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## BIFURCATION ANALYSIS OF COUPLED LASER MODES IN MUTUALLY DELAY-COUPLED LASERS

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**Abstract:** This paper considers a system of two semiconductor lasers that are mutually coupled face to face, so that they receive each other's light after a delay time  $\tau$ . The lasers are assumed to be identical, except for a possible detuning  $\Delta$  of their free-running frequencies. The coupled laser modes of a rate equation model with delay are studied with tools from bifurcation theory, especially numerical continuation. This reveals a comprehensive geometrical picture, which is organized by the unfoldings for  $\Delta \neq 0$  of pitchfork bifurcations that exists for  $\Delta = 0$ .

**Keywords:** mutually coupled lasers, time delay, symmetry properties, coupled laser modes, numerical continuation

### 1. INTRODUCTION

Semiconductor lasers are widely used today in applications such as optical data storage and optical communication. This type of laser is known to be very sensitive to external optical influences. Well known is that small reflections re-entering a semiconductor laser may completely destabilize its output. As semiconductor lasers are integrated into more complex optical systems, the question of mutual coupling has received increasing attention in recent years. The interest in coupled lasers is also fueled by the wish to utilize chaotic laser output in a sender-receiver configuration for private communication applications. In many such applications delay effects are crucial.

Today some experimental as well as theoretical studies have been performed on mutually delay-coupled lasers. Synchronization of two chaotic semiconductor lasers was reported in Mulet et al. (2002, 2004). Spontaneous symmetry breaking for zero detuning and a leader-laggard scenario for nonzero detuning was found in Heil et al. (2001). On the theoretical side, the

instantaneous-coupling limit of small  $\tau$  was studied in Yanchuk et al. (2004) and the limit of large  $\tau$  in Javaloyes et al. (2003); Viktorov et al. (2004). An approximate thermodynamic potential for two mutually coupled laser was derived in Vicente et al. (2004).

The topic of this paper is the short coupling regime where the delay time  $\tau$  is of the same order as the period of the lasers's intrinsic relaxation oscillation. Specifically, we study a rate equation model of two mutually delay-coupled semiconductor lasers that are identical but have a detuning of their free-running frequencies. Due to inherent delay in the coupling, the model equations are mathematically a delay differential equation (DDE). For the intermediate range of  $\tau$  considered here the full DDE needs to be studied.

We present a bifurcation analysis of the coupled laser modes (CLMs), which are fundamental solutions of the system where both lasers produce light of constant output intensity. The main tool is numerical continuation of the CLMs with the software package DDE-BIFTOOL; see Engelborghs et al. (2001). We concentrate here on the bifurcations as a function of the

coupling phase  $C_p$ . A representation of the CLMs in the plane of frequency versus inversion, which is often used in the laser physics literature, and experimental measurement can be found in Erzgräber et al. (2004).

## 2. THE MODEL EQUATIONS

Two mutually delay-coupled semiconductor lasers can be modelled by Lang-Kobayashi-type rate equations (compare Lang and Kobayashi (1980)) for the complex slowly-varying envelope of the optical fields  $E_{1,2}$  and the normalized inversions (electron-hole pairs)  $N_{1,2}$ ; see Fig. 1 for a sketch of the situation and Mulet et al. (2002) for more details of the derivation. Suitably rescaled, the equations can be written as

$$\frac{dE_{1,2}}{dt} = (1 + i\alpha)N_1(t)E_{1,2}(t) + \eta e^{-iC_p}E_{2,1}(t - \tau) \mp i\Delta E_{1,2}(t), \quad (1)$$

$$T \frac{dN_{1,2}}{dt} = P - N_{1,2}(t) - (1 + 2N_{1,2}(t))|E_{1,2}(t)|^2. \quad (2)$$

The two lasers are assumed to be identical, which means that they are described by the same value of the linewidth enhancement factor  $\alpha$ , the normalized carrier lifetime  $T$ , and the pump parameter  $P$ . Furthermore, the coupling is assumed to be symmetrical (as is the case in the face-to-face configuration in air), so that both lasers experience the same coupling strength  $\eta$  and delay time  $\tau$ . (Since both lasers are identical, we are not studying here the situation that one laser is used to control the other laser.) In the computations below these parameters are set to  $\alpha = 5.0$ ,  $T = 392.7$ ,  $P = 0.231$ ,  $\eta = 0.025$ , and  $\tau = 71$ . This corresponds to a physically realistic setup where the delay time is on the order of the relaxation oscillations of the two lasers; see also Erzgräber et al. (2004).

The detuning between the lasers is taken into account by the last term of (1) where  $\Delta = \frac{1}{2}(\Omega_2 - \Omega_1)$  and  $\Omega_i$  is the optical frequency of the free-running laser  $i$ . A final and important parameter is the coupling phase  $C_p = \Omega_0\tau$  with respect to the average frequency  $\Omega_0 = \frac{1}{2}(\Omega_1 + \Omega_2)$ . In this paper  $C_p$  is considered as the main independent parameter. This is reasonable because  $C_p$  can be varied in an experiment, for example, by tiny changes of  $\tau$ , which have only very little effect on the other parameters in Eqs. (1)–(2).

Equations (1)–(2) are mathematically a DDE with the single fixed delay  $\tau$ . They exhibit several symmetries which are important for what follows. First, there is the  $S^1$ -symmetry

$$(E_{1,2}, N_{1,2}) \rightarrow (cE_{1,2}, N_{1,2}) \quad (3)$$

for any  $c \in \mathbb{C}$  with  $|c| = 1$ , which is physically an invariance of both electric fields under any phase shift. The continuous symmetry group  $S^1$  is a typical feature

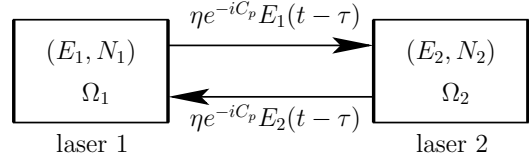


Fig. 1. Sketch of two face-to-face coupled semiconductor lasers.

of rate equations with optical feedback (see Krauskopf et al. (2000)) and must be dealt with in the numerical continuation. Second, there is the  $\mathbb{Z}_2$ -symmetry

$$(E_{1,2}, N_{1,2}, \Delta) \rightarrow (E_{2,1}, N_{2,1}, -\Delta) \quad (4)$$

of interchanging the lasers and changing the sign of  $\Delta$ . For  $\Delta = 0$  this leads to a reflectional  $\mathbb{Z}_2$ -symmetry in phase space. Finally, there are two translational symmetries involving the parameter  $C_p$ , namely

$$(E_{1,2}, N_{1,2}, C_p) \rightarrow (E_{1,2}, N_{1,2}, C_p + 2\pi), \quad (5)$$

$$(E_1, E_2, N_{1,2}, C_p) \rightarrow (E_1, -E_2, N_{1,2}, C_p + \pi). \quad (6)$$

## 3. COUPLED LASER MODES

The  $S^1$ -symmetry of Eqs. (1)–(2) means physically that when both lasers lase with some constant intensity then they must lase at the same frequency  $\omega^s$ . However, there may be a constant phase shift  $\sigma$  between the lasers. Such a solution is called a coupled laser mode (CLM). Mathematically a CLM is given as

$$\begin{cases} E_1(t) = R_1^s e^{i\omega^s t}, \\ N_1(t) = N_1^s, \\ E_2(t) = R_2^s e^{i\omega^s t + i\sigma}, \\ N_2(t) = N_2^s, \end{cases} \quad (7)$$

where  $R_{1,2}^s$ ,  $N_{1,2}^s$ ,  $\omega^s$ , and  $\sigma$  are real numbers. As is the case for external cavity modes (ECMs) for a laser with conventional optical feedback, the CLMs are group orbits of the  $S^1$ -symmetry; see Krauskopf et al. (2000); Rottschäfer and Krauskopf (2004).

Inserting the ansatz (7) into Eqs. (1)–(2) gives a set of six coupled transcendental equations for the six unknowns  $R_{1,2}^s$ ,  $N_{1,2}^s$ ,  $\omega^s$ , and  $\sigma$ . An analytical investigation of these equations in the general setting is quite a challenge and beyond this paper. However, it is quite straightforward to find the transcendental equation involving only  $\omega^s$  and  $\sigma$ , namely

$$\begin{aligned} (\omega^s)^2 = \Delta^2 + \kappa^2 & \\ & \times [\sin(C_p + \omega^s\tau + \sigma) + \alpha \cos(C_p + \omega^s\tau + \sigma)] \\ & \times [\sin(C_p + \omega^s\tau - \sigma) + \alpha \cos(C_p + \omega^s\tau - \sigma)]. \end{aligned} \quad (8)$$

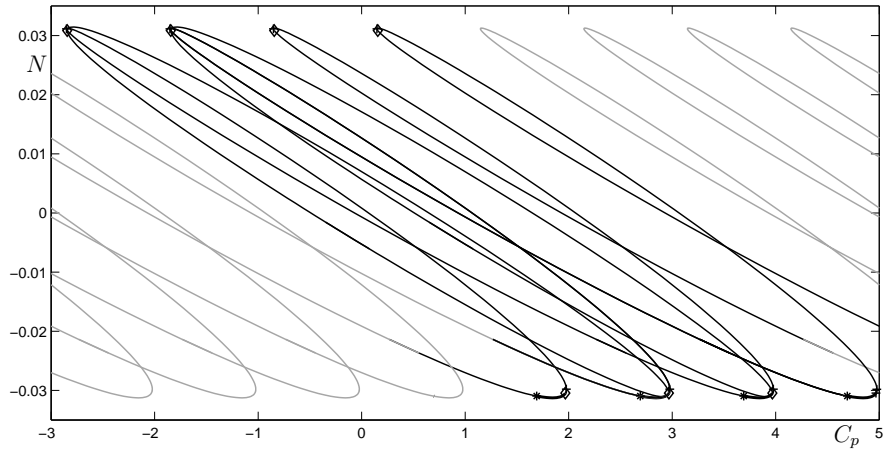


Fig. 2. The CLMs of Eqs. (1)–(2) in the  $(C_p, N)$ -plane for  $\Delta = 0$ ; shown are the inversions  $N_1$  and  $N_2$  of both lasers and  $C_p$  is in multiples of  $\pi$ . The infinitely long solid curves (relevant parts of which are highlighted) are of constant-phase CLMs; the stable regions are shown thicker. The dashed curves are ellipses of variable-phase CLMs. Saddle-node bifurcations are denoted by crosses (+), Hopf bifurcations by stars (\*), and pitchfork bifurcations by diamonds ( $\diamond$ ).

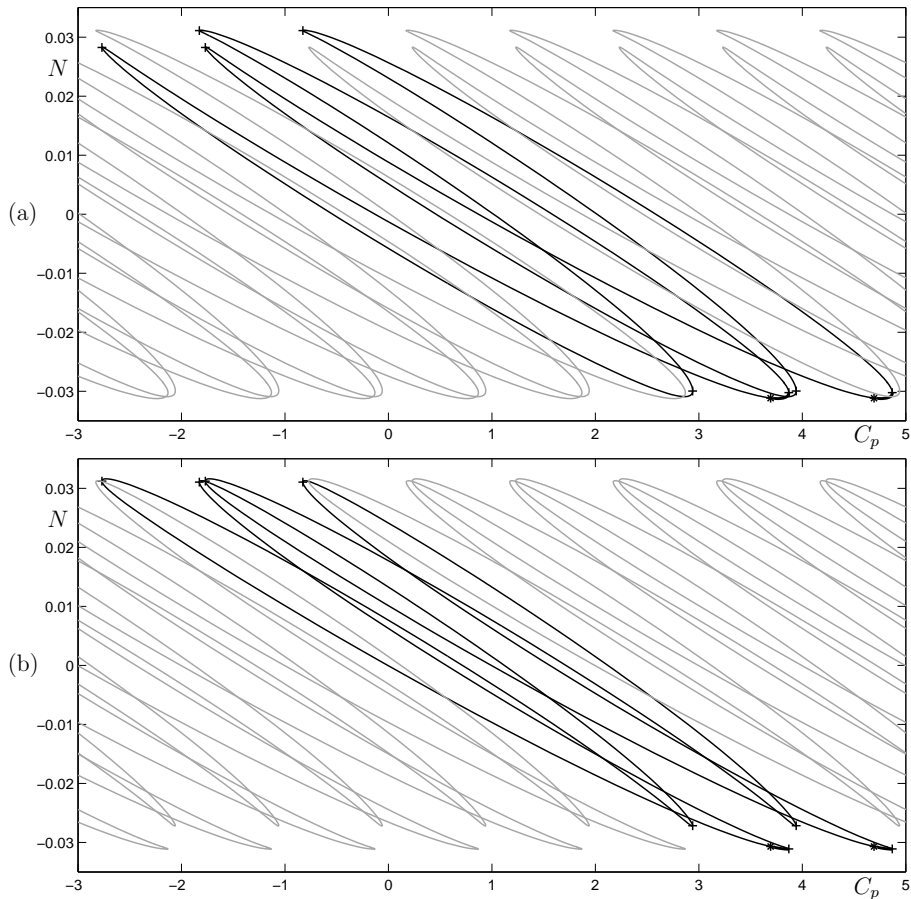


Fig. 3. The CLMs of Eqs. (1)–(2) in the  $(C_p, N)$ -plane for  $\Delta = 2.5 \times 10^{-3}$ . Panel (a) shows inversion  $N_1$  of the red laser 1 and panel (b) shows inversion  $N_2$  of the blue laser 2;  $C_p$  is in multiples of  $\pi$ . All CLMs have a variable phase and they are organised in closed self-intersecting curves, stable regions on which are shown thicker. Saddle-node bifurcations are denoted by crosses (+) and Hopf bifurcations by stars (\*).

#### 4. ZERO DETUNING

When  $\Delta = 0$  then two special types of solutions can be identified, namely those for  $\sigma = 0$  and  $\sigma = \pi$ . Then Eq. (8) reduce to

$$\omega^s = \mp [\kappa(\sin(C_p + \omega^s \tau) + \alpha \cos(C_p + \omega^s \tau))], \quad (9)$$

where the minus sign relates to  $\sigma = 0$  and the plus sign to  $\sigma = \pi$ . A solution for  $\sigma = 0$  is called an *in-phase CLM* and one for  $\sigma = \pi$  an *anti-phase CLM*, and both are referred to as *constant-phase CLMs* because  $\sigma$  is constant as a function of  $C_p$ . Furthermore, it is straightforward to see that for a constant-phase CLM one has  $R_1^s = R_2^s$  and  $N_1^s = N_2^s$ , meaning that both lasers lase with the same intensity.

The in-phase CLMs are, in fact, (ECMs of the Lang-Kobayashi equations describing the situation that a regular mirror is halfway between the two lasers. Indeed, in this case (9) is recognized as the transcendental equation for  $\omega^s$  as it is known for the Lang-Kobayashi equations; see Rottschäfer and Krauskopf (2004). The anti-phase CLMs are related to the in-phase CLMs by symmetry (6) and can, hence, also be interpreted as ECMs of the Lang-Kobayashi equations but for  $C_p + \pi$ .

However, even for  $\Delta = 0$  there are CLM for which the intensity of the two lasers is not the same so that  $R_1^s \neq R_2^s$  and  $N_1^s \neq N_2^s$ . This type of CLMs must have a phase difference  $\sigma$  that is neither 0 nor  $\pi$ , and they cannot be interpreted as ECMs of a COF laser. Because their phase varies with  $C_p$ , they are referred to as *variable-phase CLMs*.

Figure 2 shows the CLMs for  $\Delta = 0$  in the  $(C_p, N)$ -plane where the inversions  $N_1^s$  and  $N_2^s$  of both lasers are plotted. The in-phase CLMs form the curve of infinite length that has its minima near  $\pi + 2\pi k$ . Note that, since  $N_1^s = N_2^s$ , the in-phase CLM curves for laser 1 and laser 2 coincide. The curve of anti-phase CLMs looks exactly like that of the in-phase CLMs, but it is shifted by  $\pi$  due to symmetry (6), that is, it has its minima near  $2\pi k$ . To help interpret Fig. 2, a relevant part of the constant-phase CLM curves is shown as a black solid curve, while the other parts are shown only in grey. On both curves there is a small section in the low-inversion region where the constant-phase CLMs are stable, which is drawn in a thicker line style. This section is bounded on the right by a pitchfork bifurcation and on the left by a Hopf bifurcation; all other constant-phase solutions are unstable. As  $C_p$  is decreased two constant-phase CLMs are born in the saddle-node bifurcation in the low-inversion region and then finally disappear in a further saddle-node bifurcation in the high-inversion region.

Also shown in Fig. 2 are the variable-phase CLMs. They form ellipses of which two are shown as dashed curves. The ellipses intersect the constant-phase CLM in pitchfork bifurcations. One type of ellipse (left ellipse in Fig. 2) connects the in-phase CLM curve

in the low-inversion region with the anti-phase CLM curve in the high-inversion region; the other type of ellipse (right ellipse in Fig. 2) connects the anti-phase CLM curve in the low-inversion region with the in-phase CLM curve in the high-inversion region. The two types of ellipses are mapped onto each other by symmetry (6). When  $C_p$  is decreased a pair of variable-phase solutions, related to each other by the symmetry (4) of exchanging laser 1 with laser 2, are born in the pitchfork bifurcation in the low-inversion region. The inversion  $N_1$  of laser 1 traces out the upper (resp. lower) branch, while the inversion  $N_2$  of laser 2 traces out the lower (resp. upper) branch of the ellipse. The two solutions meet in the pitchfork bifurcation in the high-inversion region and disappear.

In summary, Fig. 2 shows how for  $\Delta = 0$  the variable-phase CLMs provide a connection between the in-phase and anti-phase CLMs. This is emphasized in Fig. 5(a) where the phase  $\sigma$  of all CLMs is shown over a suitable range of  $C_p$ . The horizontal lines are of constant-phase CLMs, and the variable-phase CLMs provide the connection between them. Note that Fig. 5(a) is  $\pi$  periodic in  $C_p$  and  $2\pi$ -periodic in  $\sigma$  (not all curves of CLMs are shown).

#### 5. NONZERO DETUNING

When  $\Delta \neq 0$  then Eqs. (1)–(2) no longer have the additional  $\mathbb{Z}_2$ -symmetry, so that each pitchfork bifurcation unfolds into a saddle-node bifurcation and a separate branch of solutions. Furthermore, there are no constant-phase solutions anymore, that is, the phase  $\sigma$  of all CLMs varies with  $C_p$  and also  $R_1^s \neq R_2^s$  and  $N_1^s \neq N_2^s$ . Nevertheless, the case  $\Delta = 0$  can be thought to organize the dynamics even for nonzero  $\Delta$ , as is shown now.

Figure 3 shows the situation for  $\Delta > 0$ , where panel (a) shows the inversion  $N_1$  of laser 1, the red laser that lases with a lower frequency, and panel (b) shows the inversion  $N_2$  of laser 2, the blue laser that lases with a higher frequency. The pitchfork bifurcations have disappeared and the previous constant-phase CLMs (referred to as almost constant-phase CLMs) are now connected to the previous variable-phase CLMs (still referred to as variable-phase CLMs) in two closed curves, a pair of which is shown in Fig. 4. Indeed, the two types of CLMs bifurcate in saddle-node bifurcations, two of which come from the pitchfork bifurcation for  $\Delta = 0$ . Note that the way the pitchfork bifurcation is unfolded in the high- and low-inversion regions differs for the red and the blue laser; compare panels (a) and (b) in Fig. 4. There are no longer any infinitely long curves. Nevertheless, the curves of CLMs for  $\Delta > 0$  in Fig. 3 converge to the those for  $\Delta = 0$  in Fig. 2.

Plotting the phase  $\sigma$  between the two lasers as a function of  $C_p$  results in closed curves of CLMs, as is shown in Fig. 5(b). This image is seen to contain

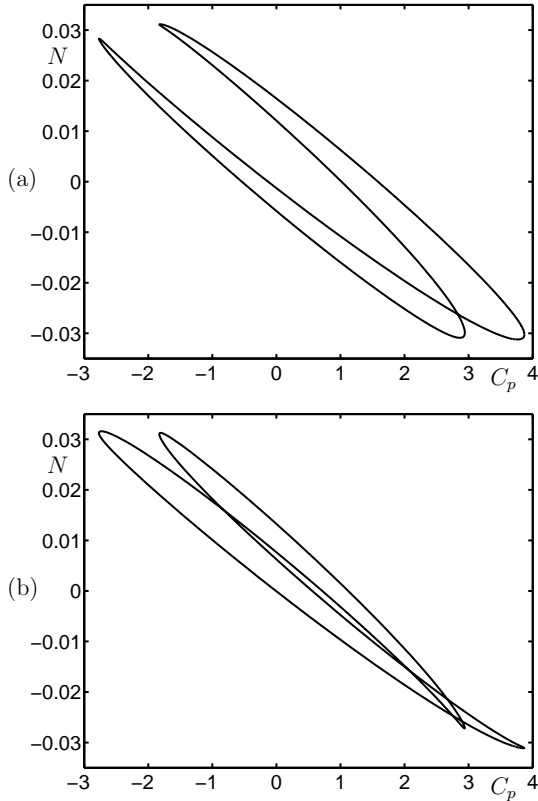


Fig. 4. A single closed curve of CLMs in the  $(C_p, N)$ -plane for  $\Delta = 2.5 \times 10^{-3}$ . Panel (a) shows inversion  $N_1$  of the red laser 1 and panel (b) shows inversion  $N_2$  of the blue laser 2;  $C_p$  is in multiples of  $\pi$ .

two branches of almost constant phase (justification to speak of almost constant-phase CLMs) which are connected by two branches of variable-phase solutions. The latter are the remainders of the variable-phase CLMs for  $\Delta = 0$ . Indeed, one realizes that Fig. 5(b) unfolds the situation shown Fig. 5(a).

## 6. CONCLUSIONS AND OUTLOOK

The geometrical structure of CLMs presented here forms the backbone of the dynamics of two mutually delay-coupled lasers in the short coupling regime. It is organized by the case of  $\Delta = 0$ , which features an additional  $\mathbb{Z}_2$ -symmetry that is broken for  $\Delta \neq 0$ .

There are several clear avenues for future research. The next step in this study will be to consider dynamics beyond the CLMs, which are stable only in small sections of  $C_p$ . Preliminary investigations of Hopf bifurcations and the bifurcating periodic orbits indicate a complicated structure of connecting bridges of periodic orbits, not unlike those found in the COF laser in Haegeman et al. (2002). Further bifurcations to chaotic dynamics will also occur.

Furthermore, while the presented structure of CLMs is structurally stable, it is nevertheless very interesting to study how it depends on the parameters of the setup,

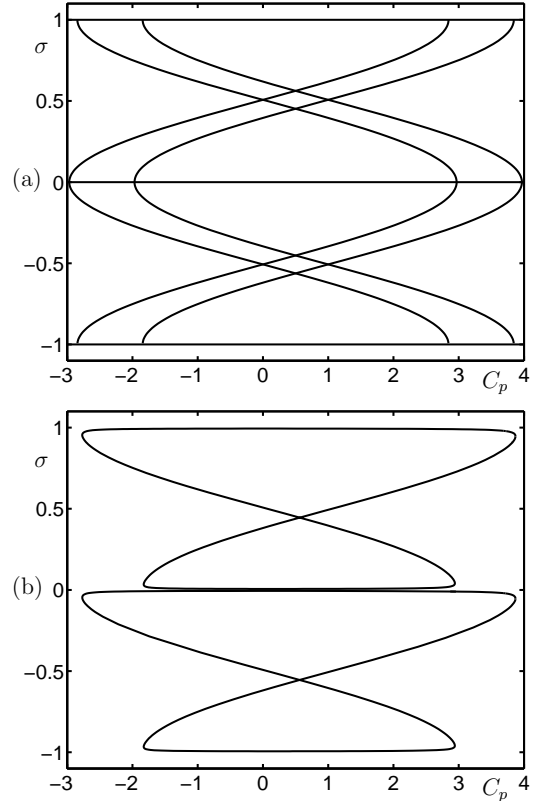


Fig. 5. The phase difference  $\sigma$  between the two lasers in the  $(C_p, \sigma)$ -plane; panel (a) shows the case  $\Delta = 0$  and panel (b) shows the case  $\Delta = 2.5 \times 10^{-3}$ ; both  $C_p$  and  $\sigma$  are in multiples of  $\pi$ .

$\eta$  and  $\tau$ , as well as on the intrinsic parameters of the lasers,  $\alpha$ ,  $T$ , and  $P$ . This is also important for matching parameters to compare theoretical results with experimental measurements. Finally, there is the question what happens when the assumption of identical lasers is dropped.

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