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TRACKING OF NONLINEAR OSCILLATIONS IN HYBRID EXPERIMENTS

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ABSTRACT

We demonstrate a method for tracking oscillations and their stability boundaries (bifurcations) in nonlinear systems. Our method does not require an underlying model of the dynamical system but instead relies on feedback stabilizability. Our method allows one to determine bifurcations of the dynamical system without the need to observe the transient oscillations for a long time to determine their decay or growth. Moreover, in the context of hybrid experiments, which couple experiments and computer simulations bidirectionally and in real-time, our method is able to overcome the presence of coupling delays (or, more generally, unknown actuator dynamics), which is a fundamental problem that is currently limiting the use of hybrid testing. We illustrate the basic ideas with a computer simulation (including coupling delays and noise) of a prototype nonlinear hybrid experiment: a real pendulum coupled at its pivot to a computer simulation of a vertically excited mass-spring-damper system.

1. INTRODUCTION

A common task in the study of nonlinear dynamical systems is to find and track oscillations and their properties (amplitude, spectrum, dynamical stability, etc.). As one varies system parameters these properties can change, for example, the oscillation may become dynamically unstable in a period doubling, or it can "disappear" in a saddle-node. These special events (and their parameter values) are called bifurcations. When a mathematical model of the system under consideration is known, for example, in the form of an ordinary differential equation (ODE), then the nonlinear oscillations and their bifurcations can be tracked, regardless of their dynamical stability, in a suitable parameter as periodic solutions of the ODE. This tracking (one also speaks

of path-following or numerical continuation) of different types of solutions and their bifurcations has emerged as a tremendously useful tool for the analysis of nonlinear dynamical systems because it reveals the boundaries of parameter regions with qualitatively different long-time behaviour. Numerical continuation can be performed with a number of freely available software packages; see, for example, the recent survey [1].

In this paper we consider the problem of tracking oscillations and stability boundaries directly in situations when a full mathematical model of the system under consideration is not available, for example in experiments. More specifically, we are interested in hybrid testing experiments of mechanical and civil engineering systems [2–5]. A hybrid experiment couples a mechanical experiment and a computer simulation bidirectionally and in real-time. One major aim of these experiments is finding and tracking stability boundaries.

We illustrate the basic ideas of control-based bifurcation tracking with a period doubling bifurcation in a prototype nonlinear hybrid experiment: a real pendulum coupled at its pivot to a computer simulation of a vertically excited mass-spring-damper (MSD) system as sketched in Figure 1. This study is still based on coupling the computer simulation to a computer simulation of the pendulum because the introduction of feedback control requires a change to the physical setup of the experiment. The original combined MSD-pendulum system (an auto-parametrically
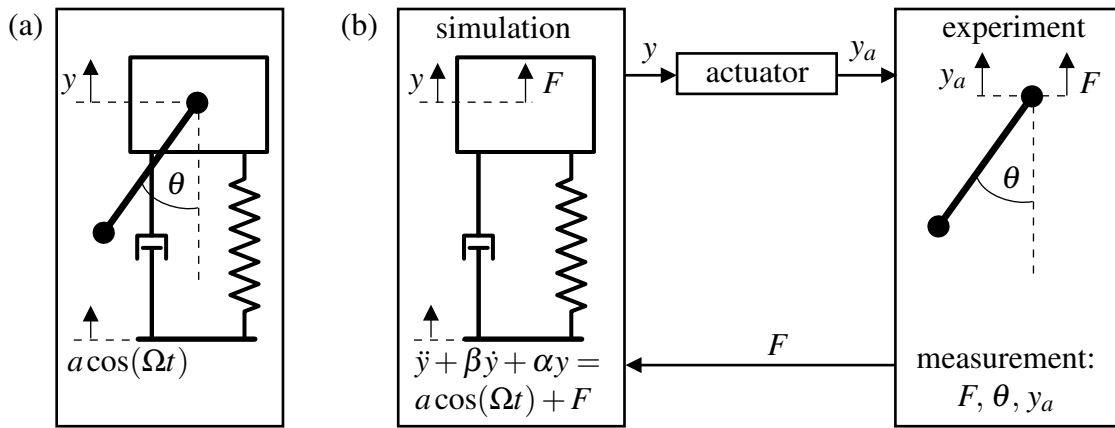

Figure 1: Sketch of decomposition of the overall mechanical system into a computer simulation of a mass-spring-damper system and a real pendulum. Panel (a): original (emulated) system, panel (b): hybrid experiment, the bidirectionally real-time coupled system as studied in [4, 5].

excited two-degree of freedom oscillator) shows a rich bifurcation structure, which can be explored systematically with the numerical methods implemented in AUTO [6] and explained in [7]. This makes the MSD-pendulum system an ideal test candidate, both, for our method and for hybrid testing of systems with nonlinear dynamical behaviour in general. In non-dimensionalized form the MSD-pendulum system shown in figure 1(a) is governed by the equations

$$(1+p)\ddot{y} + \beta\dot{y} + \alpha y + p[\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta] = a\cos(\Omega t) \tag{1}$$

$$\ddot{\theta} + \zeta\dot{\theta} + (1+\ddot{y})\sin\theta = 0 \tag{2}$$

where $p = m/M$ is the mass ratio between the mass M of the mass in the MSD subsystem and the effective mass m of the pendulum. We use the time unit $\omega^{-1} = \sqrt{l/g}$ where g is the acceleration due to gravity and l is the effective length of the pendulum.

2. ESSENTIAL INSTABILITIES IN DELAY-COUPLED MECHANICAL SYSTEMS

The basic work cycle in a hybrid experiment consists of three parts (see figure 1(b)):

1. the numerical simulation of the model (in the example, $\ddot{y} + \beta\dot{y} + \alpha y = a\cos(\Omega t) + F(t)$) with the force input $F(t)$;

2. feeding the actuator with the simulation output $y(t)$, and

3. feeding back the force measurement $F(t)$ to the simulation.

All parts of the loop need to be implemented in real time, and need to run simultaneously and in parallel to the experiment. One advantage of the hybrid setup is that one can easily and systematically vary the system parameters of the numerical subsystem (a, Ω, α, and β in our case) and still study the experimental (nonlinear) component in its original size.

The real-time requirement for the coupling between simulation and experiment via a transfer system and force measurements introduces a number of difficulties. For example, there will be a mismatch between the prescribed trajectory $y(t)$ obtained from the simulation and the output $y_a(t)$ of the transfer system; see figure 1(b). This mismatch

$$e(t) = y_a(t) - y(t) \tag{3}$$

is called the synchronisation error. It is caused by the dynamics of the actuator, which is in general not able to follow the prescribed input trajectory perfectly and instantly. Since the output of the simulation $y(t)$ is known, the synchronisation error (3) can be determined in a hybrid experiment by recording the actual motion $y_a(t)$ of the transfer system (the mechanical actuator). If the synchronisation error $e(t)$ is small then this is taken as a measure of accuracy of the whole hybrid experiment [3, 8].

In many situations the actual trajectory $y_a(t)$ of the actuator follows the prescribed trajectory $y(t)$ almost exactly, but with a fixed small pure time delay τ [4, 8, 9]. Hence, the actuator can be modelled as

$$y_a(t) = y(t - \tau). \tag{4}$$

Let us consider the system in figure 1 in the configuration where the mass m of the pendulum is larger than the mass M of the mass block in the mass-spring-damper system, that is, $p > 1$ in (1), (2). This configuration was found to be impossible to run as a hybrid experiment (that is, as a coupled system as shown in figure 1 (b); see [4]). The analysis in [4, 10] shows that this obstruction is due to the small coupling delay τ. The delay τ changes system (1), (2) to

$$\ddot{y} + p\ddot{y}(t - \tau) + \beta\dot{y} + \alpha y + p[\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta] = a\cos(\Omega t),$$
$$\ddot{\theta} + \zeta\dot{\theta} + (1 + \ddot{y}(t - \tau))\sin\theta = 0 \tag{5}$$

where we inserted the dynamics of the actuator (4) for y_a. In system (5) we have dropped the argument t from all dependent variables y and θ except for those that feature a delay. The appearance of a delayed highest derivative makes (5) a neutral delay differential equation [11]. How does system (5) behave if the mass ratio p is larger than one and the delay is small?

The interaction between a nonzero delay τ and a mass ratio $p > 1$ is easiest to understand near the hanging-down state $\theta = 0$ (which is a stable periodic state in a large part of the parameter space of the original system shown in figure 1(a)). In the linearization in $\theta = 0$ of system (5) the

second equation does not couple back into the first equation such that the linearized equation for y reads

$$\ddot{y} + p\ddot{y}(t - \tau) + \beta\dot{y} + \alpha y = a\cos(\Omega t). \quad (6)$$

The linear operator corresponding to the homogeneous part of (6) (the left-hand-side) has infinitely many eigenvalues and these eigenvalues accumulate to the eigenvalues corresponding to the simpler difference equation

$$y(t) = -py(t - \tau), \quad (7)$$

which has infinitely many eigenvalues with real part $\tau^{-1}\log p$. If $p > 1$ these eigenvalues are unstable regardless of the delay τ. The growth rate even gets larger when the delay is decreased, making the problem practically ill-posed for small delays. This instability carries over to the full system (5) linearized along any trajectory that passes through $\theta = 0$, also leading to infinitely many unstable eigenvalues for small delays $\tau > 0$; see [4, 10]. The essential spectrum of the time-τ map of (5) is located outside the unit circle. Hence, we refer to this instability as essential. This instability is, of course, not present in the emulated system (1), (2).

Moreover, none of the delay compensation schemes developed for hybrid testing is able to stabilize the infinitely many eigenvalues. The delay compensation schemes are based on polynomial extrapolation [3, 9] and have been successfully applied, for example, for the pendulum-MSD system with $p \ll 1$ and for hybrid experiments that are not split at a mass. However, any scheme based on polynomial extrapolation that is consistent (meaning that it is correct for delay $\tau = 0$) must have an essential instability for the split-mass system (5) shown in figure 1(b) if $p > 1$ and for delays arbitrarily close to τ [10].

A real actuator is not capable of supporting an instability at infinitely many frequencies. Typically, the actuator will be a stiff approximation of the idealization (4), for example,

$$\ddot{y}_a + c_s\dot{y}_a + \omega_s^2[y_a - y(t - \tau)] = F/m_s$$

for a large $c_s > 0$ and $\omega_s > 0$ where F is the force measured at the pivot. This gives rise to a regularization of the ill-posed problem (5), but still with a large number of strongly unstable eigenvalues for large c_s, ω_s and small delays, which are still technically impossible to stabilize.

3. INTERFACE MATCHING BY NEWTON ITERATION

A consequence of the arguments in the previous section is that for a mass ratio $p > 1$ it is impossible to achieve an approximation of the dynamics of the emulated system in figure 1 (a) by a bidirectional real-time coupling as in figure 1 (b). We demonstrate now that it is still possible to perform a systematic analysis of the dynamics of the emulated system. In order to achieve this we break the coupling in one direction, match the output at the interface by a Newton iteration and exploit some fundamental statements from bifurcation theory. For example, in the original emulated system (1), (2) the hanging-down state $\theta = 0$ loses its dynamical stability in a period doubling bifurcation. Standard bifurcation theory states that near this loss of stability a solution with a small harmonic amplitude of θ and period $4\pi/\Omega$ emerges [7]. Hence, in order to find the boundary of stability of the hanging-down state it is sufficient to track small period-two solutions.

Figure 2 shows how to break the bidirectional coupling. The actuator is fed with a periodic demand $\bar{y}$. In addition, the pendulum experiment is stabilized by a feedback loop with a periodic
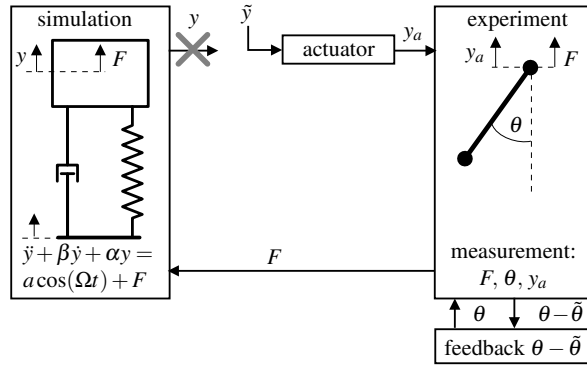
Figure 2: Sketch of the iterative hybrid experiment. The periodic input $\tilde{y}$ replaces the coupling from the simulation to the experiment. Additionally, a classical (proportional-plus-derivative) feedback loop stabilises the parametrically driven pendulum. The system has two additional periodic inputs: $\tilde{\theta}, \tilde{y}$. A model for this setup is system (8).
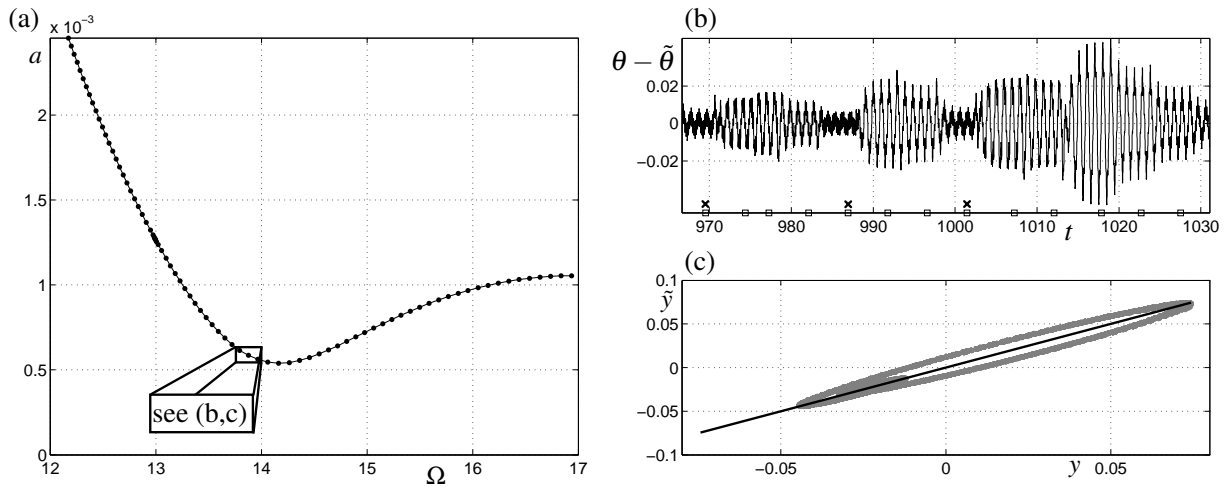

Figure 3: Results of continuation of period doubling. Panel (a): stability curve in two-parameter plane, panel (b1): control effort in feedback loop along a part of the curve, panel (b2): anticipation of y by $\tilde{y}$.

demand signal $\tilde{\theta}$. The unidirectionally coupled system would follow the model

$$
\begin{aligned}
\ddot{y} &= -\beta \dot{y} - \alpha y + a \cos(\Omega t) + F & \text{computer simulation} \\
\ddot{\theta} &= -[1 + \ddot{y}_a]\sin\theta - \zeta\dot{\theta} - \mathrm{PD}[\theta - \tilde{\theta}](t - \tau) & \text{feedback stabilized experiment} \\
F(t) &= -p\ddot{y}_a - p\left[\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right] & \text{measured force}
\end{aligned} \tag{8}
$$

where $\mathrm{PD}[x] = k_p x + k_d \dot{x}$ is a standard proportional-plus-derivative control, y_a depends only on F and $\tilde{y}$ but not on y (thus, there is no coupling from the computer simulation to the experiment). The model (8) assumes that this control is applied in the form of a torque at the pivot of the pendulum. Due to the feedback loop the unidirectionally coupled system will settle (after a transient) into a locally unique periodic state that can be measured and depends on the given demands $\tilde{y}$ and $\tilde{\theta}$ of period $4\pi/\Omega$. Its measurable outputs are the angular displacement $\theta[\tilde{\theta}, \tilde{y}]$, the motion of the pivot $y_a[\tilde{\theta}, \tilde{y}]$ and the output of the simulation $y[\tilde{\theta}, \tilde{y}, \Omega, a]$. The following system of nonlinear equations defines the period doubling bifurcation of the hanging-down state

of the original system:

$$0 = y[\tilde{\theta}, \tilde{y}, \Omega, a] - y_a[\tilde{\theta}, \tilde{y}] \qquad \text{synchronization} \qquad (9)$$

$$0 = \theta[\tilde{\theta}, \tilde{y}] - \tilde{\theta} \qquad \text{control non-invasive} \qquad (10)$$

$$r^2 = \int_0^{4\pi/\Omega} (\tilde{\theta}(t) - \theta_0)^2 \, dt \qquad \text{period doubling} \qquad (11)$$

where r is small and θ_0 is the average of $\tilde{\theta}$. The variables of this system are the two parameters a and Ω (the excitation in the simulation) and the two periodic control demands $\tilde{\theta}$ and $\tilde{y}$, which can be expressed by their first two Fourier modes:

$$\tilde{y}(t) = y_0 + y_1 \exp(it\Omega/2) + y_2 \exp(it\Omega)$$
$$\tilde{\theta}(t) = \theta_0 + \theta_1 \exp(it\Omega/2) + \theta_2 \exp(it\Omega)$$

where $y_0, \theta_0 \in \mathbb{R}$, $y_1, y_2, \theta_1, \theta_2 \in \mathbb{C}$. After a Galerkin projection of (9) and (10) onto the first two Fourier modes one obtains 11 (real-valued) equations for 12 (real-valued) variables (Ω, a, y_0, θ_0, y_1, y_2, θ_1, θ_2). This resulting system defines an implicit curve that can be found by a Newton iteration embedded into a pseudo-arclength continuation. Each evaluation of the right-hand-side of (9) and (10) requires one to run the experiments once until the transients have died. This makes function evaluations expensive compared to purely numerical bifurcation analysis as discussed in [7] and implemented in AUTO [6].

Figure 3 shows the results of a proof-of-concept computer simulation using the idealized actuator model (4) with delay $\tau = 0.01/\omega \approx 0.07$ (10 ms) and the pendulum equation (2) for the pendulum. Figure 3 (a) displays the curve defined by the Galerkin approximation of (9)–(11). Figure 3 (b) shows a typical time profile during the continuation along the period doubling curve. Note that the transients, occuring whenever the demands and parameters are changed, are typically small because demands and parameters are varied only gradually during the continuation. The squares along the time axis indicate when the system is considered to have settled down to a periodic state, giving one evaluation of the right-hand-sides of (9) and (10). The crosses along the time axis mark those function evaluations that correspond to a solution that was accepted by the tolerance of the Newton iteration as a solution of (9)–(11). Figure 3 (c) displays in the synchronization plane that $\tilde{y}$ anticipates the output of the simulation y slightly. Importantly, we achieve synchronization without expressly exploiting the knowledge about the actuator model (4).

4. CONCLUSION AND FURTHER WORK

Proof-of-concept computer simulations, including adverse effects such as coupling delay and measurement inaccuracies, propose that bifurcation analysis should be possible even for hybrid experiments that are genuinely ill-posed.

The incorporation of control-based bifurcation analysis into the hybrid experiment itself is currently in preparation. The most pressing problems for the future, apart from experimental validation, are the incorporation of other bifurcations (some have been studied in [12]), of strongly nonlinear phenomena (such as homoclinics), and of non-periodic responses.

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