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Torsional vibrations of a column of fine-grained material: A gradient elastic approach

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ABSTRACT

The gradient theory of elasticity with damping is successfully employed to explain the experimentally observed shifting in resonance frequencies during forced harmonic torsional vibration tests of columns made of fine-grained material from their theoretically computed values on the basis of the classical theory of elasticity with damping. To this end, the governing equation of torsional vibrations of a column with circular cross-section is derived both by the lattice theory and the continuum gradient elasticity theory with damping, with consideration of micro-stiffness and micro-inertia effects. Both cases of a column with two rotating masses attached at its top and bottom, and of a column fixed at its base carrying a rotating mass at its free top, are considered. The presence of both micro-stiffness and micro-inertia effects help to explain the observed natural frequency shifting to the left or to the right of the classical values depending on the nature of interparticle forces (repulsive or attractive) due to particle charge. A method for using resonance column tests to determine not only the shear modulus but also the micro-stiffness and micro-inertia coefficients of gradient elasticity for fine-grained materials is also proposed.

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1. INTRODUCTION

The most widely used laboratory test for measuring the shear modulus, G , of soils under low-strain conditions is the resonant column test [1]. To this end, the soil column (solid or hollow) is subjected to harmonic torsional vibrations and the strain amplitude is recorded for a series of loading frequencies. The lowest resonance frequency is the first natural frequency of the soil column. This frequency is used to back-calculate, on the basis of the classical wave equation governing the torsional vibrations of the column, the shear wave propagation velocity c . The shear modulus G , is then determined from c and the pre-specified mass density ρ of the soil material.

Notwithstanding the validity and usefulness of the test, one could question the accuracy of the above procedure for determining G on the basis of only the first (fundamental) natural frequency of the specimen, by arguing that use of higher natural frequencies may lead to different, possibly more accurate, values of G . Furthermore, he could point out that, since the measured resonance frequency is in reality a damped frequency, the analytical frequency equation used should include damping easily measured during the resonant column test. Finally, it has been recently observed by Richter [2] during resonant column tests involving fine-grained materials that a shift in resonance frequencies to the left or to the right of their theoretically computed values by classical elastodynamic theory occurs. This shift to the left and to the right corresponds to repulsive and attractive granular materials, respectively, depending on particle electric charge [2]. However, these interesting experimental observations could not be explained by the classical theory of elasticity.

In this paper, an effort is made to explain theoretically the experimentally observed frequency shifting and suggest a way for a more rational computation of G . This is accomplished by introducing into the aforementioned governing equation of torsional elastic vibrations with damping of a beam of circular cross-section microstructural effects. i.e., both micro-stiffness and micro-inertia effects with the aid of the lattice theory or the continuum gradient elasticity theory with two microstructural constants by following Polyzos and Fotiadis [3] and Mindlin [4], respectively. Both approaches lead to a governing equation of torsional motion including two length scale parameters, in addition to the classical shear modulus G , namely, the micro-stiffness or gradient coefficient g and the micro-inertia coefficient h . More specifically, it is shown that depending on the relation between the magnitudes of g and h , one can predict when the aforementioned frequency shifting will be to the left or to the right of the classically computed eigen-frequencies. Furthermore, since the system is characterized by three elastic constants instead of just one in the classical case, one can possibly engage the first three experimentally obtained resonance frequencies for computing G , g and h , thereby obtaining a more rational value of G than by classical means.

Generalized elasticity theories taking into account microstructural effects have been successfully employed for studying torsional vibrations of beams modeling nanotubes. In this context, Gheshlaghi et al. [5] utilized the modified couple stress theory with one length scale parameter, Kahrobaiyan et al. [6] a strain gradient theory with three length scale parameters and Lim et al. [7] a nonlocal stress theory with one length scale parameter. However, none of the above works considers micro-inertia effects, which, as demonstrated in Georgiadis et al. [8], Askes et al. [9], Papargyri-Beskou et al. [10], Fafalis et al. [11] and Dontsov et al. [12] are not only important alongside the micro-stiffness ones, but also characterize the dynamic behavior of a wide class of materials and structures. Further, none of the above works considers the effect of internal viscoelastic damping on the response. In this work, both microstructural parameters play an equally important role and help to explain the dynamic behavior of granular beams under torsional vibrations. Besides, the effect of internal viscoelastic damping on the response is considered for a more realistic treatment of the problem. Additional discussion on theoretical aspects of gradient elasticity theory is presented in section 3.

2. RESONANT COLUMN TEST RESULTS FOR FINE-GRAINED MATERIAL

In his 2006 doctoral dissertation, Richter [2] presented experimental results on the dynamic behavior of fine-grained soils under cyclic loading, which find applications in a variety of soil dynamics problems. For this purpose, he employed model materials instead of natural fine-grained soil, i.e., α -Al₂O₃ powder (hard compact particles) and Laponite (synthetic clay) representing silt and clay, respectively. A good part, but not all, of the work in [2] can also be found in Richter and Huber [13, 14].

Fine-grained materials like α -Al₂O₃ have a mean particle diameter of $d_{50}=0.8\mu\text{m}$ and exhibit a fabric depending on the surface forces between the grains, which are mainly responsible for the formation of the grain skeleton. In a fabric of attractive particles (particle charge $\text{pH}=9.1$), interparticle friction results in low density, while in a fabric of repulsive particles (particle charge $\text{pH}=4.0$) interparticle repulsion prevents friction and enhances densification, as shown in Fig. 1 taken from [2]. All these materials are, naturally, fine-grained by geotechnical standards.

Figure 1

Richter [2] reported on experimental results from resonant column tests conducted on fine-grained saturated α -Al₂O₃ columns subjected to torsional harmonic vibrations with the goal of determining the shear modulus G and the damping ratio D of these materials. The tests were conducted for small to medium values of engineering shear strains γ , i.e., for $\gamma=10^{-7}$ to 10^{-3} , for values of frequency f varying from 0 to 5600 Hz and for values of confining pressure p' varying between 20 and 320 kPa.

Figure 2

Figure 2,a represents the resonant column test apparatus used by Richter [2], while Fig. 2,b its mathematical model. The height L and the radius r of the specimen are equal to 0.10 m and 0.05 m, respectively, the polar moment of inertia of the cross-section of the specimen $I_p = \rho r^4 / 2 = 98.125 \times 10^{-3} \text{ m}^4$, while the mass moments of inertia of the top and bottom masses of the apparatus are $J_L = 0.854549 \times 10^{-3} \text{ Kg m}^2$ and $J_0 = 57.352325 \times 10^{-3} \text{ Kg m}^2$, respectively. Figures 3 and 4 contain representative results from Richter [2] corresponding to the cases of attractive ($\text{pH}=9.1$) and repulsive ($\text{pH}=4.0$) particles, respectively, for a confining pressure of 20 kPa. Both figures depict the normalized resonance factor R_t/R_b as function of frequency f , where the resonance factors R_t and R_b are defined as the ratios of the amplitudes of vibration A_t and A_b at the top and bottom, respectively, of the soil column to the static torsional angle θ_s . Furthermore, Figs 3 and 4 also depict analytical results obtained on the basis of the analytical solution due to Hardin [15]. The analytical results are based on the assumption that the material obeys the simple viscoelastic model of Kelvin-Voigt with viscosity coefficient μ equal to 686.2 Pa sec and 1119.8 Pa sec for attractive and repulsive particles, respectively. Since most soils exhibit frequency-independent damping [15, 16], $\mu\omega/G$ should be a constant (or, equivalently, μ should be analogous to $1/\omega$), where $\omega=2\pi f$ is the circular frequency of vibration. Thus, Figs 3 and 4 present the analytic solution for $\mu\omega/G=0.023$ and 0.026 for the cases of attractive and repulsive particles, respectively. In plotting the analytical results, the two unknown parameters of the model μ or $\mu\omega/G$ and G are set so that R_t/R_b and the resonance frequency of the first mode coincide. Inspection of Figs 3

and 4 reveals that the hysteretic model (with $\mu\omega/G=\text{constant}$) is much closer to the experimental results, especially for higher frequencies, than the viscous model (with $\mu=\text{constant}$), as expected [15, 16]. Also, the fact that both attractive and repulsive particles show frequency independent damping indicates that the damping character does not depend on surface forces and thus the material behavior is governed by solid particle contacts. However, the most important observation from Figs 3 and 4 is that the experimental values of resonance for all the depicted models show a shifting to the right of the analytical hysteretic ones for the case of attractive particles, and to the left of them for the case of repulsive particles. This phenomenon was not discussed by Richter [2] and, as a result, it remained unexplained.

Figure 3

Figure 4

In the following sections, an attempt will be made to theoretically explain the above phenomenon and also provide suggestions on how to obtain experimentally, from resonant column tests, the value of the shear modulus G as well as the values of microstructural parameters in a more rational and accurate way. This will be accomplished by employing a higher order (generalized) theory of elasticity or viscoelasticity.

3. A SIMPLE GRADIENT THEORY OF ELASTICITY

When the dimensions of a structure or the wavelength of dynamic disturbances become comparable to the internal length scale of its elastic material, then size effects leading to wave dispersion are observed. These microstructural effects cannot be described by the classical theory of elasticity and resort should be made instead to higher order or generalized theories of elasticity possessing internal length scale(s). Such a microstructural theory of elasticity is the one due to Mindlin [4], which in its simplified forms has been successfully used to solve a variety of boundary value problems under static or dynamic conditions involving microstructures in microelectronic and micromechanical devices and materials like foams, granular assemblies, concrete, bones and composites. Comprehensive reviews on static and dynamic gradient elasticity theory and its applications can be found in [17-20]. Microstructural effects in the theory of gradient elasticity manifest themselves in the form of increased stiffness [21, 22], size effects [17, 23], elimination or reduction of singularities [24, 25], increase of buckling loads and natural frequencies [21, 26, 27] and wave dispersion [8-12, 28, 29].

The simplest possible gradient elastic theory is the one with just one elastic constant (the gradient or micro-stiffness coefficient g with dimensions of length), in addition to the two classical elastic moduli (Young's modulus E and Poisson's ratio ν). It has been demonstrated in [8-12] that the presence of micro-inertia in dynamic microstructural problems, associated with an additional constant (the micro-inertia coefficient h with dimensions of length), is very important and has to be taken into account.

For reasons of completeness, the governing equations of motion in three-dimensional gradient elasticity with both micro-stiffness and micro-inertia effects and zero body forces, as obtained from those of Mindlin [4] under certain simplifications, are given in terms of the displacement vector \mathbf{u} as [9]

$$(1 - g^2 \nabla^2)[\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u}] = \rho(\ddot{\mathbf{u}} - h^2 \nabla^2 \ddot{\mathbf{u}}) \quad (1)$$

where λ, μ are the Lamé constants expressed in terms of E and ν as $\lambda = E\nu / [(1+\nu)(1-2\nu)]$ and $\mu = E / 2(1+\nu)$. The total and Cauchy second order stress tensor $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$, respectively, as well as the third order double stress tensor $\boldsymbol{\mu}$ are given by

$$\begin{aligned}\boldsymbol{\sigma} &= \boldsymbol{\tau} - \nabla \cdot \boldsymbol{\mu} \\ \boldsymbol{\mu} &= g^2 \nabla \boldsymbol{\tau} \\ \boldsymbol{\tau} &= 2\mu \boldsymbol{\varepsilon} + \lambda (\text{tr} \boldsymbol{\varepsilon}) \mathbf{I}\end{aligned}\tag{2}$$

with \mathbf{I} being the unit tensor and $\boldsymbol{\varepsilon}$ the strain tensor having the form

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + \mathbf{u} \nabla)\tag{3}$$

Because the gradient elastic theory increases the order of space derivatives in comparison with the classical theory, additional non-classical boundary conditions are required for the establishment of well-posed boundary value problems. These are obtained with the use of variational statements [4, 30, 3]. For a smooth boundary, these conditions consist of the displacement vector \mathbf{u} and/or the traction vector \mathbf{P} prescribed over the boundary of the domain (classical conditions) and the normal displacement vector $\partial \mathbf{u} / \partial n$ and/or the double traction vector \mathbf{R} prescribed over that boundary (non-classical conditions) with \mathbf{n} being the unit normal vector.

Recently, Polyzos and Fotiadis [3] were able to derive Mindlin's type of gradient elasticity via simple lattice models and provide expressions for the micro-stiffness and micro-inertia coefficients g and h in terms of the distance l between two successive particles of the lattice structure. This was done for the case of a rod in axial vibration without damping. Viscoelastic effects in gradient elasticity have been considered and studied in [31] via the correspondence principle in connection with the static and dynamic analysis of an axial bar.

In this work torsional vibrations of a gradient elastic bar with micro-stiffness, micro-inertia and internal viscoelastic damping are studied for the first time. The governing equations of motion and all possible boundary conditions (classical and non-classical) are obtained by both, the lattice theory of Polyzos and Fotiadis [3] and the continuum theory of Mindlin [4].

4. TORSIONAL VIBRATIONS OF A GRADIENT VISCOELASTIC BAR

Consider a circular cylindrical elastic bar experiencing torsional vibration. Its classical governing equation of motion has the form [32]

$$c^2 \frac{\partial^2 q(x,t)}{\partial x^2} = \frac{\partial^2 q(x,t)}{\partial t^2}\tag{4}$$

where $\theta(x,t)$ is the torsional angle, $c^2 = G / \rho$ is the shear wave velocity with G being the shear modulus and ρ the mass density, x is the distance along the axis of the bar and t is the time.

If the bar material is viscoelastic of the Kelvin-Voigt type, then the above equation takes the form [15]

$$c^2 \frac{\partial^2 \theta(x,t)}{\partial x^2} + \delta \frac{\partial^3 \theta(x,t)}{\partial x^2 \partial t} = \frac{\partial^2 \theta(x,t)}{\partial t^2}\tag{5}$$

where $\delta = \eta / \rho$ with δ being the kinematic viscosity in units of Length²/Time and η (equal to μ of [2, 12]) being the dynamic viscosity in units of Mass/Time.

In the following, the governing equation of torsional vibrations for the case of a cylindrical elastic bar including micro-stiffness and micro-inertia effects is derived by employing both a lattice and a continuum model in accordance with the general theories of Polyzos and Fotiadis [3] and Mindlin [4], respectively.

4.1 Lattice modeling approach

Consider a cylindrical bar of length L and cross-section area A , fixed between two cylindrical masses with moments J_0 and J_L at $x=0$ and $x=L$, respectively, in units of Mass x (Length)², as shown in Fig. 5. The bar is simulated by a lattice model consisting of equally spaced identical, rigid and massless cylindrical particles $B^{(x)}$ with very small thickness compared to the lattice size and connected to each other by simple Kelvin-Voigt systems of torsional springs and dashpots, as it is illustrated in Fig. 5. The stiffness K_l and damping C_l constants of the springs and dashpots, respectively, are defined as

$$K_l = \frac{GI_p}{l} \quad (6)$$

$$C_l = \frac{\eta I_p}{l} \quad (7)$$

with I_p being the polar moment of inertia of the cross-section in units of Length⁴ and l the lattice spacing, as shown in Fig.4.

Figure 5

Following Polyzos and Fotiadis [3], one can conclude that the influence of micro-inertia on the torsional vibration of the bar can be taken into account by considering, not massless springs as in classical lattice models, but torsional springs with uniformly distributed mass moment of inertia j . In other words, microstructure is considered in a discrete manner as a large number of very small cylindrical masses with moment of inertia j_n , uniformly distributed between two adjacent points x and $x+l$ and connected with torsional springs of stiffness k_e , as it is shown in Fig. 6. The sum of all those masses gives a moment of inertia density J that contributes to the kinetic energy of the system. Since the micro-masses are very small, the springs k_e can be replaced by one with total stiffness K_l . Thus one obtains the lattice model of Fig. 5.

Figure 6

Assuming that l is small and that a continuation process is valid, the discrete angular kinematic degrees of freedom of the masses lying at the two neighboring unit cells of point x can be expressed by the continuous variables $\theta(x-l,t)$, $\theta(x,t)$ and $\theta(x+l,t)$.

For a unit cell corresponding to the mass M with torsional angle $\theta(x,t)$, as shown in Fig.5, the strain energy density for the cell l is given by

$$U_b = \frac{1}{2} \left\{ \frac{\frac{1}{2} K_l [q(x,t) - q(x-l,t)]^2}{Al} + \frac{\frac{1}{2} K_l [q(x+l,t) - q(x,t)]^2}{Al} \right\} \quad (8)$$

Expanding $\theta(x \pm l, t)$ around the point x and considering quadratic behavior for $\theta(x, t)$, one has

$$\theta(x \pm l, t) = \theta(x, t) \pm l \frac{\partial \theta(x, t)}{\partial x} + \frac{1}{2} l^2 \frac{\partial^2 \theta(x, t)}{\partial x^2} \quad (9)$$

Inserting expressions (6) and (9) into (8) and integrating the resulting equation over the total length L of the bar, one obtains the total strain energy in the form

$$U = \int_V U_b dV = \int_0^L U_b A dx = \frac{1}{2} G I_p \int_0^L \left[\left(\frac{\partial \theta(x, t)}{\partial x} \right)^2 + \frac{l^2}{4} \left(\frac{\partial^2 \theta(x, t)}{\partial x^2} \right)^2 \right] dx \quad (10)$$

The kinematic energy density is associated only with the torsional motion of the springs and their torsional inertia, as the particle C does not experience such a motion. Following Polyzos and Fotiadis [3], one can prove that the kinetic energy density for the spring of the unit cell l is defined as

$$K_b = \frac{1}{2} \left\{ \frac{\frac{1}{2} \int_0^l J \left[\frac{\partial q^-(z, t)}{\partial t} \right]^2 dz}{Al} + \frac{\frac{1}{2} \int_0^l J \left[\frac{\partial q^+(z, t)}{\partial t} \right]^2 dz}{Al} \right\} \quad (11)$$

Where J denotes the torsional moment of inertia of spring micro-material per unit length expressed as

$$J = \rho I_p \quad (12)$$

the coordinate z indicates the distance of each point of the spring from its left end (Fig.6) and $\frac{\partial\theta^-(z,t)}{\partial t}$ and $\frac{\partial\theta^+(z,t)}{\partial t}$ represent point angular velocities of the springs with end angular velocities $\frac{\partial\theta(x-l,t)}{\partial t}$, $\frac{\partial\theta(x,t)}{\partial t}$ and $\frac{\partial\theta(x,t)}{\partial t}$, $\frac{\partial\theta(x+l,t)}{\partial t}$, respectively.

Since lattice l is very small, $\frac{\partial\theta^\pm(z,t)}{\partial t}$ are assumed linear with respect to z , i.e.,

$$\begin{aligned}\frac{\partial\theta^-(z,t)}{\partial t} &= \left[\frac{\partial\theta(x,t)}{\partial t} - \frac{\partial\theta(x-l,t)}{\partial t} \right] \left(\frac{z}{l} \right) + \frac{\partial\theta(x-l,t)}{\partial t} \\ \frac{\partial\theta^+(z,t)}{\partial t} &= \left[\frac{\partial\theta(x+l,t)}{\partial t} - \frac{\partial\theta(x,t)}{\partial t} \right] \left(\frac{z}{l} \right) + \frac{\partial\theta(x,t)}{\partial t}\end{aligned}\quad (13)$$

Expanding angular velocities $\frac{\partial\theta(x\pm l,t)}{\partial t}$ around the point x and considering linear behavior one has

$$\frac{\partial\theta(x\pm l,t)}{\partial t} = \frac{\partial\theta(x,t)}{\partial t} \pm l \frac{\partial^2\theta(x,t)}{\partial x\partial t} \quad (14)$$

Equations (13), in view of Eq. (14), become

$$\begin{aligned}\frac{\partial\theta^-(z,t)}{\partial t} &= \frac{\partial^2\theta(x,t)}{\partial x\partial t} z + \frac{\partial\theta(x,t)}{\partial t} - l \frac{\partial^2\theta(x,t)}{\partial x\partial t} \\ \frac{\partial\theta^+(z,t)}{\partial t} &= \frac{\partial^2\theta(x,t)}{\partial x\partial t} z + \frac{\partial\theta(x,t)}{\partial t}\end{aligned}\quad (15)$$

Thus, substituting Eq. (12) and (15) in (11) and integrating the resulting equation over the total length L of the bar, one obtains its total kinetic energy in the form

$$K = \int_v K_b dV = \int_0^L K_b A dx = \frac{1}{2} \rho I_p \int_0^L \left[\left(\frac{\partial\theta(x,t)}{\partial t} \right)^2 + \frac{l^2}{3} \left(\frac{\partial^2\theta(x,t)}{\partial x\partial t} \right)^2 \right] dx \quad (16)$$

The power density absorbed by the two dashpots joined at point x , is for the unit cell

$$D_i = \frac{1}{2} \left\{ \frac{\frac{1}{2} C_l \left[\frac{\partial q(x,t)}{\partial t} - \frac{\partial q(x-l,t)}{\partial t} \right]^2}{Al} + \frac{\frac{1}{2} C_l \left[\frac{\partial q(x+l,t)}{\partial t} - \frac{\partial q(x,t)}{\partial t} \right]^2}{Al} \right\} \quad (17)$$

Utilizing the asymptotic relations (14) and the relation (7) in (17), the expression of the absorbed power density for the unit cell obtains the form

$$D_t = \frac{1}{2} \eta \frac{I_p}{A} \left(\frac{\partial^2 \theta(x,t)}{\partial x \partial t} \right)^2 \quad (18)$$

Thus the absorbed power density for the whole bar will be

$$D = \int_V D_t dV = \int_0^L D_t A dx = \frac{1}{2} \eta I_p \int_0^L \left(\frac{\partial^2 \theta(x,t)}{\partial x \partial t} \right)^2 dx \quad (19)$$

Denoting derivatives with respect to x and t by primes and overdots, respectively, and taking into account the contribution of the attached cylindrical masses with moment of inertia J_0 and J_L (with units of Kgm^2) to the kinetic energy of the system, the strain and energy densities U , K from (10) and (16), respectively and the absorbed power density D from (19) can be written as

$$\begin{aligned} U &= \frac{1}{2} G I_p \int_0^L \left[(q')^2 + \frac{l^2}{4} (q'')^2 \right] dx \\ K &= \frac{1}{2} r I_p \int_0^L \left[(\dot{q})^2 + \frac{l^2}{3} (\dot{q}')^2 \right] dx + \frac{1}{2} J_0 [\dot{q}(0,t)]^2 + \frac{1}{2} J_L [\dot{q}(L,t)]^2 \\ D &= \int_0^L \bar{D} [(\dot{q}')^2] dx = \int_0^L \frac{1}{2} h I_p (\dot{q}')^2 dx \end{aligned} \quad (20)$$

The governing equation of torsional motion of the bar as well as all possible boundary conditions (classical and non-classical) can be determined with the aid of Hamilton's variational principle valid for a non-conservative system (Kim et al. [33])

$$\int_{t_0}^{t_1} d(K - U) dt + \int_{t_0}^{t_1} dW dt = \int_{t_0}^{t_1} \left[\int_0^L \frac{\partial \bar{D}}{\partial \dot{q}'} dq' \right] dt \quad (21)$$

where U , K and D are provided by (20) and W stands for the work done by external moment tractions M and double moment tractions T acting at both ends of the bar. It is easy to see that

$$\int_{t_0}^{t_1} \left[\int_0^L \frac{\partial \bar{D}}{\partial \dot{q}'} dq' \right] dt = \int_{t_0}^{t_1} \left\{ \int_0^L \frac{1}{2} h I_p \frac{\partial (\dot{q}')^2}{\partial \dot{q}'} dq' dx \right\} dt = \int_{t_0}^{t_1} \left\{ h I_p \left[- \int_0^L \dot{q}'' dq dx + [\dot{q}' dq]_0^L \right] \right\} dt \quad (22)$$

$$\int_{t_0}^{t_1} \delta W dt = \int_{t_0}^{t_1} [M(0,t) \delta \theta(0,t) + M(L,t) \delta \theta(L,t)] dt + \int_{t_0}^{t_1} [T(0,t) \delta \theta'(0,t) + T(L,t) \delta \theta'(L,t)] dt \quad (23)$$

The difference $K-U$ can be written on account of (20)_{1,2} as

$$\begin{aligned}
K - U &= \int_0^L \frac{1}{2} \left\{ I_p \left[r \left((\dot{q})^2 + \frac{l^2}{3} (\dot{q}')^2 \right) - G \left((q')^2 + \frac{l^2}{4} (q'')^2 \right) \right] \right\} dx \\
&+ \frac{1}{2} J_0 [\dot{q}(0,t)]^2 + \frac{1}{2} J_L [\dot{q}(L,t)]^2 = \\
&= \int_0^L F(q', q'', \dot{q}, \dot{q}') dx + \frac{1}{2} J_0 [\dot{q}(0,t)]^2 + \frac{1}{2} J_L [\dot{q}(L,t)]^2
\end{aligned} \tag{24}$$

Following Lanczos [34] and taking into account that according to Hamilton's principle $dq(x, t_0) = 0$ and $dq(x, t_1) = 0$, one obtains with the aid of (24)

$$\begin{aligned}
\int_{t_0}^{t_1} d(K - U) dt &= \int_{t_0}^{t_1} \int_0^L dF(q', q'', \dot{q}, \dot{q}') dx dt + J_0 \dot{q}(0,t) d\dot{q}(0,t) + J_L \dot{q}(L,t) d\dot{q}(L,t) = \\
&= \int_{t_0}^{t_1} \left\{ \int_0^L \left[\frac{\partial^2}{\partial x^2} \left(\frac{\partial F}{\partial q''} \right) - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial q'} \right) - \frac{\partial}{\partial t} \left(\frac{\partial F}{\partial \dot{q}} \right) + \frac{\partial^2}{\partial x \partial t} \left(\frac{\partial F}{\partial \dot{q}'} \right) \right] dq dx \right\} dt + \\
&+ \int_{t_0}^{t_1} \left[\left[\frac{\partial F}{\partial u'} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial q''} \right) - \frac{\partial}{\partial t} \left(\frac{\partial F}{\partial \dot{q}'} \right) \right] dq + \frac{\partial F}{\partial q''} dq' \right]_0^L dt + J_0 \dot{q}(0,t) d\dot{q}(0,t) + J_L \dot{q}(L,t) d\dot{q}(L,t)
\end{aligned} \tag{25}$$

Thus, on account of (22) – (25), Eq. (21) yields

$$\begin{aligned}
&\int_{t_0}^{t_1} I_p \int_0^L \left[G \left(q'' - \frac{l^2}{4} q'''' \right) - r \left(\ddot{q} - \frac{l^2}{3} \ddot{q}'' \right) + h \dot{q}'' \right] dq dx dt + \\
&+ \int_{t_0}^{t_1} \left\{ -I_p \left[\left[G \left(q' - \frac{l^2}{4} q''' \right) + r \frac{l^2}{3} \ddot{q}' + h \dot{q}' \right] dq \right]_0^L - J_0 \ddot{q}(0,t) dq(0,t) - J_L \ddot{q}(L,t) dq(L,t) \right\} dt \\
&+ \int_{t_0}^{t_1} \left\{ M(0,t) dq(0,t) + M(L,t) dq(L,t) \right\} dt \\
&+ \int_{t_0}^{t_1} I_p \left[\left(G \frac{l^2}{4} q'' \right) dq' \right]_0^L dt \\
&+ \int_{t_0}^{t_1} \left\{ T(0,t) dq'(0,t) + T(L,t) dq'(L,t) \right\} dt = 0
\end{aligned} \tag{26}$$

The vanishing of the first integral in (26) provides the equation of torsional motion of the bar in the form

$$G \left(q'' - \frac{l^2}{4} q'''' \right) + h \dot{q}'' = r \left(\ddot{q} - \frac{l^2}{3} \ddot{q}'' \right) \tag{27}$$

or

$$c^2 \left(1 - \frac{l^2}{4} \frac{\partial^2}{\partial x^2} \right) \left(\frac{\partial^2 \theta(x,t)}{\partial x^2} \right) + \delta \frac{\partial^3 \theta(x,t)}{\partial x^2 \partial t} = \left(1 - \frac{l^2}{3} \frac{\partial^2}{\partial x^2} \right) \left(\frac{\partial^2 \theta(x,t)}{\partial t^2} \right) \quad (28)$$

The vanishing of the second and third integrals in (26) provides the classical boundary conditions indicating that either $\theta(0,t)$ and $\theta(L,t)$ are prescribed or $M(0,t)$ and $M(L,t)$ of the form

$$\begin{aligned} \left[GI_p \left(q\dot{\theta} - \frac{l^2}{4} q\dot{\theta}\dot{\theta} \right) + rI_p \frac{l^2}{3} \ddot{q}\dot{\theta} + hI_p \dot{q}\dot{\theta} \right]_{x=0} - J_0 \ddot{q}(0,t) &= -M(0,t) \\ \left[GI_p \left(q\dot{\theta} - \frac{l^2}{4} q\dot{\theta}\dot{\theta} \right) + rI_p \frac{l^2}{3} \ddot{q}\dot{\theta} + hI_p \dot{q}\dot{\theta} \right]_{x=L} + J_L \ddot{q}(L,t) &= M(L,t) \end{aligned} \quad (29)$$

are prescribed, or $q(0,t)$ and $M(L,t)$ are prescribed, or finally $q(L,t)$ and $M(0,t)$ are prescribed.

Finally, the vanishing of the fourth and fifth integral of (26) provides the non-classical boundary conditions indicating that either $\theta'(0,t)$ and $\theta'(L,t)$ are prescribed or $T(0,t)$ and $T(L,t)$ of the form.

$$\begin{aligned} GI_p \frac{l^2}{4} \theta''(0,t) &= -T(0,t) \\ GI_p \frac{l^2}{4} \theta''(L,t) &= T(L,t) \end{aligned} \quad (30)$$

are prescribed or $\theta'(0,t)$ and $T(L,t)$ are prescribed, or finally $\theta'(L,t)$ and $T(0,t)$ are prescribed.

The just described lattice model is able to explain rather the behavior of a resonant column made of repulsive particles for which micro-inertia is greater than micro-stiffness ($l/\sqrt{3} > l/2$), than of attractive ones. In the case of fine-grained materials with attractive particles, non-local interaction between the particles appears leading to a low density formation of the grain skeleton. This property can be simulated in the aforementioned lattice model by considering “non-local” torsional springs K_{2l} without micro-inertia effects connecting the particle $B^{(x)}$ with the particles $B^{(x-2l)}$ and $B^{(x+2l)}$, as it is illustrated in Fig. 7.

Figure 7

In that case, taking into account the potential energy density of the non-local springs of stiffness K_{2l} in the above procedure, the micro-stiffness parameter $l^2/4$ is replaced by the

expression $b^2 \frac{l^2}{4}$ where

$$b^2 = \frac{1 + 16 \frac{G_n}{G}}{1 + 4 \frac{G_n}{G}}, \quad (31)$$

with G_n denoting the shear modulus corresponding to the nonlocal springs of stiffness K_{2l} .

From (31) it is apparent that for $G_n / G > 0.03125$, the micro-stiffness parameter $b^2 \frac{l^2}{4}$ for attractive particles is greater than the micro-inertia parameter $l^2/3$ or $b > 2/\sqrt{3} = 1.56$.

4.2 Continuum modeling approach

The above results can also be derived by employing directly the simplified Mindlin's [4] form II version of gradient elasticity with two microstructural constants, as described in Section 3, which is a continuum theory. For a gradient viscoelastic circular cylindrical bar of length L , cross-sectional area A , mass density ρ , shear modulus G , Poisson ratio $\nu=0$ and microstructural stiffness and inertia constants g and h , respectively, one can derive with the aid of [4, 32] its strain energy, kinetic energy and energy of dissipation as follows:

Employing a Cartesian system of axes (x, y, z) with x along the bar length and y and z on a cross-section of the bar, one has that under torsional deformation the only nonzero components of Cauchy and double stress tensors $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ obtained from Eqs. (2) are of the form

$$\begin{aligned} t_{xy} &= 2G e_{xy}, & t_{xz} &= 2G e_{xz} \\ m_{xy} &= g^2 \nabla t_{xy} / \nabla x, & m_{xz} &= g^2 \nabla t_{xz} / \nabla x \end{aligned} \quad (32)$$

where the strains ε_{xy} and ε_{xz} are expressed in terms of the torsion angle θ as [32]

$$\begin{aligned} 2\varepsilon_{xy} &= -z(\partial\theta / \partial x) \\ 2\varepsilon_{xz} &= y(\partial\theta / \partial x) \end{aligned} \quad (33)$$

The nonzero displacement components u_y and u_z along the y and z axes, respectively, are given in terms of the torsion angle θ as [32]

$$\begin{aligned} u_y &= -z\theta \\ u_z &= y\theta \end{aligned} \quad (34)$$

Strain gradients and displacement gradients can be obtained from (33) and (34) in the form

$$\begin{aligned} 2(\partial\varepsilon_{xy} / \partial x) &= -z(\partial^2\theta / \partial x^2), & 2(\partial\varepsilon_{xz} / \partial x) &= y(\partial^2\theta / \partial x^2) \\ \partial u_y / \partial x &= -z(\partial\theta / \partial x), & \partial u_z / \partial x &= y(\partial\theta / \partial x) \end{aligned} \quad (35)$$

Thus, the strain energy of the bar can be obtained as

$$U = \frac{1}{2} \int_V 2(\tau_{xy}\varepsilon_{xy} + \tau_{xz}\varepsilon_{xz} + \mu_{xy}\varepsilon'_{xy} + \mu_{xz}\varepsilon'_{xz}) dV \quad (36)$$

where V indicates the bar volume and primes indicate differentiation with respect to x . Substituting stresses in terms of strains with the aid of (32) in Eq. (36), one obtains

$$U = \frac{1}{2} \int_V 2 \left[2G(\varepsilon_{xy})^2 + 2G(\varepsilon_{xz})^2 + 2Gg^2(\varepsilon'_{xy})^2 + 2Gg^2(\varepsilon'_{xz})^2 \right] dV \quad (37)$$

Substituting strains and their gradients in (37) in terms of the torsion angle derivatives, as given by (33) and (35)_{1,2} one receives

$$U = \frac{1}{2} GI_p \int_0^L \left[(\theta')^2 + g^2(\theta'')^2 \right] dx \quad (38)$$

where I_p is the polar moment of inertia in the form

$$I_p = \int_A (y^2 + z^2) dA \quad (39)$$

The above expression for U is exactly the same with that in (20)₁ provided that $g = l/2$.

The kinetic energy of the bar can be obtained as

$$K = \frac{1}{2} \int_V \left[(r\dot{u}_y^2 + r\dot{u}_z^2) + h^2 (r(\dot{u}'_y)^2 + r(\dot{u}'_z)^2) \right] dV \quad (40)$$

where overdots indicate differentiation with respect to time t and the terms inside the integral which are multiplied by h^2 represent the effect of micro-inertia. Substituting velocities and velocity gradients in (40) by their expressions in (34) and (35)_{3,4} after differentiation with respect to time t , one obtains Eq. (40) in the form

$$K = \frac{1}{2} r I_p \int_0^L \left[(\dot{q})^2 + h^2 (\dot{q}')^2 \right] dx \quad (41)$$

The above expression for K , augmented by the inertial energies of the two end masses, is the same as that in (20)₂ provided that $h = l/\sqrt{3}$.

The dissipation energy in the bar is due to the presence of viscous effects, which are assumed not to be influenced by the material microstructure. For the simple case of the Kelvin-Voigt viscoelastic model, the viscous components of the stresses τ_{xy}^v and τ_{xz}^v are assumed to be of the form

$$t_{xy}^v = 2h\dot{\varepsilon}_{xy}, \quad t_{xz}^v = 2h\dot{\varepsilon}_{xz} \quad (42)$$

and thus the dissipation energy in the bar takes the form

$$D = \frac{1}{2} \int_V 2 \left(t_{xy}^v \dot{\varepsilon}_{xy} + t_{xz}^v \dot{\varepsilon}_{xz} \right) dV \quad (43)$$

Substituting in (43) viscous stresses and velocities of strain in terms of derivatives of θ with the aid of (42) and (33), one can receive D in the form

$$D = \frac{1}{2} h I_p \int_0^L (\dot{q}')^2 dx \quad (44)$$

The above expression is exactly the same as the one in (20)₃ obtained by the lattice theory.

Thus, the continuum approach provides the same expressions for U, K and D obtained by the lattice theory approach provided that the micro-stiffness and micro-inertia coefficients g and h are equal to $l/2$ and $l/\sqrt{3}$, respectively, where l is the lattice spacing. With these U, K and D one can employ Hamilton's variational method as before and obtain the same governing equation (27) and boundary conditions (29) and (30) with $l^2/4$ and $l^2/3$ replaced by g^2 and h^2 , respectively.

In conclusion, the governing equation of torsional motion of a circular cylindrical bar with end masses and gradient viscoelastic material behavior is of the form

$$c^2 \left(1 - g^2 \frac{\partial^2}{\partial x^2} \right) \left(\frac{\partial^2 q(x,t)}{\partial x^2} \right) + d \frac{\partial^3 q(x,t)}{\partial x^2 \partial t} = \left(1 - h^2 \frac{\partial^2}{\partial x^2} \right) \left(\frac{\partial^2 q(x,t)}{\partial t^2} \right) \quad (45)$$

where in view of Eqs. (28) and (31)

$$\begin{aligned} g^2 &= l^2 / 4 \\ h^2 &= l^2 / 3 \end{aligned} \quad (46)$$

for repulsive particles and

$$\begin{aligned} g^2 &= b^2 l^2 / 4 \\ h^2 &= l^2 / 3 \end{aligned} \quad (47)$$

for attractive particles. This equation is accompanied by the classical and non-classical boundary conditions (29) and (30), respectively, with $l^2/4$, $b^2 l^2/4$ and $l^2/3$ replaced by g^2 and h^2 as indicated by (46) and (47). One can observe that i) Eqs. (46) and (47) provide expressions for determining the phenomenological coefficients g and h in terms of the geometry of the microstructure of both repulsive and attractive particles and ii) Eq. (45) reduces to the classical form of Eq. (5) for $g = h = 0$.

At this point it is interesting to study the propagation of harmonic with time torsional waves in a gradient elastic bar in order to see the influence of the two non-classical constants g and h on its torsional motion. Thus, assuming torsional waves of the form

$$q(x,t) = A e^{i(kx - \omega t)} \quad (48)$$

where A is the amplitude, k the wave number and ω the circular frequency, Eq. (45) with $\delta = 0$ takes the form

$$\left(1 + h^2 k^2 \right) \omega^2 - \frac{G}{r} k^2 \left(1 + g^2 k^2 \right) = 0 \quad (49)$$

Solving Eq. (49) for ω , one can compute the phase velocity of the propagation V of the torsional waves in the bar in the form

$$V = \omega / k = c \left[\left(1 + g^2 k^2 \right) / \left(1 + h^2 k^2 \right) \right]^{1/2} \quad (50)$$

where $c = \sqrt{G / \rho}$ is the classical wave propagation velocity. Equation (50) is the dispersion relation indicating variation of V with k or ω . For $g = h = 0$ (classical case) one obtains from

(50) $V=c$, indicating that there is no dispersion and the velocity of propagation is constant. Figure 8 depicts the variation of V versus k for various combinations of g and h . For $g > h$ one has the case of attractive particles for which $V > c$, while for $g < h$ one has the case of repulsive particles for which $V < c$. These results are the same with those observed for wave propagation in an infinitely extended gradient elastic medium or in a bar under axial motion [10].

5. FREE TORSIONAL VIBRATION OF A GRADIENT VISCOELASTIC BAR

Consider the governing equation (45) of torsional vibrations of the gradient viscoelastic bar of Fig. 4 subject to the classical and non-classical boundary conditions (29) and (30) with $M=0$, $T=0$ and $l^2/4$ and $l^2/3$ being replaced by g^2 and h^2 , respectively. Assuming a time harmonic solution of the form

$$q(x,t) = \bar{q}(x)e^{i\omega t} \quad (51)$$

where $\bar{\theta}(x)$ represents the amplitude of the torsional angle, ω the circular frequency of vibration and $i = \sqrt{-1}$, one can obtain the governing equation of motion and the associated boundary conditions in the form

$$-c^2 g^2 \bar{q}''''(x) + (c^2 + i\omega d - h^2 \omega^2) \bar{q}''(x) + \omega^2 \bar{q}(x) = 0 \quad (52)$$

$$GI_p [\bar{q}'(0) - g^2 \bar{q}'''(0)] - \omega^2 r I_p h^2 \bar{q}'(0) + i\omega h I_p \bar{q}'(0) = -J_0 \omega^2 \bar{q}(0) \quad (53)$$

$$GI_p [\bar{q}'(L) - g^2 \bar{q}'''(L)] - \omega^2 r I_p h^2 \bar{q}'(L) + i\omega h I_p \bar{q}'(L) = J_L \omega^2 \bar{q}(L) \quad (54)$$

$$GI_p g^2 \bar{q}''(0) = 0 \quad (55)$$

$$GI_p g^2 \bar{q}''(L) = 0$$

Introducing the dimensionless parameters

$$\tilde{g} = g/L, \quad \tilde{h} = h/L, \quad x = x/L, \quad a = \omega L/c \quad (56)$$

$$D = d/(cL), \quad \bar{J}_0 = rLI_p/J_0, \quad \bar{J}_L = rLI_p/J_L$$

one can rewrite Eqs. (52)-(55) in the form

$$-\tilde{g}^2 \bar{q}''''(x) + (1 + iaD - \tilde{h}^2 a^2) \bar{q}''(x) + a^2 \bar{q}(x) = 0 \quad (57)$$

$$-\tilde{g}^2 \bar{q}'''(0) + (1 + iaD - \tilde{h}^2 a^2) \bar{q}'(0) = -\left(a^2 / \bar{J}_0\right) \bar{q}(0) \quad (58)$$

$$-\tilde{g}^2 \bar{q}'''(1) + (1 + iaD - \tilde{h}^2 a^2) \bar{q}'(1) = -\left(a^2 / \bar{J}_L\right) \bar{q}(1) \quad (59)$$

$$\bar{q}''(0) = \bar{q}''(1) = 0 \quad (60)$$

The solution of Eq. (57) has the form

$$\bar{q}(x) = A_1 \sin px + B_1 \cos px + A_2 \sinh qx + B_2 \cosh qx \quad (61)$$

where

$$p = \sqrt{\frac{-1 - ig + \tilde{h}a^2 + \sqrt{(1 + ig - \tilde{h}a^2)^2 + 4\tilde{g}^2a^2}}{2\tilde{g}^2}} \quad (62)$$

$$q = \sqrt{\frac{1 + ig - \tilde{h}^2a^2 + \sqrt{(1 + ig - \tilde{h}^2a^2)^2 + 4\tilde{g}^2a^2}}{2\tilde{g}^2}} \quad (63)$$

with γ being equal to $aD = wd/c^2 = wh/G$ for frequency dependent or viscous damping and equal to 2β (β =constant damping coefficient) for frequency independent or hysteretic damping.

For the case of a column fixed at its base, one has $\bar{J}_0 = 0$ and $\bar{J}_L > 1$. For this case the boundary conditions (58)-(60) reduce to

$$\bar{q}'''(0) = \bar{q}'''(1) = \bar{q}(0) = 0 \quad (64)$$

$$-\tilde{g}^2\bar{q}'''(1) + (1 + ig - \tilde{h}^2a^2)\bar{q}'(1) = (a^2/\bar{J}_L)\bar{q}(1) \quad (65)$$

One can now obtain numerical results for the following eight cases corresponding to various combinations of values of the parameters \tilde{g} , \tilde{h} and γ :

5.1 Classical elasticity without damping

In this case one has $\tilde{g} = \tilde{h} = g = 0$ and thus the governing equation (57) and boundary conditions (64) and (65) become

$$\bar{q}''''(x) + a^2\bar{q}(x) = 0 \quad (66)$$

$$\bar{\theta}(0) = 0, \quad \bar{\theta}'(1) = (\alpha^2/\bar{J}_L)\bar{\theta}(1) \quad (67)$$

leading to the frequency equation

$$\alpha \tan \alpha = \bar{J}_L \quad (68)$$

which for $\bar{J}_L = 10$ can provide the first four eigenfrequencies, as shown in Table 1.

Table 1

5.2 Classical elasticity with damping

In this case one has $\tilde{g} = \tilde{h} = 0$, $\gamma \neq 0$ and thus the governing equation (57) and boundary conditions (64) and (65) become

$$(1 + ig)\bar{q}''''(x) + a^2\bar{q}(x) = 0 \quad (69)$$

$$\bar{\theta}(0) = 0, \quad (1 + i\gamma)\bar{\theta}'(1) = (\alpha^2 / \bar{J}_L)\bar{\theta}(1) \quad (70)$$

leading to the frequency equation

$$\alpha \tan\left(\frac{\alpha}{\sqrt{1+i\gamma}}\right) = \bar{J}_L \sqrt{1+i\gamma} \quad (71)$$

which for $\bar{J}_L = 10$ can provide the first four eigenfrequencies for the case of $\gamma = 2\beta$ (hysteretic damping), as shown in Table 2.

Table 2

5.3 Gradient elasticity with micro-inertia and without damping

In this case one has $\tilde{h} \neq 0$, $\tilde{g} = g = 0$ and thus the governing equation (57) and boundary conditions (64) and (65) become

$$(1 - \tilde{h}^2 a^2)\bar{q}''(x) + a^2\bar{q}(x) = 0 \quad (72)$$

$$\bar{q}(0) = 0, \quad (1 - \tilde{h}^2 a^2)\bar{q}'(1) = (a^2 / \bar{J}_L)\bar{q}(1) \quad (73)$$

leading to the frequency equation

$$a \tan\left(\frac{a}{\sqrt{1 - \tilde{h}^2 a^2}}\right) = \bar{J}_L \sqrt{1 - \tilde{h}^2 a^2} \quad (74)$$

It is apparent from (72) that vibration occurs only when $1 - \tilde{h}^2 a^2 > 0$ or $a < 1/\tilde{h}$. Thus, for the typical values of $\tilde{h} = 0.01$ and $\tilde{h} = 0.05$ one has that $a < 100$ and $a < 20$, respectively. Table 3 shows the first four eigenfrequencies for this case when $\bar{J}_L = 10$ and $\tilde{h} = 0.0$, $\tilde{h} = 0.01$, $\tilde{h} = 0.05$, while Fig. 9 depicts the first four natural frequencies versus \tilde{h} . One can observe from that figure that for increasing values of \tilde{h} , the frequencies decrease, especially for higher modes. This is because micro-inertia effects are here significant and are associated with the much more dense arrangement of repulsive particles.

Table 3

Figure 9

5.4 Gradient elasticity with micro-inertia and damping

In this case one has $\tilde{h} \neq 0$, $\gamma \neq 0$, $\tilde{g} = 0$ and thus the governing equation (57) and boundary conditions (64) and (65) become

$$(1 + ig - \tilde{h}^2 a^2) \bar{q}'''(x) + a^2 \bar{q}(x) = 0 \quad (75)$$

$$\bar{q}(0) = 0, \quad (1 + ig - \tilde{h}^2 a^2) \bar{q}'(1) = \left(a^2 / \bar{J}_L \right) \bar{q}(1) \quad (76)$$

leading to the frequency equation

$$a \tan \left(\frac{a}{\sqrt{(1 + ig - \tilde{h}^2 a^2)}} \right) = \bar{J}_L \sqrt{1 + ig - \tilde{h}^2 a^2} \quad (77)$$

For $\bar{J}_L = 10$ and $\tilde{h} = 0.0, \tilde{h} = 0.01, \tilde{h} = 0.05$, Table 4 provides the first four eigenfrequencies for $2\beta = 0.01$.

Table 4

5.5 Gradient elasticity with micro-stiffness and without damping

In this case one has $\tilde{g} = 0, \tilde{h} = g = 0$ and thus the governing equation (57) and boundary conditions (64) and (65) become

$$-\tilde{g}^2 \bar{q}''''(x) + \bar{q}''(x) + a^2 \bar{q}(x) = 0 \quad (78)$$

$$\bar{q}''(0) = \bar{q}''(1) = \bar{q}(0) = 0 \quad (79)$$

$$-\tilde{g}^2 \bar{q}''(1) + \bar{q}'(1) = \left(a^2 / \bar{J}_L \right) \bar{q}(1) \quad (80)$$

leading to the frequency equation

$$p \cos p(\tilde{g}^2 p^2 + 1) + \frac{p^2 \sin p \cosh q}{q \sinh q} (-\tilde{g}^2 q^2 + 1) - \frac{a^2}{\bar{J}_L} \sin p \left(1 + \frac{p^2}{q^2} \right) = 0 \quad (81)$$

Table 5 gives the first four eigenfrequencies for this case for $\bar{J}_L = 10$ and $\tilde{g} = 0.0, \tilde{g} = 0.01, \tilde{g} = 0.05$, while Fig. 10 depicts the first four natural frequencies versus \tilde{g} . One can observe from that figure that for increasing values of \tilde{g} , the frequencies increase, especially for higher modes. This is because micro-stiffness effects are here significant and are associated with the much less dense but stiffer arrangement of attractive particles.

Table 5

Figure 10

5.6 Gradient elasticity with micro-stiffness and damping

In this case one has $\tilde{g} = 0, \gamma \neq 0, \tilde{h} = 0$ and thus the governing equation (57) and boundary conditions (64) and (65) become

$$-\tilde{g}^2 \bar{q}''''(x) + (1 + ig) \bar{q}'''(x) + a^2 \bar{q}''(x) = 0 \quad (82)$$

$$\bar{q}''(0) = \bar{q}''(1) = \bar{q}'(0) = 0 \quad (83)$$

$$-\tilde{g}^2 \bar{q}''''(1) + (1 + ig) \bar{q}'''(1) = \left(a^2 / \bar{J}_L \right) \bar{q}''(1) \quad (84)$$

leading to the frequency equation

$$p \cos p(\tilde{g}^2 p^2 + 1 + ig) + \frac{p^2 \sin p \cosh q}{q \sinh q} (-\tilde{g}^2 q^2 + 1 + ig) - \frac{a^2}{\bar{J}_L} \sin p \left(1 + \frac{p^2}{q^2} \right) = 0 \quad (85)$$

For $\bar{J}_L = 10$ and $\tilde{g} = 0.0$, $\tilde{g} = 0.01$, $\tilde{g} = 0.05$, Table 6 gives the first four eigenfrequencies for $2\beta = 0.01$.

Table 6

5.7 Gradient elasticity with micro-inertia, micro-stiffness and without damping

In this case one has $\tilde{g} = 0$, $\gamma = 0$, $\tilde{h} = 0$ and thus the governing equation (57) and boundary conditions (64) and (65) become

$$-\tilde{g}^2 \bar{q}''''(x) + (1 - \tilde{h}^2 a^2) \bar{q}'''(x) + a^2 \bar{q}''(x) = 0 \quad (86)$$

$$\bar{q}''(0) = \bar{q}''(1) = \bar{q}'(0) = 0 \quad (87)$$

$$-\tilde{g}^2 \bar{q}''''(1) + (1 - \tilde{h}^2 a^2) \bar{q}'''(1) = \left(a^2 / \bar{J}_L \right) \bar{q}''(1) \quad (88)$$

leading to the frequency equation

$$p \cos p(\tilde{g}^2 p^2 + 1 - \tilde{h}^2 a^2) + \frac{p^2 \sin p \cosh q}{q \sinh q} (-\tilde{g}^2 q^2 + 1 - \tilde{h}^2 a^2) - \frac{a^2}{\bar{J}_L} \sin p \left(1 + \frac{p^2}{q^2} \right) = 0 \quad (89)$$

For $\bar{J}_L = 10$, Tables 7(a) and 7(b) provide the first four eigenfrequencies for $\tilde{h} = 0.01$ and $\tilde{g} = 0.01$ and $\tilde{g} = 0.05$ and for $\tilde{g} = 0.01$ and $\tilde{h} = 0.01$ and $\tilde{h} = 0.05$, respectively.

Table 7

5.8 Gradient elasticity with micro-inertia, micro-stiffness and damping

In this case one has $\tilde{g} = 0$, $\gamma \neq 0$, $\tilde{h} = 0$ and thus the governing equation (57) and boundary conditions (64) and (65) lead to the frequency equation

$$p \cos p(\tilde{g}^2 p^2 + 1 + ig - \tilde{h}^2 a^2) + \frac{p^2 \sin p \cosh q}{q \sinh q} (-\tilde{g}^2 q^2 + 1 + ig - \tilde{h}^2 a^2) - \frac{a^2}{\bar{J}_L} \sin p(1 + \frac{p^2}{q^2}) = 0 \quad (90)$$

Tables 8 provides the first four eigenfrequencies for $\bar{J}_L = 10$ and $2\beta=0.01$ for the following combinations of \tilde{g} and \tilde{h} : $\tilde{g} = \tilde{h} = 0.01$, $\tilde{g} = 0.01$ and $\tilde{h} = 0.05$ and $\tilde{g} = 0.05$ and $\tilde{h} = 0.01$.

Table 8

6. RESONANT FREQUENCY SHIFTING AND ELASTIC CONSTANTS

In Section 2, the resonant column test results for a fine-grained material (α -Al₂O₃ powder) as obtained by Richter [2] were outlined. In this Section an effort is made to explain those results by utilizing the theoretical results presented in Section 5. More specifically, the observed in [2] (see Figs 3 and 4 as examples) shifting of the theoretically computed natural frequencies to the left and to the right of the experimentally obtained ones for the cases of attractive ($pH=9.1$) and repulsive ($pH=4.0$) particles, respectively, will be explained on the basis of the results of the gradient viscoelasticity theory. It will be shown that this 3-10% frequency shifting is the result of the inadequacy of the classical theory of viscoelasticity to take into account microstructural effects, which appear in granular materials, like the α -Al₂O₃ powders [4]. In other words, it will be shown here that there are differences 3-10% between the natural frequency values of the gradient and classical theories of viscoelasticity, thereby indicating that the results of gradient viscoelasticity are almost the same with those of the experiments.

Consider, as a simple, yet representative example, the cases 5.4 and 5.6 corresponding to the results of Tables 4 and 6 for which one has $\tilde{g} = 0$, $\tilde{h} = 0.05$, $2\beta = 0.01$ and $\tilde{g} = 0.05$, $\tilde{h} = 0$, $2\beta = 0.01$, respectively. The results of these two cases will be compared against those of case 5.2 (Table 2) for which $\tilde{g} = \tilde{h} = 0$, $2\beta = 0.01$. One should notice that only hysteretic damping is considered here as the only realistic one for granular materials, as explained in Section 2. Since for the comparison in [2] (see Figs. 3 and 4) the resonance frequency of the first mode of the tests and that of the classical theory of viscoelasticity coincide and the observed shifting refers to the higher modes, the results of Tables 4 and 6 are modified in order to agree with this fact on the assumption that the test results are almost the same with those of the gradient viscoelasticity theory.

Table 9 shows the first three natural frequencies (rounded to the first three decimal digits) for cases 5.2, 5.4 and 5.6 before (a) and after (b) modification. One can observe from the second row of Table 9.b that $4.213 < 4.306$ and $6.802 < 7.228$ with corresponding shifts 2.21% and 6.26%, respectively. This is the case of repulsive particles ($pH=4.0$) with α_i of tests \approx gradient theory $<$ α_i of classical theory. For this case $\tilde{g} = 0$ and $\tilde{h} = 0.05$ indicating that there are only micro-inertia effects.

Table 9

Indeed for repulsive particles one has a density increase and hence inertia increase. One can also observe from the third row of Table 9.b that $4.379 > 4.306$ and $7.601 > 7.228$ with corresponding shifts of 1.67% and 4.91%, respectively. This is the case of attractive particles ($pH=9.1$) with α_i of tests \approx gradient theory $>$ α_i of classical theory. For this case $\tilde{g} = 0.05$ and $\tilde{h} = 0$ indicating that there are only micro-stiffness effects. Indeed for attractive particles one

has a density decrease and hence inertia decrease (zero in this case). One can reach the same conclusions for the more general case 5.8 with $\tilde{g} = 0.01$, $\tilde{h} = 0.05$ and $\tilde{g} = 0.05$, $\tilde{h} = 0.01$ and thus explain again the natural frequency shifting phenomenon.

The above argument that the resonant frequency shifting can be attributed to microstructural effects was essentially a qualitative one. In the following, an attempt will be made to use the experimentally obtained results and on the basis of the gradient viscoelastic theory developed in Section 4, determine the elastic constants G , g and h , thereby indirectly providing a quantitative proof of the appearance of this frequency shifting as a result of microstructural effects and simultaneously suggest a method for the experimental determination of these constants. To this end one first has to determine the mass densities ρ_a and ρ_r of the attractive and repulsive particles, respectively. From Richter [2] one has that the void ratios e_a and e_r for attractive and repulsive particles are 1.4 and 0.54, respectively. Thus, since the particle density $\rho_p = 3900 \text{ kg/m}^3$ [2] and the mass density $\rho_{a,r} = \rho_p / (1 + e_{a,r})$, one has $\rho_a = 1625 \text{ kg/m}^3$ and $\rho_r = 2532 \text{ kg/m}^3$. The hysteretic damping for the case of confining pressure $p' = 20 \text{ kPa}$ is from Richter [2] $2\beta = 0.023$ and $2\beta = 0.026$ for attractive and repulsive particles, respectively.

Using the values of J_o , J_L and I_p of Section 2 and the above computed values of mass densities ρ_a and ρ_r , one can obtain from (56)_{6,7}

$$\begin{aligned} \bar{J}_{oa} &= 0.02772337477, & \bar{J}_{La} &= 1.86063057823 \\ \bar{J}_{or} &= 0.043241490, & \bar{J}_{Lr} &= 2.90211561888 \end{aligned} \quad (91)$$

for attractive and repulsive particles, respectively.

Consider first the case of attractive particles. According to the method used in practice for the experimental determination of G , one measures the first resonant frequency f_1 and computes G from the relation $G = \rho c^2$, where $\rho = 1625 \text{ Kg/m}^3$ and $c = \omega_1 L / \alpha_1 = 2\pi f_1 L / \alpha_1$ (Eq. (56)₄) with α_1 being the first root of the frequency equation for the case of classical elasticity without damping reading

$$\left(a^2 - \bar{J}_o \bar{J}_L \right) \tan a = a (\bar{J}_o + \bar{J}_L) \quad (92)$$

and obtained from Eq. (61) under boundary conditions (58)-(60) with $\tilde{g} = \tilde{h} = g = 0$. With $f_1 = 290 \text{ Hz}$ (Fig. 3) and J_{oa} , J_{La} values those of (91)_{1,2}, one has $\alpha_1 = 1.07337$ and hence $G_1 = 46.82 \text{ MPa}$. Using now the measured resonant frequencies of the next three modes $f_2 = 1015 \text{ Hz}$, $f_3 = 1815 \text{ Hz}$ and $f_4 = 2670 \text{ Hz}$ (Fig. 3) in conjunction with the roots $\alpha_2 = 3.62361$, $\alpha_3 = 6.56364$ and $\alpha_4 = 9.61874$ of Eq. (92), one obtains $G_2 = 50.33 \text{ MPa}$, $G_3 = 49.05 \text{ MPa}$ and $G_4 = 49.43 \text{ MPa}$, indicating that there is a mode effect increasing G by up to 7.5%.

Since the measured natural frequencies include the effect of damping, the correct determination of G should be done on the basis of α_i ($i=1-4$) not obtained from (92) but from the frequency equation for the case of classical elasticity with damping, *i.e.*, from

$$\left(\frac{a^2}{1+i\gamma} - \bar{J}_o \bar{J}_L \right) \tan \left(\frac{a}{\sqrt{1+i\gamma}} \right) = \frac{a}{\sqrt{1+i\gamma}} (\bar{J}_o + \bar{J}_L) \quad (93)$$

obtained from Eq. (61) under boundary conditions (58)-(60) with $\tilde{g} = \tilde{h} = 0$. With \bar{J}_{oa} and \bar{J}_{La} values those of (91)_{1,2} and $g = 0.023$, one can obtain from (93) the real part of α 's as $\alpha_1 = 1.07344$, $\alpha_2 = 3.62385$, $\alpha_3 = 6.56407$ and $\alpha_4 = 9.61937$ resulting in $G_1 = 46.82 \text{ MPa}$, $G_2 = 50.327 \text{ MPa}$, $G_3 = 49.04 \text{ MPa}$ and $G_4 = 49.42 \text{ MPa}$ and indicating that there is practically no effect of damping when determining the shear modulus G for any mode.

Consider now the case of gradient elasticity with micro-stiffness and damping with frequency equation the one obtained from Eq. (61) under boundary conditions (58)-(60) with

$\tilde{h} = 0$, $g = 0.023$ and \bar{J}_{oa} and \bar{J}_{La} those of (91)_{1,2}. This is a characteristic case for attractive particles and on the basis of the measured resonant frequencies $f_1=290$ Hz, $f_2=1015$ Hz, $f_3=1815$ Hz and $f_4=2670$ Hz (Fig. 3) one can obtain the following values of G and g :

$$G = 46.50 \text{ MPa}, \quad g = 0.0085 \text{ m} \quad (94)$$

$$G = 46.52 \text{ MPa}, \quad g = 0.0035 \text{ m} \quad (95)$$

$$G = 46.52 \text{ MPa}, \quad g = 0.0026 \text{ m} \quad (96)$$

by solving for $c=2\pi fL/\alpha$ and $\tilde{g} = g/L$ the systems of two nonlinear algebraic equations resulting from the above-mentioned frequency equation in turn for f_1 and f_2 , f_1 and f_3 and f_1 and f_4 , respectively. It is observed that the effects of the micro-stiffness consist of practically providing a single value of G , very close to the one obtained in the classical way, and independently of the modes considered and of determining values of g decreasing with increasing modes as a result of grain rearrangement at higher modes.

Consider now the case of repulsive particles for which the measured resonant frequencies are $f_1=350$ Hz, $f_2=1080$ Hz, $f_3=1900$ Hz and $f_4=2800$ Hz (Fig. 4), the mass density $\rho=2532$ Kg/m³, the hysteretic damping $\gamma=0.026$ and the values of \bar{J}_{or} and \bar{J}_{Lr} are given by (91)_{3,4}. Working exactly as in the previous case of attractive particles, one can obtain the following pairs of α and G for the first four modes associated with classical elasticity without damping

$$\begin{aligned} a_1 &= 1.21113, & G_1 &= 83.48 \text{ MPa} \\ a_2 &= 3.8046, & G_2 &= 80.54 \text{ MPa} \\ a_3 &= 6.69848, & G_3 &= 80.42 \text{ MPa} \\ a_4 &= 9.71939, & G_4 &= 82.96 \text{ MPa} \end{aligned} \quad (97)$$

and with damping

$$\begin{aligned} a_1 &= 1.21123, & G_1 &= 83.46 \text{ MPa} \\ a_2 &= 3.80492, & G_2 &= 80.53 \text{ MPa} \\ a_3 &= 6.69904, & G_3 &= 80.41 \text{ MPa} \\ a_4 &= 9.72021, & G_4 &= 82.94 \text{ MPa} \end{aligned} \quad (98)$$

One can observe from (97) and (98) that there is no damping effect but there is a mode effect (up to 6.23%) as far as the determination of G in the classical way is concerned, exactly as in the case of attractive particles.

Considering now for repulsive particles the case of gradient elasticity with micro-inertia and damping with frequency equation the one obtained from Eq. (61) under boundary conditions (58)-(60) with $\tilde{g} = 0$, $g = 0.026$ and \bar{J}_{or} and \bar{J}_{Lr} those of (91)_{3,4}. Working as in the case of attractive particles, one can obtain the following values of G and h :

$$G = 83.70 \text{ MPa}, \quad h = 0.0053 \text{ m} \quad (99)$$

$$G = 83.57 \text{ MPa}, \quad h = 0.0030 \text{ m} \quad (100)$$

$$G = 83.47 \text{ MPa}, \quad h = 0.0008 \text{ m} \quad (101)$$

for the combinations of f_1 and f_2 , f_1 and f_3 , f_1 and f_4 , respectively. It is observed that the effects of micro-inertia consist of practically providing a single value of G , very closed to the one obtained in the classical way, and independently of the modes considered and of determining values of h decreasing with increasing modes as a result of grain rearrangements at higher modes, as in the case of attractive particles.

In conclusion, through the experimental results of Figs 3 and 4 [2], it has been possible with the use of the gradient elastic theory to (i) determine the microstructural constants g and h for attractive and repulsive particles as $g=0.0095-0.0026$ m and $h=0.0053-0.0008$ m leading with the aid of relations (46) and (47) to lattice spacing $l=0.012-0.0033$ m and $l= 0.0092-0.0014$ m for attractive and repulsive particles, respectively; (ii) determine the shear modulus $G=46.52$ MPa and $G=83.58$ MPa for attractive and repulsive particles, respectively, very close to its values obtained in the classical way.

Finally, consider the most general case of having to determine all three elastic constants G , g and h of a fine-grained $\alpha\text{-Al}_2\text{O}_3$ material from resonant column test results, like those of Figs 3 and 4 for attractive and repulsive particles, respectively. In order to accomplish this, one has to form the frequency equation coming from the general solution (61) under the boundary conditions (58)-(60), consider $a_i=2\pi f_i L/c$, where f_i are the three of the measured resonant frequencies and solve the resulting system of three nonlinear algebraic equations for c , \tilde{g} and \tilde{h} from which $G=\rho c^2$, $g = \tilde{g}L$ and $h = \tilde{h}L$ can be evaluated.

Thus, for the case of attractive particles and use of the first three measured resonant frequencies of Fig. 3 one can determine

$$G = 46.5 \text{ MPa}, \quad g = 0.004 \text{ m} \quad h = 0.0005 \text{ m} \quad (102)$$

Use of the first, third and fourth resonant frequencies of Fig.3 leads to

$$G = 46.5 \text{ MPa}, \quad g = 0.003 \text{ m}, \quad h = 0.0005 \text{ m} \quad (103)$$

For the case of repulsive particles one can determine

$$G = 83.67 \text{ MPa}, \quad g = 0.0008 \text{ m}, \quad h = 0.004 \text{ m} \quad (104)$$

by using the first three measured resonant frequencies of Fig. 4 and

$$G = 83.5 \text{ MPa}, \quad g = 0.0001 \text{ m}, \quad h = 0.0015 \text{ m} \quad (105)$$

by using the first, third and fourth measured resonant frequencies of Fig. 4. One can observe from both cases of attractive and repulsive particles that (i) the shear modulus G is independent of the two used frequency combinations and close to the classically obtained value and (ii) there is not a single solution of the nonlinear system of equations for g and h for both measured frequency combinations because at higher frequencies the microstructure of the material changes due to grain rearrangements.

7. CONCLUSIONS

On the basis of the preceding developments, one can draw the following conclusions:

1) A gradient viscoelastic theory with micro-stiffness and micro-inertia has been developed for the description of torsional vibrations of a viscoelastic bar with microstructure.

2) The governing equations of motion and the classical and non-classical boundary conditions of the problem have been derived both by a lattice theory and a continuum gradient viscoelastic theory.

3) Free torsional vibrations of a viscoelastic microstructured bar have been studied and the frequency equations for eight cases involving various combinations of the parameters g (micro-stiffness), h (micro-inertia) and 2β (hysteretic damping) have been presented and solved for the first five eigenfrequencies.

4) Use of the above theory and results enables one to explain the observed during tests natural frequency shift to the left or to the right of the classical frequency values for the cases of repulsive and attractive particles, respectively, of fine-grained equivalent soil models.

5) A method for determining the material parameters g , h and the shear modulus G of a fine-grained material by measuring the first four resonance frequencies of torsional vibrations of a column made of this material has been proposed.

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