beyond those which are represented by classical tautologies and contradictions. On the other hand, when this is not the case then the functionality of complete bounded supervaluations provides computational advantages similar to those of Kleene valuations whilst preserving an underlying classical framework.

The relationship between Kleene and complete bounded supervaluations is carried over to belief pairs, as described in section 6. In this context an interesting case is that in which there is only uncertainty about the level of vagueness at which sentences should be interpreted. For Kleene belief pairs the resulting measures on $S\mathcal{L}$ turn out to be fully truth functional as based on the minimum and maximum operators (see theorem 25). Under the same assumptions supervaluation belief pairs are necessity and possibility measures on the sentences of \mathcal{L} . Making the further restriction to complete bounded supervaluation pairs results in a subclass of necessity and possibility measures which follow both the minimum and maximum rules for conjunction and disjunction respectively, on a significant fragment of the language (see corollary 29). This relationship between Kleene and complete bounded supervaluation belief pairs may then perhaps explain some of the confusion about the difference between fuzzy logic and possibility theory as, for example, outlined in [2].

The use of conditional probability in the definition of conditional belief pairs, as described in section 7, seems natural given the probabilistic treatment of uncertainty that we have adopted. Furthermore, the resulting measures allow for conditioning on knowledge relating to the full range of truth values of the sentences of \mathcal{L} . For example, according to the proposed model it is possible to condition on the knowledge that a sentence is true, that it is not false or that it is borderline. It remains unclear, however, what is the exact relationship between this form of conditional measures and the various implication

¹¹For example, this could hold if the concepts *middle class* and *rich* were defined only by income.

operators used in fuzzy logics. In particular, theorem 36 and 37 show that even in the case when all valuations with non-zero probability form a nested sequence ordered by semantic precision, the resulting conditional belief degrees do not coincide with any of the commonly used implication operators. Perhaps the reason for this can be found in an extension to the analysis of Lewis [18] comparing conditional probabilities with probabilities of material conditionals. Such a study could be illuminating with regard to the general issue of conditioning in a non-classical setting and should be part of future research into this topic.

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