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The time of graph bootstrap percolation

Karen Gunderson, Sebastian Koch, and Michał Przykucki*

K. Gunderson Heilbronn Institute for Mathematical Research School of Mathematics University of Bristol Bristol BS8 1TW United Kingdom e-mail: karen.gunderson@bristol.ac.uk

S. Koch Department of Pure Mathematics and Mathematical Statistics University of Cambridge Cambridge CB3 0WB United Kingdom e-mail: sk629@cam.ac.uk

> M. Przykucki St Anne's College University of Oxford Oxford OX2 6HS United Kingdom e-mail: przykucki@maths.ox.ac.uk

Abstract: Graph bootstrap percolation, introduced by Bollobás in 1968, is a cellular automaton defined as follows. Given a "small" graph H and a "large" graph $G = G_0 \subseteq K_n$, in consecutive steps we obtain G_{t+1} from G_t by adding to it all new edges e such that $G_t \cup e$ contains a new copy of H. We say that G percolates if for some $t \ge 0$, we have $G_t = K_n$.

For $H = K_r$, the question about the smallest size of percolating graphs was independently answered by Alon, Frankl and Kalai in the 1980's. Recently, Balogh, Bollobás and Morris considered graph bootstrap percolation for G = G(n, p) and studied the critical probability $p_c(n, K_r)$ for the event that the graph percolates with high probability. In this paper, using the same setup, we determine up to a logarithmic factor the critical probability for percolation by time t for all $1 \le t \le C \log \log n$.

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1. Introduction

Cellular automata, introduced by von Neumann [17] after a suggestion of Ulam [19], are dynamical systems acting on graphs using local and homogeneous update rules. The *H*-bootstrap percolation process is one example of such an automaton and can be described as follows. Given a fixed graph *H* and a graph $G \subset K_n$ set $G_0 = G$ and then, for each t = 0, 1, 2, ..., let

$$G_{t+1} = G_t \cup \{ e \in E(K_n) : \exists H \text{ with } e \in H \subset G_t \cup e \}.$$
(1)

Let $\langle G \rangle_H = \bigcup_{t=0}^{\infty} G_t$ denote the *closure* of *G* under *H*-bootstrap percolation. We say that *G* percolates (or *H*-percolates) in the *H*-bootstrap process if $\langle G \rangle_H = K_n$. (See Figure 1).

The notion of *H*-percolation, introduced by Bollobás in 1968 [6] under the name of *weak satura*tion, has been extensively studied in the case where *H* is a complete graph. Initially, the extremal properties of the *H*-bootstrap process attracted the most attention. Alon [1], Frankl [13] and Kalai [16] independently confirmed a conjecture of Bollobás and proved that the smallest K_r -percolating graphs on *n* vertices have size $\binom{n}{2} - \binom{n-r+2}{2}$.

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FIG 1. An example of the K₄-bootstrap percolation process. Dashed edges are added to the graph on the next time step.

Recently, Balogh, Bollobás and Morris [4] observed a strong connection between weak saturation and r-neighbour bootstrap percolation, a dynamical process suggested in 1979 by Chalupa, Leath and Reich [11]. For an integer $r \ge 2$, the r-neighbour bootstrap process on a graph G = (V, E) with an 'initial set' of vertices $A \subset V$ is defined by setting $A_0 = A$ and for $t = 0, 1, 2, \ldots$, defining

$$A_{t+1} = A_t \cup \{ v \in V : |N(v) \cap A_t| \ge r \},$$
(2)

where N(v) is the set of neighbours of v in G. The set $\langle A \rangle = \bigcup_{t=0}^{\infty} A_t$ is the closure of A and we say that A percolates if $\langle A \rangle = V$. Often, the vertices in the set A_t are called 'infected' and the remaining vertices are 'healthy'. The usual question asked in the context of r-neighbour bootstrap percolation is the following: if the vertices of G are initially infected independently at random with probability p, for what values of p is percolation likely to occur? The probability of percolation is clearly non-decreasing in p hence it is natural to define the *critical probability* $p_c(G, r)$ as

$$p_c(G, r) = \inf\{p : \mathbb{P}_p(\langle A \rangle = V(G)) \ge 1/2\}.$$
(3)

The study of critical probabilities has brought numerous and often very sharp results for various graphs G and the values of the infection threshold. For example, van Enter [12] and Schonmann [18] studied *r*-neighbour bootstrap percolation on \mathbb{Z}^d , Holroyd [14] and Balogh, Bollobás, Duminil-Copin and Morris [3] analysed finite grids, while Balogh and Pittel [5], Janson, Łuczak, Turova and Vallier [15] and Bollobás, Gunderson, Holmgren, Janson and Przykucki [8] worked with random graphs.

Motivated by this approach, Balogh, Bollobás and Morris defined the critical probability for Hbootstrap percolation on K_n to be

$$p_c(n, H) = \inf\{p : \mathbb{P}_p(\langle G_{n,p} \rangle_H = K_n) \ge 1/2\},$$
(4)

where $G_{n,p}$ is the Erdős-Rényi random graph, obtained by choosing every edge of K_n independently at random with probability p. In [4], they showed that taking $\lambda(r) = \left(\binom{r}{2} - 2\right)/(r-2)$, for some c > 0and $n \in \mathbb{N}$ large enough,

$$\frac{n^{-1/\lambda(r)}}{c\log n} \le p_c(n, H) \le n^{-1/\lambda(r)}\log n.$$
(5)

In this paper we focus on a different question related to K_r -bootstrap percolation. Namely, for what values of p is percolation likely to occur by time t? Defining K_r -bootstrap percolation as in (1), let $T = T(G_0, n) = \min\{t : G_t = K_n \text{ in the } K_r\text{-boots. process}\}$. Let the critical probability for percolation by time t be defined as

$$p_c(n, r, t) = \inf\{p : \mathbb{P}_p(T \le t) \ge 1/2\}.$$
 (6)

For notational convenience, set

$$\tau = \tau(r) = {r \choose 2} - 1, \quad e_t = \tau^t, \text{ and } \quad v_t = (r-2)\frac{(\tau^t - 1)}{(\tau - 1)} + 2.$$
(7)

The following theorem is the main result of this paper.

Theorem 1.1. Let $r \ge 4$ and $t = t(n) \le \frac{\log \log n}{3 \log \tau}$. Let $(p_n)_{n=1}^{\infty}$ be a sequence of probabilities, let $\omega(n) \to \infty$ and let T = T(n). Under the K_r -bootstrap process,

- (i) if, for all $n, p(n) \ge n^{-(v_t-2)/e_t} \log n$, then $\mathbb{P}_{p_n}(T \le t) \to 1$ as $n \to \infty$ and (ii) if, for all $n, p(n) \le n^{-(v_t-2)/e_t}/\omega(n)$, then $\mathbb{P}_{p_n}(T \le t) \to 0$ as $n \to \infty$.

Thus, Theorem 1.1 shows that for all $r \ge 4$ and $1 \le t \le \frac{\log \log n}{3 \log \tau}$, and $\omega(n) \to \infty$, then for $n \le 1$ sufficiently large,

$$n^{-(v_t-2)/e_t}/\omega(n) \le p_c(n,r,t) \le n^{-(v_t-2)/e_t}\log n.$$
(8)

Similar questions related to the time of r-neighbour bootstrap percolation on grids have recently been studied by Bollobás, Holmgren, Smith and Uzzell [9], Bollobás, Smith and Uzzell [10] and by Bollobás, Balister and Smith [2].

The proofs of both statements of Theorem 1.1 rely on the properties of a family of graphs, denoted $\{F_t: t \geq 1\}$, that are described in detail in Section 3. For each t, there is a pair of vertices in $V(F_t)$ so that if F_t occurs as a subgraph of G_0 , then that pair is guaranteed to be added to the graph by time t. The graph F_t is thought of as 'anchored on' that special pair of vertices.

To prove Statement (i) of Theorem 1.1, Janson's inequality is used to bound from below the probability that a particular pair $\{x, y\}$ is contained as the anchor vertices in some copy of F_t . To establish a bound in this way, estimates are needed on the probability that two overlapping copies of F_t occur in G_0 . This amounts to determining the minimum possible ratio of edges to vertices in some non-trivially overlapping pair. It turns out that the minimum ratio is not obtained for one of the extreme cases, i.e., neither for two copies of F_t that share only one vertex, nor for two copies that share all but one vertex. Even though we do not prove it directly, our proof suggests that as $t \to \infty$, the two overlapping copies of F_t that minimise the edge-to-vertex ratio share an approximately 4/((r+1)(r-2)) fraction of the vertex set. Bounding this ratio from below for all possible configurations of two such copies is the main challenge in the proof of the upper bound on $p_c(n, r, t)$, and is dealt with in detail in Section 3.1.

To prove Statement (ii) of Theorem 1.1 we employ two extremal results about graphs that add eto the graph in at most t time steps: one of them to bound the number of their vertices from above, and one (a corollary from a highly nontrivial result in [4]) to bound their edge density from below. Then, for p as in Statement (ii) of Theorem 1.1, we show that with high probability no such graph can be found in $G_{n,p}$. This completes the proof of our main result.

The remaining sections of the paper are organised as follows. In Section 2 we briefly discuss the K_3 bootstrap percolation process which behaves differently from K_r -bootstrap processes when $r \geq 4$. In Section 3, we introduce the graphs, F_t , that are the main focus of the proofs to come and prove some key properties. In Section 3.1, which is the crucial part of our argument, we prove some properties of graphs consisting of two overlapping copies of F_t . In Sections 4 and 5 we prove Statements (i) and (ii) of Theorem 1.1 respectively. Finally, in Section 6 some open problems are stated.

2. K_3 -bootstrap percolation

In this section we discuss the case r = 3. Observe that a graph G percolates in K_3 -bootstrap percolation if and only if G is connected. Also, at every time step each non-edge between vertices at distance 2 is added to the graph. Therefore, if G is a connected graph with diameter d, then the diameter of the graph obtained from G after one step of the K_3 -bootstrap process is $\lfloor d/2 \rfloor$. Hence, G percolates in $\lceil \log_2 d \rceil$ time steps.

In [7] Bollobás proved the following theorem.

Theorem 2.1. Let $G_{n,p}$ be the Erdős-Rényi random graph.

1. Suppose $p^2n - 2\log n \to \infty$ and $n^2(1-p) \to \infty$. Then $G_{n,p}$ has diameter 2 whp.

2. Suppose the functions $d = d(n) \ge 3$ and $0 satisfy <math>(\log n)/d - 3\log \log n \to \infty$, $p^d n^{d-1} - 2\log n \to \infty$ and $p^{d-1}n^{d-2} - 2\log n \to -\infty$. Then $G_{n,p}$ has diameter d whp.

Let $\omega(n) = o(\log n)$ tend to infinity arbitrarily slowly. Clearly, if $p \ge 1 - 1/(n^2 \omega(n))$ then whp. $G_{n,p} = K_n$ which has diameter 1. Simplifying a bit, Theorem 2.1 implies that if

$$\sqrt{\frac{2\log n + \omega(n)}{n}} \le p \le 1 - \frac{1}{\omega(n)}$$

then $G_{n,p}$ has diameter 2, and that for $3 \le d \le \log n/4 \log \log n$, if

$$p(n) \in \left((2\log n + \omega(n))^{\frac{1}{d}} n^{-\frac{d-1}{d}}, (2\log n - \omega(n))^{\frac{1}{d-1}} n^{-\frac{d-2}{d-1}} \right)$$

then the random graph $G_{n,p(n)}$ has diameter d whp. This answers our question about the time of K_3 -bootstrap percolation.

3. Adding an edge to the graph using sparse subgraphs

Throughout the following sections, fix $r \ge 4$. For simplicity, r is often omitted from the notation. We define a family $\{F_t : t \ge 1\}$ of graphs that add a given pair as an edge to the graph exactly at time t in the K_r -bootstrap process. We prove that these are the "sparsest" minimal such graphs (i.e., they minimise the number of edges to the number of vertices ratio). Finally, in Section 3.1 we prove a lower bound on the edge-density of two non-disjoint copies of the graph F_t . This bound is the key element of arguments to come.

The graph F_t is defined recursively and the fixed edge that we add to the graph at time t using F_t will always be denoted by $e_0 = \{1, 2\}$.

For t = 1, set $F_1 = K_r - e_0$, an *r*-clique missing one edge.

For each $t \ge 1$, given F_t , for each $e \in E(F_t)$, let V(e) be a new set of r-2 vertices and let K(e) be a copy of $K_r - e$, an *r*-clique missing one edge, on $V(e) \cup e$. Then, F_{t+1} is defined to be the graph with vertex set

$$V(F_{t+1}) = \left(\bigcup_{e \in E(F_t)} V(e)\right) \cup V(F_t)$$

and edge set

$$E(F_{t+1}) = \bigcup_{e \in E(F_t)} E(K(e))$$

(see Figure 2). Recall that we define $\tau = \binom{r}{2} - 1$. By induction on t, for every $t \ge 1$,

$$e_t = e(F_t) = |E(F_t)| = \tau^t \text{ and}$$
(9)

$$v_t = v(F_t) = |V(F_t)| = |V(F_{t-1})| + e_{t-1}(r-2) = 2 + (r-2)\frac{\tau^t - 1}{\tau - 1}.$$
(10)

Lemma 3.1. In the K_r -bootstrap process started from F_t the edge e_0 is added to the graph in exactly t steps.

Proof. We prove this fact by induction of t. The statement is trivial for t = 1 as $F_1 = K_r - e_0$. Assume that the Lemma holds for $t = k \ge 1$. Note that after one step of the process started from F_{k+1} we obtain a copy of F_k in our graph since F_{k+1} is obtained from F_k by placing a copy of K_r minus an edge on every edge of F_k . Thus e_0 is added to the graph after at most k + 1 steps of the process started from F_{k+1} .



FIG 2. Construction of the graph F_t . Note that every edge in F_t is adjacent to at least one vertex in $V(F_t) \setminus V(F_{t-1})$.

The construction of F_{k+1} can be also seen as placing a copy of F_k on each of the τ edges of $F_1 = K_r - e_0$. By induction we know that these copies of F_k on their own add the respective edges of F_1 in k time steps. This process could possibly accelerate if some interaction between two different copies of F_k occurred early in the process, say, before the F_k 's add their respective anchor edges. Let therefore $F^1, F^2, \ldots, F^{\tau}$ be the different copies of F_k in F_{k+1} . By construction of F_{k+1} we have that for all i < j the F^i and F^j share at most one vertex.

Let F' and F'' be two different copies of F_k in F_{k+1} and let w be the vertex shared by F' and F''if it exists. Let $u \in F', v \in F''$ with $u, v \neq w$ be such that the edge $e = \{u, v\}$ is added first among the edges not induced by any F^i for $1 \leq i \leq \tau$. Since $r \geq 4$, the copy of $K_r - e$ that adds e, without loss of generality, contains a vertex $z \notin F'', z \neq u$. Then the edge $e' = \{z, v\}$ is either not induced by any F^i in which case, by the choice of e, it cannot be added before e, or e' is one of the edges of F_1 that $F^1, F^2, \ldots, F^{\tau}$ are anchored on. However, if that is the case, by the choice of e we know that e'can only be added at time k (because e' is added before any edge not induced by any F^i does) and hence e can only be added at time k + 1. Thus e_0 cannot be added by F_{k+1} before time k + 1.

Recall that we denote

$$\lambda = \frac{\binom{r}{2} - 2}{r - 2} = \frac{r + 1}{2} - \frac{1}{r - 2}$$
(11)

and that τ can be written

$$\tau = \binom{r}{2} - 1 = \frac{(r+1)(r-2)}{2}.$$
(12)

Let also $c_t = 1/(\tau^t - 1)$. Note that, using (10),

$$\frac{e_t}{v_t - 2} = \frac{\tau^t}{(r - 2)\frac{\tau^t - 1}{(r - 2)\lambda}} = \lambda \left(1 + c_t\right) = \lambda + \frac{\tau - 1}{r - 2}\frac{1}{\tau^t - 1} = \lambda + \frac{1}{v_t - 2}.$$
(13)

Equation (13) is used throughout this section to show that F_t is the sparsest minimal graph that adds e_0 to the graph in t time steps of the K_r -bootstrap process.

Let us recall the following Witness-Set Algorithm introduced in [4]. Given a graph G, we assign a graph $F = F(e) \subset G$ to each edge $e \in \langle G \rangle_{K_r}$ as follows:

- 1. If $e \in G$ then set $F(e) = \{e\}$.
- 2. Choose an order in which to add the edges of $\langle G \rangle_{K_r}$, and at each step identify which *r*-clique was completed (if more than one is completed then choose one).
- 3. Add the edges one by one. If e is added by the r-clique K, then set

$$F(e) := \bigcup_{e \neq e' \in K} F(e').$$

A graph F is an *r*-witness set if there exists a graph G, an edge e, and a realization of the Witness-Set Algorithm (i.e., a choice as in Step 2) such that F = F(e). The following highly nontrivial extremal result occurs as Lemma 9 in [4], which is stated here without repeating the proof.

Lemma 3.2. Let F be a graph and $r \ge 4$, and suppose that F is an r-witness set. Then

$$|E(F)| \ge \lambda(|V(F)| - 2) + 1.$$

We say that a graph G is a minimal graph adding e if $e \in \langle G \rangle_{K_r}$ but for all proper subgraphs $G' \subsetneq G$ of G we have $e \notin \langle G' \rangle_{K_r}$. It's an immediate observation that every minimal graph adding e to G is an r-witness set. Hence we have the following corollary.

Corollary 3.3. Let $r \ge 4$ and let F be a minimal graph adding e to the graph for some $e \in \langle F \rangle_{K_r}$. Then

$$|E(F)| \ge \lambda(|V(F)| - 2) + 1.$$
(14)

We now show that F_t maximises the number of vertices among all minimal graphs that add e_0 to the graph in exactly t time steps of the K_r -bootstrap process.

Lemma 3.4. Let $r \ge 4, t \ge 1$ and let F be a minimal graph adding e_0 at to the graph time t in the K_r -bootstrap process. Then $|V(F)| \le v_t = (r-2)\frac{\tau^t-1}{\tau-1} + 2$ and $|E(H)| \le e_t = \tau^t$.

Proof. We prove the lemma by induction on t. For t = 1 the lemma is trivial as $K_r - e_0$ is the only minimal graph adding e_0 on the first time step. Hence assume that the lemma holds for some $t \ge 1$ and consider a minimal graph F such that e_0 is added at time t + 1 in the K_r -bootstrap process started from F.

After one step of the process we obtain a graph F' containing some minimal subgraph F'' that adds e_0 in t additional time steps. By induction we have $|V(F'')| \leq (r-2)\frac{\tau^t-1}{\tau-1}+2$ and $|E(F'')| \leq \tau^t$. Now, since F was a minimal graph adding e_0 in time t+1, to maximise the number of vertices and edges in F we should in the first step of the process add every edge e of F'' using a copy of $K_r - e$ disjoint from the copies adding other edges in F''. This shows that $|E(F)| \leq \tau |E(F'')|$ and $|V(F)| \leq |V(F'')| + (r-2)|E(F'')|$. This completes the induction and the lemma follows. \Box

The proof of Lemma 3.4 immediately shows a further extremal result.

Corollary 3.5. For any $t \leq 1$, up to isomorphism, F_t is the only minimal graph on v_t vertices adding e_0 to the graph in exactly t time steps.

As usual, for any graph G and $A, B \subset V(G)$, let $E(A, B) = \{\{a, b\} \in E(G) : a \in A, b \in B\}$ and e(A, B) = |E(A, B)|.

In the proofs to come, results on edge-densities of subsets of the graphs $\{F_t : t \ge 1\}$ are proved by induction on t. To make the notation clearer, let us use $E_t(A, B)$ to denote the edges between A and B in the graph F_t and $e_t(A, B) = |E_t(A, B)|$. As usual, $\delta(G) = \min\{\deg_G(v) : v \in V(G)\}$ is used for the minimum degree of G. We shall find the following simple estimate useful in our studies of F_t .

Lemma 3.6. For any $t \ge 2$ and any set $L \subseteq V(F_t)$

$$e_t(L, F_t) \ge \frac{r-1}{2} |L|.$$

Proof. Note that for $t \ge 2$, $\delta(F_t) = r - 1$. Thus

$$(r-1)|L| \le \sum_{v \in L} \deg_{F_t}(v) = 2e_t(L,L) + e_t(L,L^c) \le 2e_t(L,F_t).$$

3.1. Overlapping copies of F_t

To prove Statement (i) of Theorem 1.1 we shall show that if p is large enough then with high probability there is a copy of F_t anchored on every edge of $G_{n,p}$. Towards this aim, we shall show that a measure of the variance of the number of such copies of F_t anchored on a fixed edge e_0 is not too large compared to their expected number. In Section 4, this fact together with Janson's inequality is used to deduce the desired result. Hence, we need to prove that it is significantly "harder" (in terms of the ratio of the number of edges to the number of vertices) to find two different such copies of F_t that overlap in at least one vertex (other than $1, 2 \in e_0$) than it is to find two disjoint such copies.

In particular, as the main result in this subsection, it is shown that for any $L \subseteq V(F_t) \setminus \{1, 2\}$,

$$\frac{e_t(L, F_t)}{|L|} \ge \frac{e_t}{v_t - 2} \tag{15}$$

with equality only when $L = V(F_t) \setminus \{1, 2\}$. With this in mind, define ε_t to be such that

$$1 + \varepsilon_t = \left(\frac{v_t - 2}{e_t}\right) \min\left\{\frac{e_t(L, F_t)}{|L|} : L \subsetneq V(F_t) \setminus \{1, 2\}\right\}.$$
(16)

From the definition above, there is no guarantee that ε_t is non-negative. Using induction on t, we shall prove that this is the case by first giving a weak upper bound on ε_t in Lemma 3.7 and then using it to prove a relatively good lower bound on ε_t for all $t \ge 1$.

Lemma 3.7. For all $r \ge 4$ and $t \ge 1$ we have $\varepsilon_t \le \frac{1}{r+1}$.

Proof. First consider the case t = 1. For all $L \subseteq V(F_1) \setminus \{1, 2\}$ the vertices in $F_1 \setminus L$ are connected to each other and to both 1 and 2. Hence, for $1 \leq \ell \leq r-3$ and $|L| = \ell$, the vertices in $F_1 \setminus L$ induce $\binom{r-\ell}{2} - 1$ edges, which gives

$$\frac{e_1(L, F_1)}{|L|} = \frac{1}{\ell} \left(\binom{r}{2} - 1 - \binom{r-\ell}{2} + 1 \right) \\ = \frac{1}{2\ell} \left(r^2 - r - r^2 + 2r\ell - \ell^2 + r - \ell \right) \\ = \frac{2r - \ell - 1}{2} \\ \ge \frac{r+2}{2},$$

with equality for $\ell = r - 3$. Hence

$$\varepsilon_{1} = \left(\frac{r+2}{2}\right) \left(\frac{v_{1}-2}{e_{1}}\right) - 1$$

$$= \left(\frac{r+2}{2}\right) \left(\frac{r-2}{\binom{r}{2}-1}\right) - 1$$

$$= (r-2) \left(\frac{r-2}{r^{2}-r-2}\right) - 1$$

$$= \frac{r+2}{r+1} - 1 = \frac{1}{r+1}$$
(17)

which proves the lemma for t = 1.

Now assume that $t \ge 2$. Then in F_t there is a vertex v connected to 1 and not to 2. Let $L = V(F_t) \setminus \{1, 2, v\}$. This implies $|L| = v_t - 3$ and $e_t(L, F_t) = e_t - 1$. Thus, for $t \ge 2$ and $r \ge 4$,

$$\varepsilon_t \le \left(\frac{v_t - 2}{e_t}\right) \left(\frac{e_t - 1}{v_t - 3}\right) - 1$$

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$$= \frac{e_t v_t - 2e_t - v_t + 2 - e_t v_t + 3e_t}{e_t (v_t - 3)}$$
$$= \frac{e_t - v_t + 2}{e_t (v_t - 3)}$$
$$< \frac{1}{v_t - 3} \le \frac{1}{v_2 - 3}$$
$$= \frac{1}{(r - 2)\binom{r}{2} - 1}$$
$$\le \frac{1}{(r - 2)6 - 1} \le \frac{1}{r + 1}.$$

This completes the proof of Lemma 3.7.

The following lemma gives us another result in a similar direction and is used in this section to show that one need only consider certain choices for $L \subseteq V(F_t)$ in order to determine ε_t .

Lemma 3.8. For all $r \ge 4$ and $t \ge 2$ we have

$$\min\left\{\frac{e_t(L,F_t)}{|L|}: L \subsetneq V(F_t) \setminus \{1,2\}\right\} < \frac{r+1}{2}.$$

Proof. We prove the lemma by giving an example of a simple set $L \subsetneq V(F_t) \setminus \{1, 2\}$ that satisfies the inequality. Let $v \in V(F_{t-1}) \setminus V(F_{t-2})$. Then, let

$$L = \{v\} \cup \{u \in V(F_t) \setminus V(F_{t-1}) : \{v, u\} \in E(F_t)\}.$$

We have

$$|L| = 1 + \deg_{F_t}(v) = 1 + (r-1)(r-2)$$

and

$$e_t(L, F_t) = (r-1)\left(\binom{r}{2} - 1\right) = \frac{(r-1)(r+1)(r-2)}{2}.$$

Hence

$$\frac{e_t(L,F_t)}{|L|} = \frac{(r+1)}{2} \frac{(r-1)(r-2)}{1+(r-1)(r-2)} < \frac{(r+1)}{2}.$$

Note that, in general, Lemma 3.8 yields a worse upper bound on ε_t than that given by Lemma 3.7, but the form is useful in the proof of Lemma 3.11 to come.

The next Theorem is the main tool in the proof of Statement (i) of Theorem 1.1 to come. Here we give a lower bound on ε_t that holds for all $t \ge 1$.

Theorem 3.9. For all $r \ge 4$ and $t \ge 1$,

$$\varepsilon_t \ge \frac{1}{r+1} \left(\frac{2}{r^2 - 2}\right)^{t-1}.$$
(18)

Before proving Theorem 3.9, a few auxiliary lemmas are stated and proved below, along with an outline of the proof. These are then used to establish Theorem 3.9 by induction on t.

Recall that the graph F_{t+1} is constructed by placing an independent copy of $K_r - e$ on every edge e of F_t . Further recall that for each $e \in E(F_t)$ we write V(e) to denote this new set of r-2 vertices.

For the induction step from t to (t+1) we will fix a set $L_t \subseteq V(F_t) \setminus \{1,2\}$ and look for the smallest possible edge densities $\frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|}$ among sets of the form $L_t \cup M$ where $M \subseteq V(F_{t+1}) \setminus V(F_t)$ (see Figure 3).

In the following lemma we first deal with the case $L_t = \emptyset$, showing that no set contained entirely in $V(F_{t+1}) \setminus V(F_t)$ can minimise the edge density.



Fig 3. Sets L_t and M in F_{t+1} together with the edges counted in $e_{t+1}(L_t \cup M, F_{t+1})$.

Lemma 3.10. For all $r \ge 4$ and $t \ge 2$ we have

$$\min\left\{\frac{e_t(L,F_t)}{|L|}: L \subseteq V(F_t) \setminus V(F_{t-1})\right\} \ge \frac{r+1}{2}.$$

Proof. Let $L \subset V(F_t) \setminus V(F_{t-1})$. Hence for every $v \in L$ we have $\deg_{F_t}(v) = r - 1$ and at most r - 3 neighbours of v are also in L. Thus $e_t(L, F_t)$, the number of edges adjacent to L, satisfies

$$e_t(L, F_t) \ge |L|\left(\frac{r-3}{2}+2\right) = |L|\frac{r+1}{2}$$

and the lemma follows.

Thus, by Lemma 3.8, the minimum in equation (16) is not attained with $L \subseteq V(F_t) \setminus V(F_{t-1})$.

We now show that if for some $e \in E(F_t)$ certain conditions are fulfilled then moving all vertices from V(e) into M does not increase the density. The details of this are given in the following lemma.

Lemma 3.11. Let $t \ge 2$, $L_t \subseteq V(F_t)$ with $L_t \ne \emptyset$, and $M \subseteq V(F_{t+1}) \setminus V(F_t)$. For every $e = \{x, y\} \in E(F_t)$ with $\{x, y\} \cap L_t \ne \emptyset$ then

$$\frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|} \ge \frac{e_{t+1}(L_t \cup M \cup V(e), F_{t+1})}{|L_t \cup M \cup V(e)|}.$$

Proof. Set $m = |V(e) \setminus M|$ and note that the conclusion is trivially true for m = 0. Thus, assume $1 \le m \le r-2$.

We consider two different cases. Suppose first that both $x, y \in L_t$. Then $e_{t+1}(L_t \cup M \cup V(e), F_{t+1}) - e_{t+1}(L_t \cup M, F_{t+1}) = \binom{m}{2}$ and hence

$$\frac{e_{t+1}(L_t \cup M \cup V(e), F_{t+1})}{|L_t \cup M \cup V(e)|} = \frac{e_{t+1}(L_t \cup M, F_{t+1}) + \binom{m}{2}}{|L_t \cup M| + m}$$

$$= \frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|} \cdot \frac{|L_t \cup M|}{|L_t \cup M| + m} + \frac{\binom{m}{2}}{m} \frac{m}{|L_t \cup M| + m}$$

$$= \frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|} \cdot \frac{|L_t \cup M|}{|L_t \cup M| + m} + \frac{m-1}{2} \frac{m}{|L_t \cup M| + m}$$

$$\leq \frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|} \cdot \frac{|L_t \cup M|}{|L_t \cup M| + m} + \frac{r-3}{2} \frac{m}{|L_t \cup M| + m}$$

$$\leq \frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|}. \quad \text{(by Lemma 3.6)}$$

The case when $x \in L_t$ and $y \notin L_t$ or when $x \notin L_t$ and $y \in L_t$, is similar. In this case, $e_{t+1}(L_t \cup M \cup V(e), F_{t+1}) - e_{t+1}(L_t \cup M, F_{t+1}) = \binom{m+1}{2}$, and thus

$$\frac{e_{t+1}(L_t \cup M \cup V(e), F_{t+1})}{|L_t \cup M \cup V(e)|} = \frac{e_{t+1}(L_t \cup M, F_{t+1}) + \binom{m+1}{2}}{|L_t \cup M| + m} \\ = \frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|} \cdot \frac{|L_t \cup M|}{|L_t \cup M| + m} + \frac{m+1}{2} \frac{m}{|L_t \cup M| + m} \\ \le \frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|} \cdot \frac{|L_t \cup M|}{|L_t \cup M| + m} + \frac{r-1}{2} \frac{m}{|L_t \cup M| + m} \\ \le \frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|}. \quad \text{(by Lemma 3.6)}$$

This completes the proof.

On the other hand, when the edge e does not satisfy the conditions in Lemma 3.11, the following lemma holds.

Lemma 3.12. Let $t \ge 2$, $L_t \subseteq V(F_t)$ with $L_t \ne \emptyset$, $e \in E_t(L_t^c, L_t^c)$ and $M \subseteq V(F_{t+1}) \setminus V(F_t)$. If $|V(e) \cap M| > 0$ then

$$\frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|} > \min\left\{\frac{e_{t+1}(L, F_{t+1})}{|L|} : L \subsetneq V(F_{t+1}) \setminus \{1, 2\}\right\}.$$

Proof. Set $m = |V(e) \cap M| > 0$ and recall that we have $m \le r-2$. Since $L_t \ne \emptyset$, we have $L_t \cup M \setminus V(e) \ne \emptyset$. Thus,

$$\begin{aligned} \frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|} &= \frac{e_{t+1}(L_t \cup M \setminus V(e), F_{t+1}) + \binom{m}{2} + m(r-m)}{|L_t \cup M \setminus V(e)| + m} \\ &= \frac{e_{t+1}(L_t \cup M \setminus V(e), F_{t+1})}{|L_t \cup M \setminus V(e)|} \cdot \frac{|L_t \cup M \setminus V(e)|}{|L_t \cup M \setminus V(e)| + m} \\ &+ \frac{\binom{m}{2} + m(r-m)}{m} \frac{m}{|L_t \cup M \setminus V(e)| + m} \\ &= \frac{e_{t+1}(L_t \cup M \setminus V(e), F_{t+1})}{|L_t \cup M \setminus V(e)|} \cdot \frac{|L_t \cup M \setminus V(e)| + m}{|L_t \cup M \setminus V(e)| + m} \\ &+ \left(r - \frac{m+1}{2}\right) \frac{m}{|L_t \cup M \setminus V(e)| + m} \\ &\geq \frac{e_{t+1}(L_t \cup M \setminus V(e), F_{t+1})}{|L_t \cup M \setminus V(e)|} \cdot \frac{|L_t \cup M \setminus V(e)|}{|L_t \cup M \setminus V(e)| + m} \\ &+ \frac{r+1}{2} \frac{m}{|L_t \cup M \setminus V(e)| + m}. \end{aligned}$$

If $\frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|} \ge \frac{r+1}{2}$ then the claim holds by Lemma 3.8. Otherwise we have that

$$\frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|} > \frac{e_{t+1}(L_t \cup M \setminus V(e), F_{t+1})}{|L_t \cup M \setminus V(e)|}$$

This completes the proof.

Recall that by Lemma 3.10 any set $L_t \cup M$ that minimises the ratio $\frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|}$ has $L_t \neq \emptyset$. Furthermore, Lemma 3.12 tells us that any set $L_t \cup M$ with $L_t \neq \emptyset$ that minimises the ratio

 $\frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|} \text{ has } |V(e) \cap M| = 0 \text{ for every edge } e \in E_t(L_t^c, L_t^c). \text{ Let us fix } L_t \subset V(F_t) \text{ and take } M \text{ to be maximal such that } L_t \cup M \text{ minimises the edge density.}$

Assume first that $L_t \neq V(F_t) \setminus \{1,2\}$ with $L_t \neq \emptyset$. By Lemma 3.11 we see that we then have $|M| = (r-2)e_t(L_t, F_t)$. Since all edges incident to $L_t \cup M$ in F_{t+1} are, by the construction of F_{t+1} , incident to M, and since M is a union of $e_t(L_t, F_t)$ disjoint cliques on r-2 vertices each, such that every vertex in M has exactly two neighbours outside M, we have

$$\frac{e_{t+1}(L_t \cup M, F_{t+1})}{L_t \cup M} = \frac{e_t(L_t, F_t)\left(\binom{r}{2} - 1\right)}{|L_t| + (r-2)e_t(L_t, F_t)}.$$
(19)

Note that when $L_t = V(F_t) \setminus \{1, 2\}$ this choice of M would result in having $L_t \cup M = V(F_{t+1}) \setminus \{1, 2\}$, i.e., the edge density is minimised by taking the whole graph. As we want to minimise among all possible proper vertex subsets of the whole graph F_{t+1} , the case $L_t = V(F_t) \setminus \{1, 2\}$ requires some further consideration.

Lemma 3.13. Let $t \ge 2$, $L_t \subseteq V(F_t)$ with $L_t \ne \emptyset$ and $M \subseteq V(F_{t+1}) \setminus V(F_t)$. Let $e = \{x, y\} \in E(F_t)$ with $\{x, y\} \cap L_t \ne \emptyset$ and let $w \in V(e) \setminus M$. We have

$$\frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|} \ge \frac{e_{t+1}(L_t \cup M \cup (V(e) \setminus \{w\}), F_{t+1})}{|L_t \cup M \cup V(e) \setminus \{w\}|}.$$

Proof. We prove this lemma analogously to the proof of Lemma 3.11 to obtain that if $x \in L_t, y \notin L_t$ then

$$\frac{e_{t+1}(L_t \cup M \cup (V(e) \setminus \{w\}), F_{t+1})}{|L_t \cup M \cup V(e) \setminus \{w\}|} \le \frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|} \cdot \frac{|L_t \cup M|}{|L_t \cup M| + m - 1} + \frac{r - 1}{2} \frac{m - 1}{|L_t \cup M| + m - 1} \le \frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|}, \quad \text{(by Lemma 3.6)}$$

while if $x, y \in L_t$ then

$$\frac{e_{t+1}(L_t \cup M \cup (V(e) \setminus \{w\}), F_{t+1})}{|L_t \cup M \cup V(e) \setminus \{w\}|} \le \frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|} \cdot \frac{|L_t \cup M|}{|L_t \cup M| + m - 1} + \frac{r - 3}{2} \frac{m - 1}{|L_t \cup M| + m - 1} \le \frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|}, \quad \text{(by Lemma 3.6)}$$

Assume therefore that $L_t = V(F_t) \setminus \{1, 2\}$. This implies that for all $e \in E(F_t)$ we have $e \in E_t(L_t, F_t)$. Let us again take M to be maximal such that $L_t \cup M$ minimises the ratio $\frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|}$.

Since M is maximal, we use Lemma 3.11 to show that there is exactly one edge $e \in E(F_t)$ such that $|V(e) \cap M| < r-2$. We then use Lemma 3.13 to show that for this edge we have $|V(e) \cap M| = r-3$. Let $\{v\} = V(e) \setminus M$. Since $V(F_{t+1}) \setminus (L_t \cup M) = \{1, 2, v\}$, v can have at most one neighbour, i.e., either vertex 1 or 2, not in $L_t \cup M$. This implies that we have

$$\frac{e_{t+1}(L_t \cup M, F_{t+1})}{L_t \cup M} \ge \frac{e_{t+1} - 1}{v_{t+1} - 3}.$$
(20)

We are now ready to prove Theorem 3.9.

Proof of Theorem 3.9. The proof proceeds by induction on t. We have already seen in the proof of Lemma 3.7 that $\varepsilon_1 = \frac{1}{r+1}$ so the claim holds for t = 1. Thus assume the statement is true for some value of $t \ge 1$. We now proceed with establishing a recursive lower bound on $e_{t+1}(L_{t+1}, F_{t+1})/|L_{t+1}|$. As before, we consider sets of the form $L_{t+1} = L_t \cup M$, where $L_t \subseteq V(F_t) \setminus \{1, 2\}$ and $M \subseteq V(F_{t+1}) \setminus V(F_t)$, and we write $\ell = |L_t|$.

Thus, if there exists a set $L_t \cup M$ that minimises the ratio $\frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|}$ for which we have $L_t \neq V(F_t) \setminus \{1, 2\}$ then by (19), we have

$$\frac{e_{t+1}(L_t \cup M, F_{t+1})}{\ell + |M|} \ge \frac{\frac{(r-2)(r+1)}{2}e_t(L_t, F_t)}{\ell + (r-2)e_t(L_t, F_t)} \\
\ge \frac{\frac{(r-2)(r+1)}{2}\ell\lambda(1+c_t)(1+\varepsilon_t)}{\ell + (r-2)\ell\lambda(1+c_t)(1+\varepsilon_t)}$$
(by ind. hyp. and (13))
$$= \lambda(1+c_{t+1})\frac{\frac{(r-2)(r+1)}{2}\frac{(1+c_t)}{(1+c_{t+1})}(1+\varepsilon_t)}{1+(r-2)\lambda(1+c_t)(1+\varepsilon_t)}.$$

Hence, using the fact that $c_t = 1/(\tau^t - 1)$, we see that ε_{t+1} is at least

$$\frac{\frac{(r-2)(r+1)}{2}\frac{(1+c_t)}{(1+c_{t+1})}(1+\varepsilon_t)}{1+(r-2)\lambda(1+c_t)(1+\varepsilon_t)} - 1 = \frac{(1+c_t)(1+\varepsilon_t)\left(\frac{(r-2)(r+1)}{2}\frac{1}{1+c_{t+1}} - (r-2)\lambda\right) - 1}{1+(r-2)\lambda(1+c_t)(1+\varepsilon_t)}$$
$$= \frac{(1+c_t)(1+\varepsilon_t)\left(1-\frac{1}{\tau^t}\right) - 1}{1+(r-2)\lambda(1+c_t)(1+\varepsilon_t)}$$
$$= \frac{(1+c_t)(1+\varepsilon_t)\frac{1}{1+c_t} - 1}{1+(r-2)\lambda(1+c_t)(1+\varepsilon_t)}$$
$$= \frac{\varepsilon_t}{1+(r-2)\lambda(1+c_t)(1+\varepsilon_t)}.$$
(21)

Using the bound $\varepsilon_t \leq 1/(r+1)$ in Lemma 3.7, we can bound the expression in (21) from below by

$$\frac{\varepsilon_t}{1+(r-2)\lambda(1+c_t)\frac{r+2}{r+1}} = \frac{\varepsilon_t}{1+(r-2)\lambda\frac{\tau^t}{\tau^t-1}\frac{r+2}{r+1}}$$
$$= \frac{\varepsilon_t}{1+(\tau-1)\frac{\tau^t}{\tau^t-1}\frac{r+2}{r+1}}$$
$$\geq \frac{\varepsilon_t}{1+\tau\frac{r+2}{r+1}}$$
$$= \frac{\varepsilon_t}{1+\frac{(r+2)(r-2)}{2}}$$
$$= \frac{2}{r^2-2}\varepsilon_t.$$

Thus in this case we obtain a lower bound on ε_{t+1} given by

$$\varepsilon_{t+1} \ge \frac{2}{r^2 - 2} \varepsilon_t \ge \frac{1}{r+1} \left(\frac{2}{r^2 - 2}\right)^t.$$

If on the other hand there exists a set $L_t \cup M$ with minimal ratio for which $L_t = V(F_t) \setminus \{1, 2\}$ then by (20) we have

$$\frac{e_{t+1}(L_t \cup M, F_{t+1})}{|L_t \cup M|} \ge \frac{e_{t+1} - 1}{v_{t+1} - 3}$$

$$= \frac{\tau^{t+1} - 1}{(r-2)\frac{(\tau^{t+1}-1)}{\tau-1} - 1}$$
$$= \frac{e_{t+1}}{v_{t+1} - 2} \left(\frac{1 - \frac{1}{\tau^{t+1}}}{1 - \frac{(\tau-1)}{(r-2)(\tau^{t+1}-1)}} \right)$$

Thus, we see that in this case ε_{t+1} is at least

$$\begin{aligned} \frac{1 - \frac{1}{\tau^{t+1}}}{1 - \frac{(\tau-1)}{(r-2)(\tau^{t+1}-1)}} &- 1 = \frac{\frac{\tau-1}{(r-2)(\tau^{t+1}-1)} - \frac{1}{\tau^{t+1}}}{1 - \frac{(\tau-1)}{(r-2)(\tau^{t+1}-1)}} \\ &= \frac{\tau^{t+2} - (r-1)\tau^{t+1} + (r-2)}{\tau^{t+1}(r-2)(\tau^{t+1}-1)} \cdot \frac{1}{1 - \frac{(\tau-1)}{(r-2)(\tau^{t+1}-1)}} \\ &> \frac{\tau^{t+2} - (r-1)\tau^{t+1}}{\tau^{t+1}(r-2)(\tau^{t+1}-1)} \\ &> \frac{\tau - (r-1)}{(r-2)\tau^{t+1}} \\ &= \frac{1}{\tau^{t+1}} \left(\frac{r+1}{2} - \frac{r-1}{r-2}\right) \\ &= \frac{1}{\tau^{t+1}} \left(\frac{r+1}{2} - \frac{1}{r-2} - 1\right) \\ &= \frac{\lambda - 1}{\tau^{t+1}}. \end{aligned}$$

It remains to show that for all $t \ge 0$ we have

$$\frac{1}{r+1} \left(\frac{2}{r^2 - 2}\right)^t \le \frac{\lambda - 1}{\tau^{t+1}}.$$
(22)

To see that the inequality in (22) holds, note that for t = 0, since $r \ge 4$,

$$\frac{1}{r+1} = \frac{r-2}{(r+1)(r-2)} \le \frac{\frac{r+1}{2} - \frac{1}{r-2} - 1}{\frac{(r+1)(r-2)}{2}} = \frac{\lambda - 1}{\tau},$$

and moreover

$$\frac{2}{r^2 - 2} < \frac{2}{r^2 - r - 2} = \frac{1}{\tau}.$$

This completes the proof of Theorem 3.9.

Let us conclude this section by commenting on the sharpness of the bound in (18). We know that $\varepsilon_1 = 1/(r+1)$ is obtained by taking $L = L_1$ of size r-3, i.e., by leaving just one vertex in $V(F_1) \setminus \{1, 2\}$ outside L_1 . If we then continue by, for $2 \le i \le t$, taking L_i to be the union of L_{i-1} and all vertices in $V(F_i) \setminus V(F_{i-1})$ that are adjacent to at least one vertex in L_{i-1} , then the resulting set L_t has $1 + 2(r-2)\frac{\tau^{t-1}-1}{\tau-1}$ vertices and $\tau^{t-1}(\tau-2)$ edges adjacent to it. This shows that for some $C_r > 0$, one can obtain a bound $\varepsilon_t \le C_r/\tau^t$. Since $\tau = (r^2 - r - 2)/2$, this implies that our lower bound on ε_t is relatively sharp.

We have thus shown that the edge density of all proper subsets of $V(F_t)$ is strictly bounded below by $e_t/(v_t - 2)$. As we will see in Section 4, we are now equipped with the necessary means to prove Statement (i) of Theorem 1.1.

4. Upper bound on the critical probability

In this section we prove Statement (i) of Theorem 1.1. We shall use the following form of the Janson's inequality.

Theorem 4.1. Let R be a set and let $S \subset R$ be a random subset of R, where each $r \in R$ is in S independently with probability p. Let $\{B_1, \ldots, B_m\}$ be a collection of finite subsets of R and let C_i be the event that $B_i \subset S$. Let $Z = \sum_{i=1}^m \mathbb{1}_{\{C_i\}}$ and let $\mu = \sum_{i=1}^m \mathbb{P}_p(C_i) = \mathbb{E}[Z]$. For $1 \leq i, j \leq m$, $i \neq j$, let $i \sim j$ if $B_i \cap B_j \neq \emptyset$, i.e., if the events C_i and C_j are dependent. Let $\Delta = \sum_{i \sim j} \mathbb{P}(C_i \cap C_j)$. Then

$$\mathbb{P}_p(Z=0) \le e^{-\mu + \Delta/2}.$$
(23)

In this section we show that if $p(n) \ge n^{-\frac{v_t-2}{e_t}} \log n$ then we have $\mathbb{P}_{p_n}(e_{12} \notin E(G_t)) \le n^{-3}$. By the union bound this implies $\mathbb{P}_{p_n}(T \le t) \to 1$.

Proof of Statement (i) of Theorem 1.1. Fix n, sufficiently large, and $t = t(n) \leq \frac{\log \log n}{3 \log \tau}$. As always, fix two vertices 1 and 2 and let $e_0 = \{1, 2\}$. Given any of the $\binom{n-2}{v_t-2}$ subsets $X_i \subset \{3, 4, \ldots, n\}$ of size $(v_t - 2)$, let $F_t(X_i)$ be an arbitrary fixed copy of the graph F_t on $X_i \cup \{1, 2\}$ that adds the edge e_0 to the graph G in t time steps. We shall apply Theorem 4.1 to bound the probability of e_0 not being added to the graph by time t from above.

Let $p(n) = n^{-(v_t-2)/e_t} \log n$, as in Statement (i) and let $G = G_{n,p(n)}$. In Theorem 4.1 we shall take $R = [n]^{(2)}, S = E(G)$ and for $i = 1, \ldots, \binom{n-2}{v_t-2}$ let $B_i = E(F_t(X_i))$. We define C_i , as well as Z, as in Theorem 4.1. We clearly have $\mu = \sum_{i=1}^{m} \mathbb{P}_p(C_i) = \binom{n-2}{v_t-2} p^{e_t}$.

Lemma 3.9 can be used to give an upper bound on Δ in Theorem 4.1 as follows,

$$\begin{split} \Delta &= \sum_{i \sim j} \mathbb{P}(C_i \cap C_j) \\ &= \sum_{1 \leq |X_i \cap X_j| \leq v_{t-3}} \mathbb{P}(B_i \subset S \text{ and } B_j \subset S) \\ &\leq \binom{n-2}{v_t-2} \sum_{k=1}^{v_t-3} \binom{n-v_t}{k} p^{e_t + k\frac{e_t}{v_t-2}(1+\varepsilon_t)} \qquad \text{(by Thm. 3.9)} \\ &= \mu \sum_{k=1}^{v_t-3} \binom{n-v_t}{k} p^{k\frac{e_t}{v_t-2}(1+\varepsilon_t)} \\ &\leq \mu \sum_{k=1}^{v_t-3} n^k p^{k\frac{e_t}{v_t-2}(1+\varepsilon_t)}. \end{split}$$

Note that, by the definition of ε_t and by Lemma 3.8,

$$\frac{e_t}{v_t - 2}(1 + \varepsilon_t) \le \frac{r+1}{2} = \lambda + \frac{1}{r-2} < 2\lambda.$$

Now, using the fact that $p(n) = n^{-(v_t-2)/e_t} \log n$ we have since ,

$$\sum_{k=1}^{v_t-3} n^k p^{k \frac{\varepsilon_t}{v_t-2}(1+\varepsilon_t)} \le \sum_{k=1}^{v_t-3} n^k (\log n)^{2\lambda k} n^{-k(1+\varepsilon_t)}$$
$$= \sum_{k=1}^{v_t-3} (\log n)^{2\lambda k} n^{-k\varepsilon_t}.$$

Since $\varepsilon_t \geq \frac{1}{r+1}(2/(r^2-2))^{t-1}$, we deduce, using $t \leq \frac{\log \log n}{3 \log \tau}$, that $\varepsilon_t > (\log n)^{-1/2}$. Indeed, we have

$$\varepsilon_t \ge \frac{1}{r+1} \left(\frac{2}{r^2 - 2}\right)^{t-1} > \frac{1}{r+1} \left(\frac{1}{\tau} \frac{2\tau}{r^2 - 2}\right)^{\frac{\log\log n}{3\log \tau}} = \frac{1}{r+1} (\log n)^{-1/3} \left(\frac{r^2 - r - 2}{r^2 - 2}\right)^{\frac{\log\log n}{3\log \tau}}.$$

Now, since for all $r \ge 4$ we have $\tau \ge 5$,

$$\left(\frac{r^2-r-2}{r^2-2}\right)^{\frac{1}{3\log 5}} > 0.93 > e^{-1/10}$$

and the bound $\varepsilon_t \geq (\log n)^{-1/2}$ follows for *n* large enough. Consequently, for *n* large enough we have that $(\log n)^{2\lambda} n^{-\varepsilon_t} < 1/2$. Hence continuing the string of inequalities from (24),

$$\Delta \le \mu \sum_{k=1}^{v_t - 3} \left((\log)^{2\lambda} n^{-\varepsilon_t} \right)^k \le \mu \sum_{k=1}^{\infty} \left(\frac{1}{2} \right)^k = \mu.$$

Thus, using Theorem 4.1 and the fact that for $p \ge n^{-\frac{v_t-2}{e_t}} \log n$ we have

$$\mu = \binom{n-2}{v_t - 2} p^{e_t} \ge \frac{(\log n)^{e_t}}{(v_t - 2)^{v_t - 2}} \ge \left(\frac{\log n}{\tau^t}\right)^{\tau^t},$$

we obtain

$$\mathbb{P}(Z=0) \le \exp\left(-\mu + \Delta/2\right)$$
$$\le \exp\left(-\mu/2\right)$$
$$\le \exp\left(-\frac{1}{2}\left(\frac{(\log n)}{\tau^t}\right)^{\tau^t}\right).$$

Note that the function $x \mapsto \left(\frac{\log n}{x}\right)^x$ is increasing for $x \in (0, \frac{\log n}{e}]$. When t = 1, for n sufficiently large,

$$\left(\frac{\log n}{\tau}\right)^{\tau} = \log n \frac{(\log n)^{\tau-1}}{\tau^{\tau}} \ge 6\log n.$$

For $t \leq \frac{\log \log n}{3 \log \tau}$ we have $\tau \leq \tau^t \leq (\log n)^{1/3} < \frac{\log n}{e}$ and so,

$$\left(\frac{\log n}{\tau^t}\right)^{\tau^t} \ge \left(\frac{\log n}{\tau}\right)^{\tau} \ge 6\log n$$

when n is sufficiently large. Thus,

$$\mathbb{P}(X=0) \le \exp\left(-\frac{1}{2}\left(\frac{(\log n)}{\tau^t}\right)^{\tau^t}\right) \le \exp(-3\log n) = n^{-3}.$$

and applying the union bound yields

$$\mathbb{P}_{p_n}(T \le t) \ge 1 - \frac{1}{n}.$$

This completes the proof of Statement (i) of Theorem 1.1.

5. Lower bound on the critical probability

In this section we prove Statement (ii) of Theorem 1.1. More precisely, we show that if $p(n) = o\left(n^{-\frac{v_t-2}{e_t}}\right)$ then the fixed pair $e_0 = \{1, 2\}$ is not added to the graph by time t with high probability.

Proof of Statement (ii) of Theorem 1.1. We use Lemma 3.4 to bound the number of vertices of subgraphs adding e_0 after a particular number of steps from above, Corollary 3.3 to bound the number of edges of these graphs from below, and Corollary 3.5 to bound the number of graphs on v_t vertices that add e_0 in exactly t steps. Recall that $E(G_t)$ denotes the edges of the graph after t time steps and so

$$\mathbb{P}_{p_n}(T \le t) \le \mathbb{P}_{p_n}(e_{12} \in E(G_t)) \\
\le \sum_{i=0}^t \mathbb{P}_{p_n}(e_{12} \text{ is added at time } i) \\
\le p_n + \sum_{i=1}^{t-1} \sum_{j=r-2}^{v_i-2} n^j 2^{j^2} n^{-(\lambda j+1)\frac{v_t-2}{e_t}} + \sum_{j=r-2}^{v_t-3} n^j 2^{j^2} n^{-(\lambda j+1)\frac{v_t-2}{e_t}} \\
+ n^{v_t-2} n^{-e_t \frac{v_t-2}{e_t}} / \omega(n)^{e_t}$$
(25)

For all $t \geq 1$ we have

$$v_{t-1} - 2 = (r-2)\frac{\tau^{t-1} - 1}{\tau - 1} < \frac{r-2}{\tau}\frac{\tau^t - 1}{\tau - 1} = \frac{v_t - 2}{\tau}.$$

Using (13) it follows that for $i \leq t-1$ we can bound the powers of n in the second term of (25) by

$$\begin{split} j - (\lambda j + 1) \frac{(v_t - 2)}{e_t} &= j - j \frac{\lambda(v_t - 2)}{e_t} - \frac{v_t - 2}{e_t} \\ &= j \left(1 - \frac{1}{1 + c_t} \right) - \frac{1}{\lambda(1 + c_t)} \\ &= \left(jc_t - \frac{1}{\lambda} \right) \frac{1}{1 + c_t} \\ &\leq \left((v_{t-1} - 2)c_t - \frac{1}{\lambda} \right) \frac{1}{1 + c_t} \\ &= \left(\frac{r - 2}{\tau - 1} (\tau^{t-1} - 1) \frac{1}{\tau^t - 1} - \frac{r - 2}{\tau - 1} \right) \frac{1}{1 + \frac{1}{\tau^t - 1}} \\ &= \frac{r - 2}{\tau - 1} \frac{\tau^{t-1} - \tau^t}{\tau^t - 1} \frac{\tau^t - 1}{\tau^t} \\ &= -\frac{r - 2}{\tau}. \end{split}$$

Analogously, for i = t and $j \le v_t - 3$, we can bound the powers of n in the third term of (25) by

$$j - (\lambda j + 1) \frac{(v_t - 2)}{e_t} = \left(jc_t - \frac{1}{\lambda}\right) \frac{1}{1 + c_t}$$
$$\leq \left((v_t - 3)c_t - \frac{1}{\lambda}\right) \frac{1}{1 + c_t}$$
$$= \left(\frac{(\tau^t - 1)}{\lambda} \frac{1}{\tau^t - 1} - c_t - \frac{1}{\lambda}\right) \frac{1}{1 + c_t}$$

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$$= \frac{-c_t}{1+c_t} = \frac{-1/(\tau^t - 1)}{1+1/(\tau^t - 1)}$$
$$= \frac{-1}{\tau^t}.$$

We can use these estimates and the fact that, for any $i, v_i - 2 = \frac{\tau^i - 1}{\lambda} < \tau^i / \lambda$, to bound $\mathbb{P}_{p_n}(T \le t)$ from above. Indeed,

$$\mathbb{P}_{p_n}(T \le t) \le p_n + t(v_{t-1} - 2)2^{(v_{t-1} - 2)^2} n^{-(r-2)/\tau} + (v_t - 2)2^{(v_t - 2)^2} n^{-1/\tau^t} + \omega(n)^{-e_t} \le p_n + t \frac{\tau^{t-1}}{\lambda} 2^{\tau^{2(t-1)}/\lambda^2} n^{-(r-2)/\tau} + \frac{\tau^t}{\lambda} 2^{\tau^{2t}/\lambda^2} n^{-1/\tau^t} + \omega(n)^{-e_t}.$$
(26)

There is some constant $C'_r > 0$ such that for all $t \ge C'_r$ we have

$$2^{\tau^{2(t-1)}/\lambda^2} \geq \frac{t\tau^{t-1}}{\lambda} \qquad \text{and} \qquad 2^{\tau^{2t}(\lambda^2-1)/\lambda^2} \geq \frac{\tau^t}{\lambda}.$$

For $t < C'_r$ all four terms in (26) tend to 0 as $n \to \infty$ and we clearly have $\mathbb{P}_{p_n}(T \leq t) = o(1)$. For $t \geq C'_r$ we continue (26) to obtain

$$\mathbb{P}_{p_n}(T \le t) \le p_n + 2^{2\tau^{2(t-1)}/\lambda^2} n^{-(r-2)/\tau} + 2^{\tau^{2t}} n^{-1/\tau^t} + \omega(n)^{-e_t} \\ \le p_n + \exp\left(\frac{2\tau^{2t}}{\lambda^2} \log 2 - \frac{(r-2)}{\tau} \log n\right) + \exp\left(\tau^{2t} \log 2 - \frac{1}{\tau^t} \log n\right) + \omega(n)^{-e_t}.$$

Thus, for $C'_r \leq t \leq \frac{\log \log n}{3 \log \tau}$, with all logarithms having base e,

$$\mathbb{P}_{p_n}(T \le t) \le p_n + \exp\left(\frac{2(\log n)^{2/3}}{\lambda^2}\log 2 - \frac{(r-2)}{\tau}\log n\right) + \exp\left((\log n)^{2/3}\log 2 - (\log n)^{2/3}\right) + \omega(n)^{-e_t} = o(1).$$

This completes the proof of Theorem 1.1.

6. Open problems

In this paper we determine the critical probability for percolation by time t in K_r -bootstrap percolation up to a logarithmic factor. The first obvious problem to consider is the following.

Problem 6.1. Close the gap between Statement (i) and Statement (ii) in Theorem 1.1.

The second open problems we pose here is of extremal nature. Lemma 3.4 tells us that minimal graphs adding e_0 to the graph at time t have at most $(r-2)\frac{\tau^t-1}{\tau-1}+2$ vertices and τ^t edges.

Problem 6.2. How small, both in terms of the size of the vertex set and the edge set, can minimal graphs adding e_0 at time t be?

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