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A simple model of impact dynamics in many dimensional systems, with applications to heat exchangers

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Abstract

In this paper we shall present a simple hybrid model of impact dynamics in heat exchangers. The method, based on graph theory and probability theory, enables us to model the variation in global dynamics as measurable local parameters are changed. We find a sudden jump from no to many repeated impacts, in agreement with numerical and experimental evidence.

1 Introduction

A heat exchanger typically consists of a large number of thin pipes (around 200) contained within an overall pressurised vessel. The pipes are held in position by spacers spread at suitable intervals along their length. The spacers have a small clearance to allow for thermal expansion and contraction of the pipes. Hot fluid is pumped in one direction through the thin pipes, while cool fluid passes through the main vessel in the other direction. Thus in normal operation, heat is transferred between the two fluids without need for mixing (highly desirable if, for example, the hotter fluid is radioactive). An unwanted characteristic of these systems occurs when the fluid motion and local boiling cause the thin pipes to vibrate and repeatedly impact both each other and the fixed spacers, leading to wear both along pipe lengths and around pipe circumferences. If allowed to proceed unchecked, pipework will fail and expensive plant shutdown will follow [53].

The behaviour of the heat exchanger system is not completely understood. The complexity of the conditions inside the vessel: fully turbulent flow with heat transfer, cavitation, localised boiling and impacts, for example, make it almost impossible to formulate a model that includes all of the relevant effects (although large scale computer simulations are carried out [2, 50]). Neither is easy to take measurements inside a functioning vessel, the extreme conditions making the practicalities of data collection very difficult. Analytic studies generally consider one single pipe [21, 25], which misses any non-trivial collective behaviour, or treat the pipes as a continuum [6, 39], which is difficult to compare with real experiments.

We shall attempt to overcome these difficulties by considering a simple model of the system which retains the discrete nature of the pipes, while not attempting to model the fluid behaviour or dynamics of the pipes in great detail. We shall consider only impacts between pipes, neglecting fretting and interactions between pipes and spacers. We are particularly keen to explain an experimental observation that, as the flow speed is increased, there is a sudden qualitative change in the behaviour of the heat exchanger. At low flow speeds, there are no impacts, while above some critical speed there is a sudden jump to many impacts [33, 42]. This effect has also been observed in computer simulations [2].

A heat exchanger is a very large dimensional version of a *piecewise smooth* dynamical system. These systems may be described by equations of the form

$$\dot{x} = f(x, t, \mu) \tag{1}$$

where $f : \mathbb{R}^{m+p+1} \to \mathbb{R}^m$ is a piecewise smooth function, $\mu \in \mathbb{R}^p$ is a vector of parameters and $x \in \mathbb{R}^m$. Low dimensional versions of such systems have become of increasing interest in recent years, with even the simplest having extremely rich dynamics. They are used to model a huge variety of physical systems in engineering and applied science. Examples include engineering systems with impacts [3, 21, 30, 52, 59], power electronics [10, 13, 24], earthquake engineering [26], structural engineering [8, 14, 35], models of locomotive walking [40], systems with friction [51, 54], rail-wheel dynamics [32,34,48] and neural dynamics [5]. Simpler 'toy' models are also much studied, such as the impact oscillator [7, 17–19, 47], or bouncing ball model [28, 38, 57, 61]. Much recent research has concentrated on bifurcations unique to such systems [7,9,11,12,15–20,31,36,43–46,62].

In previous work [27, 29] we have already demonstrated how graph theory [4] may be used to find periodic orbits in general piecewise smooth dynamical systems. The key idea is to represent the piecewise smooth dynamical system as a directed graph. The method then provides a means of predicting, classifying and counting periodic motion in the dynamical system.

The main idea of this paper is that the totality of these periodic motions is the source of the observed repeated impacts. Each of the individual periodic motions occurs with a certain probability; depending on the parameters in the model it is possible that motions with few impacts are more likely to occur that those with many impacts, or vice versa. The jump in the total number of impacts occurs after a certain critical parameter value is exceeded, just as in percolation theory [22, 58].

The paper is organised as follows. We begin, in section 2, by recapping on the basic ideas of graph theory and their relationship to piecewise smooth systems, and go on to describe the graph for the model of the heat exchanger. We then proceed, in section 3, to count simple periodic orbits, exploiting the nature of our new graph by using techniques from combinatorics. A key idea, the introduction of an impact probability to the graph, is described in section 4. We then proceed in section 5 to derive a distribution for the number of pipes undergoing repeated impacts, and examine how statistics such as the mode vary as the probability of an impact changes. This suggests the possibility of a sudden jump in the number of pipe-pipe impacts, a claim which we prove in section 6. In section 7 we describe a method of relating measurable physical parameters to the impact probability, and hence make direct comparison between graph theory predictions and numerical simulations, showing reasonable qualitative agreement. In section 8 we propose an extension to the method, and show how this improves the fit between theory and numerics. Finally, in section 9, we provide some concluding remarks.

2 Graph theory and the heat exchanger

The language of graph theory has many dialects. We choose to follow [4] for definiteness. A graph $G = (V, E)$ is a collection of vertices $V = \{V_1, V_2, \ldots, V_{N_V}\}$ and edges $E = \{e_1, e_2, \ldots, e_{N_e}\}$. In our case we shall be assigning a direction to each edge and so we are actually dealing with a directed graph (or digraph). However since we only deal with digraphs in what follows, we shall simply call them graphs. A graph may also be defined by its *adjacency matrix*, A. A is a $N_V \times N_V$ matrix; the entries $[A]_{ij}$ are defined to be the number of different edges with initial vertex V_i and final vertex V_j. A *circuit* in the graph is a sequence of edges starting and finishing at the same vertex. A circuit make up of k edges is said to have *length* k. A circuit is called *simple* if it passes through no vertex more than once. Note that, in this paper, all graphs are *connected*, that is for any two vertices, it is always possible to find a path beginning at one and ending at the other.

Graph theory is already used in a large variety of problems (see, for example, [4] for an indication of its widespread use), from theoretical combinatorial dynamics [1] to multibody mechanical systems [41].

In this work we use a directed graph to represent the periodic motions in a dynamical system. The relationship between dynamical system and graph can be summarised as:

- the surfaces in phase space where f or its derivatives are discontinuous are interpreted as vertices of a directed graph of the system,
- a possible phase space trajectory between two discontinuity surfaces is then an edge of the graph between the corresponding vertices, and
- the direction along an edge corresponds to increasing time.

Circuits in the graph are central to our method in that:

- every periodic orbit in the dynamical system can be represented as a circuit in the graph,
- every circuit may be uniquely expressed as a sum of fundamental circuits, and
- the fundamental circuits may be found algorithmically.

We consider a simplified model of the heat exchanger, which retains impacting behaviour, but does not attempt to model the dynamics of the fluid flow in detail. For definiteness, we restrict ourselves to one space dimension (although there is no bar to this method being extended to higher space dimensions). The vibrating pipes may be modelled most simply by a 'bouncing ball' system (similar to [28]), in which n masses oscillate parallel to the x-axis; see figure 1. Their equilibrium

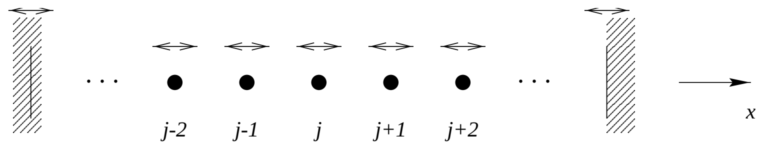


Figure 1: Simple one space dimensional model of heat exchanger

positions are uniformly spread (at a distance d) along this axis. The pipes can only impact nearest neighbours. Between impacts, we model the displacement of the jth pipe, x_j, to be governed by the differential equation

$$m_j \frac{\mathrm{d}^2 x_j}{\mathrm{d}t^2} + \omega_j^2 (x_j - jd) = F_j(t) \tag{2}$$

where m_j is the mass per unit length of the jth pipe, and ω_j is its natural frequency. The forcing is supplied by the influence of the surrounding fluid, which we assume to be random, to capture its complicated behaviour. We may expand F_j as a Fourier series with random coefficients

$$F_j(t) = \sum_{i=1}^{\infty} \alpha_{i,j} \cos \Omega_i t + \beta_{i,j} \sin \Omega_i t \tag{3}$$

where $\alpha_{i,j}$ and $\beta_{i,j}$ are independent random variables; an approach which has been successfully used in other random systems [37, 55, 56]. Damping is provided by impacts between pipes, where we apply Newton's experimental law of restitution.

We now describe a graph representing the periodic motions in the heat exchanger model. The vertices of the graphs correspond to the the discontinuity surfaces in the dynamical system. Therefore, in the heat exchanger model, vertices represent impacts; the edges of the graph are the possible phase space trajectories between them. In a one dimensional heat exchanger with n pipes, there are $n+1$ possible impacts; namely $n-1$ pipe-pipe impacts, and 2 pipe-wall impacts. Since vertices in the graph correspond to impacts in the dynamical system, a graph for a 1-d heat exchanger model with n pipes has $n+1$ vertices. Moreover, there is no restriction on the order of impacts: after an impact (say) between pipes j and $j+1$, the next impact may be between pipes j and $j+1$ again, or any of the other pairs of pipes. Thus, starting at any one vertex, there must be an edge leading to each and every other vertex (including itself). So the graph representing the periodic motions in the dynamical system is *fully connected*; that is there is an edge from each vertex to every other vertex. A fully connected graph with $(n+1)$ vertices may equivalently be represented by an $(n+1) \times (n+1)$ adjacency matrix with all entries equal to one. An picture of such a graph for a system with three pipes is shown in figure 2.

As mentioned above, circuits in this graph correspond to periodic orbits in the dynamical system. Therefore, a circuit of length k represents a periodic orbit with k impacts per period, and a circuit that passes through l distinct vertices means that there are $l+1$ diffrerent pipes undergoing repeated impact.

The graph described here represents an improvement to our previous approach [27]. Higher order circuits, where there is more than one impact per pipe per period, are represented easily as subgraphs, with no modification of the graph. The counting approach presented in [27], while exploiting the symmetry of the system, is difficult to modify to include orbits with arbitrary numbers of impacts or arrays of pipes.

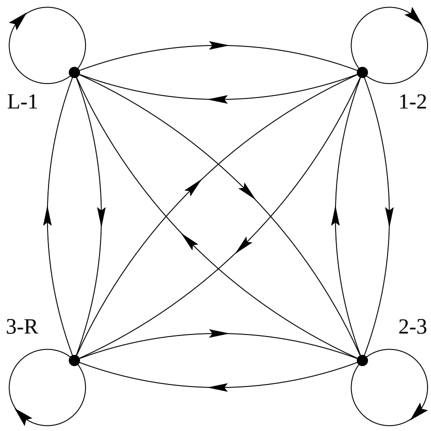


Figure 2: Graph describing the periodic motions in a heat exchanger model with three pipes. L-1 denotes an impact between the left hand wall and first pipe, etc.

It is, however, straightforward to extend the method described above to represent a two dimensional array of pipes. We need only prescribe and count the possible impacts; since any sequence of these possible impacts is theoretically permissible, the graph of the system will be fully connected, with the number of vertices equal to the number of possible impacts. Say, for example that the pipes are arranged on a N by N square lattice, and that the pipe width to spacing ratio permits impacts only between pipe (i, j) and its eight nearest neighbours. Then there are $2(2N-1)(N-1)$ possible pipe-pipe impacts, and $4N$ possible pipe-wall impacts, and so $4N^2 - 2N + 2$ possible impacts. Thus the graph representing this system is fully connected, with $4N^2 - 2N + 2$ vertices. We shall only consider the one dimensional case in this paper, which shows many interesting phenomena that we believe to be generic, while remaining analytically and numerically tractable.

With this new representation of the problem we are in a position to try and understand the impacting dynamics of the system. Periodic orbits in the heat exchanger correspond to proper circuits in the graph, which we can find algorithmically. If we assume that all the circuits are equally likely to occur, we can form a distribution of the number of impacting pipes, and discover what is the most likely to occur. Furthermore we can also introduce the effect of turbulent motion affecting each pipe, and take account of varying parameters in the system, by arguing that impacts occur with a certain probability $p \in [0, 1]$, which we can associate with each edge of the graph. This leads to a non-uniform distribution of circuits in the graph, and hence a variation of the most likely number of impacts as p varies. Finding a distribution of circuits in the graph relies upon counting and classifying the circuits, a task we shall describe in the next section.

3 Counting simple circuits

Our first step is to count the circuits in the graph. We divide the circuits into classes to facilitate this task. The first class we consider are the simple circuits, where no vertex in the graph is visited more than once. We shall return to the subject of circuits that pass through a vertex more than once in section 8.

There are numerous ways that the simple circuits may be enumerated. We may use the fundamental circuits, as in our previous paper [27], or the property that the diagonal entries of A^k (where A is the adjacency matrix), $[A^k]_{ii}$, are the number of paths of length k starting and finishing at vertex i. Alternatively, we may exploit the nature of the particular graph for the heat exchanger to our advantage, and use combinatoric techniques to count the circuits.

In a fully connected graph with N vertices, the number of distinct paths of length k with initial and final vertices i is

$$(N-1)(N-2)\dots(N-(k-1)) = \frac{(N-1)!}{(N-k)!}, \tag{4}$$

since to complete a valid path, we must choose $k-1$ additional vertices, all different. Thus the number of distinct simple circuits of length $k \in \{1, \dots, N\}$ in a graph with N vertices, $\phi_N(k)$, is given by

$$\phi_N(k) = \frac{N!}{k(N-k)!}, \tag{5}$$

since there are N choices of the initial vertex i, with each circuit repeated k times (once for each vertex along its length). Each of these circuits corresponds to a possible periodic orbit with $k+1$ pipes undergoing repeated impact.

If N is large, the ratio

$$\frac{\phi_N(k)}{\phi_N(N)} = \frac{N}{k(N-k)!} \ll 1 \tag{6}$$

for all $k \lessapprox N$. Thus the huge majority of circuits have length $\approx N$, the size of the whole system. This effect is demonstrated in figure 3; a graph of the relative number of simple circuits versus length for a fully connected graph with 100 vertices. Were all these circuits equally likely, we would

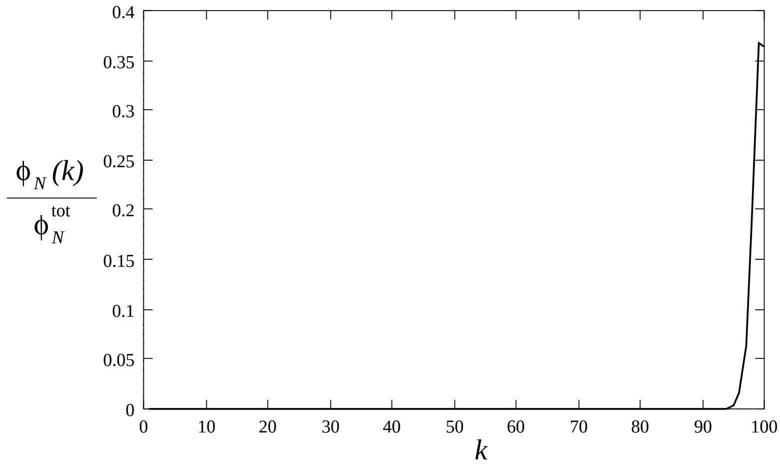


Figure 3: The number of simple circuits plotted against length for a fully connected graph with 100 vertices. The ordinate is normalised by the total number of simple circuits, $\phi_N^{\rm tot} = \sum_{j=1}^{N} \phi_N(j)$

expect to see only periodic orbits where almost every pipe undergoes impact. Experimental and numerical evidence show that this is not the case, but that there is a sudden jump from no to many impacts as the fluid forcing is increased. In the next section we shall attempt to explain this phenomenon by including the effect of turbulent motion forcing each pipe to vibrate.

4 Introducing probability

We shall now consider a method to include the effect of a turbulent forcing on each pipe, and hence a parameter variation, into the graph theory approach, and show how this provides a mechanism for a sudden jump in the number of impacts.

We assume that the forcing function $F_j(t)$ is random and may be expanded as a Fourier series with random coefficients, and takes the form of equation (3). Therefore we know some statistics of the forcing but not its explicit functional form. This approach has been used successfully to study random fluctuations, for example of water waves [37] or noise [55, 56]. Including a random forcing function naturally prompts the introduction of probability theory.

Linking probability theory and graph theory is well known in the areas of percolation theory and random graphs [22, 49, 58]. Both exhibit the phenomenon of a sudden jump from small to large scale behaviour as a parameter is varied continuously, motivating our hybrid method.

We associate a weight $p_{ij} \in (0, 1)$ with each edge of the graph. Since edges represent a transition between impacts, we can think of p_{ij} as the probability of an impact between pipes i and j in the heat exchanger. We can then form a probabilistic adjacency matrix, A_p, with entries p_{ij} if there is an edge from vertex i to vertex j with weight p_{ij}, and zero otherwise. We shall assume for simplicity that the p_{ij} are equal to some constant value p. We also assume that impacts occur independently, so that having an impact at any particular pipe does not affect the location of the next impact. This is an obvious area for further work; perhaps the occurrence of an impact might increase the probability of an impact locally, or that pipe-wall impacts are less likely than pipe-pipe impacts. By examining the data from our numerical simulations (which we describe in more detail in section 7), these assumptions seem reasonable at least as a first guess; the occurrence of an impact between a pair of pipes does not seem to greatly influence the location of the next impact. Despite these seemingly restrictive assumptions, we are able to find behaviour in the graph theory method that is qualitatively similar to that of experimental and numerical observations.

Having counted the circuits in the previous section, we can now go on to find the relative likelihood of each circuit, and hence a distribution of number of impacts. Since we assume impacts occur independently, the probability of observing any particular circuit of length k is proportional to p^k. The proportionality constant will be determined by the set of circuits we choose to form the sample space. In the next section we take the simple circuits as the sample space, and derive a probability distribution function for the number of impacts in the heat exchanger.

For the moment, p will be an arbitrary parameter that we can vary at will. Small p corresponds to a low forcing on each pipe, large p to strong forcing. We will discuss how to relate p to measurable quantities in the model in section 7.

5 Distribution functions

We have already found the number of simple circuits of length k in a graph with N vertices, $\phi_N(k)$ (recall that a graph with N vertices corresponds to a heat exchanger with $N - 1$ pipes). Using the independence assumption, the probability of a simple circuit of length k, that is with k different pipe-pipe impacts per period, is proportional to p^k multiplied by $\phi_N(k)$. Initially, we take the sample space to be the simple circuits, so that if X is the number of different pipe-pipe impacts, it has the distribution

$$P(X=k) = \frac{p^k \phi_N(k)}{\sum_{j=1}^{N} p^j \phi_N(j)} \tag{7}$$

We label the numerator of this expression as $\Phi_{N,p}(k)$ and the denominator as $\Gamma_{N,p}$, so that

$$\Phi_{N,p}(k) = \frac{p^k N!}{k(N-k)!}, \qquad \Gamma_{N,p} = \sum_{j=1}^{N} \frac{p^j N!}{j(N-j)!}. \tag{8}$$

Note that the normalising factor $\Gamma_{N,p}$ ensures that $\sum_{k=1}^{N} P(X=k)$ is equal to one.

Figure 4 shows graphs of the probability distribution (7), so the abscissa is the number of different pipe-pipe impacts, for various values of the parameter p. It appears that for small values of p we expect to see very few impacts, while for very small increase in p the number of impacts increases rapidly towards the size of the whole system, where each pipe undergoes an impact.

To clarify this variation, we study the most likely number of impacting pipes, $k^\star$: the value of k which maximises $\Phi_{N,p}$ for fixed N and p, or the mode of the distribution X. Figure 5 shows a plot of the most likely number of impacting pipes plotted against p. We do indeed see a very rapid rise in the most likely number of impacts as p increases. Note that $k^\star(p)$ is a step function, since the most likely number of impacting pipes is integer valued. Closer investigation of the small p region suggests a large discontinuity (i.e. much larger than 1) in this curve, in qualitative agreement with experiment and numerical simulations. In the next section we shall prove that this sudden jump in the number of impacting pipes does indeed exist.

6 Discontinuity in most likely number of impacts

In order to prove that there is a sudden jump in the most likely number of impacting pipes, we must find $k^\star$ as a function of p. Since $k^\star$ is the location of the maximum of $\Phi_{N,p}$ as a function of k, we investigate the difference between two neighbouring points on the curve $\Phi_{N,p}(k)$:

$$\Phi_{N,p}(k+1) - \Phi_{N,p}(k) = \frac{p^k N!}{(N-k)!k(k+1)} \left[-pk^2 + (Np-1)k - 1\right]. \tag{9}$$

Thus $\Phi_{N,p}$ increases or decreases as the quadratic polynomial

$$f_{N,p}(k) = -pk^2 + (Np-1)k - 1 \tag{10}$$

is positive or negative respectively, and so the turning points of $\Phi_{N,p}$ correspond to the zeros of $f_{N,p}$, given by

$$k_\pm = \frac{1}{2p}\left[(Np-1) \pm \sqrt{(Np-1)^2 - 4p}\right]. \tag{11}$$

We also have that

$$f_{N,p}(0) = -1 \tag{12}$$

$$f_{N,p}(N) = -(N+1). \tag{13}$$

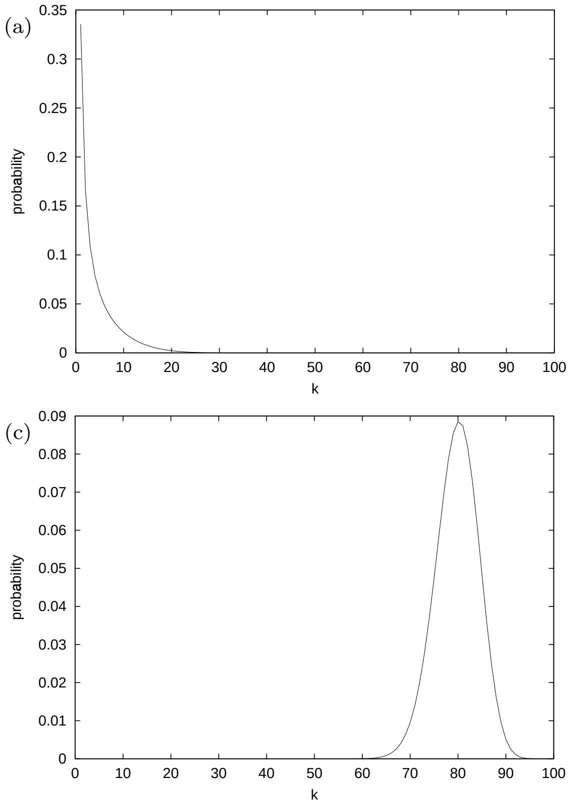


Figure 4: Distributions of the number of impacting pipes (k) in a heat exchanger with 100 pipes, with parameter values (a) $p = 0.01$, (b) $p = 0.02$, (c) $p = 0.05$ and (d) $p = 0.1$.

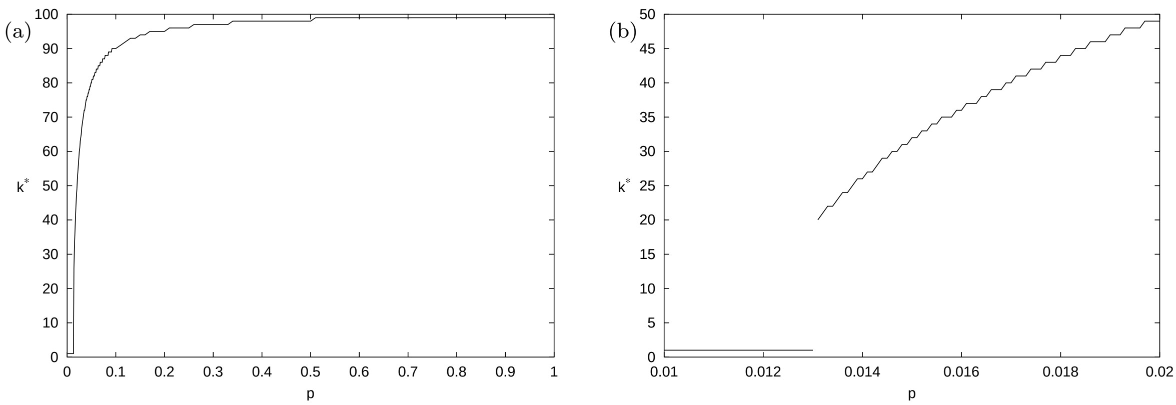


Figure 5: (a) Most likely number of impacting pipes ($k^\star$) plotted against p, for a heat exchanger with 100 pipes, and (b) blowup of small p region

Moreover

$$f_{N,0}(k) = -(k+1) < 0 \tag{14}$$

and

$$\frac{\partial}{\partial p} f_{N,p}(k) = k(N-k) > 0 \tag{15}$$

for all $k \in (0, N)$, so for fixed N and k, $f_{N,p}(k)$ increases through zero as p increases from zero.

Therefore at $p = 0$, $f_{N,p}$ is negative for all $k \in (0, N)$, thus $\Phi_{N,p}$ is monotone decreasing, and the most likely number of impacting pipes is 1. As p increases to

$$p = \left(\frac{1+\sqrt{N+1}}{N}\right)^2, \tag{16}$$

a repeated root appears in the interval $k \in (0, N)$; the other root of p which renders the discriminant of equation (11) equal to zero corresponds to a repeated root with $k < 0$. At this value of p, $\Phi_{N,p}$ has a stationary inflection, so the maximum of $\Phi_{N,p}$, and hence the most likely number of impacts, is still 1. Increasing p further produces two zeros of $f_{N,p}$ at $k = k_\pm$; hence $\Phi_{N,p}$ has a local minimum at $k_{\min} = [k_-]$ and a local maximum at $k_{\max} = [k_+]$, where $[x]$ denotes the largest integer greater than x. The global maximum is then either at $k = 1$ or $k = k_{\max}$; at some point the global maximum will switch to $k = k_{\max}$, giving rise to the sudden jump, as required.

This behaviour is perhaps best explained pictorially: figure 6 shows a sequence of sketches of $f_{N,p}$ and $\Phi_{N,p}$ as p increases from zero. Recall that the most likely number of impacting pipes, $k^\star$, is the value of k which maximises $\Phi_{N,p}$. In figure 6(a), $p = 0$, thus $f_{N,p}(k)$ is always negative, $\Phi_{N,p}$ is decreasing, and so the most likely number of impacting pipes is one. Increasing p slightly leads to a repeated root of $f_{N,p}$ (shown in figure 6(b)), so $\Phi_{N,p}$ has a stationary inflexion, but is still decreasing, and hence the most likely number of impacts is still 1. Increasing p further, as shown in figure 6(c), leads to $f_{N,p}$ having two roots, so $\Phi_{N,p}$ has a local minimum and a local maximum. Initially, the value of $\Phi_{N,p}$ at the local maximum is less than at $k = 1$, and so the most likely number of impacts is still 1. As p increases further, there is some critical value at which the local maximum of $\Phi_{N,p}$ becomes a global maximum, at this value of p there is a sudden jump in the most likely number of impacting pipes to a value much larger than 1, as shown in figure 6(d).

To summarise, despite the many assumptions we have made, our simple graph theory and probability method shows the existence of a sudden jump in the most likely number of impacting pipes, in qualitative agreement with experimental evidence. We believe this is the first time that such a jump has been demonstrated in any theoretical model of heat exchanger dynamics. We would now like to discover if the method can provide any quantitative agreement with our numerical simulations of the heat exchanger model, which we describe in the next section.

7 Numerical simulations

We now wish to test the theoretical prediction of the occurrence of a sudden jump against numerical simulations. In our simulations we choose the simplest possible parameters, so that the differential equation governing the trajectory of pipe j between impacts is

$$\frac{\mathrm{d}^2 x_j}{\mathrm{d}t^2} + (x_j - j) = F_j(t) \tag{17}$$

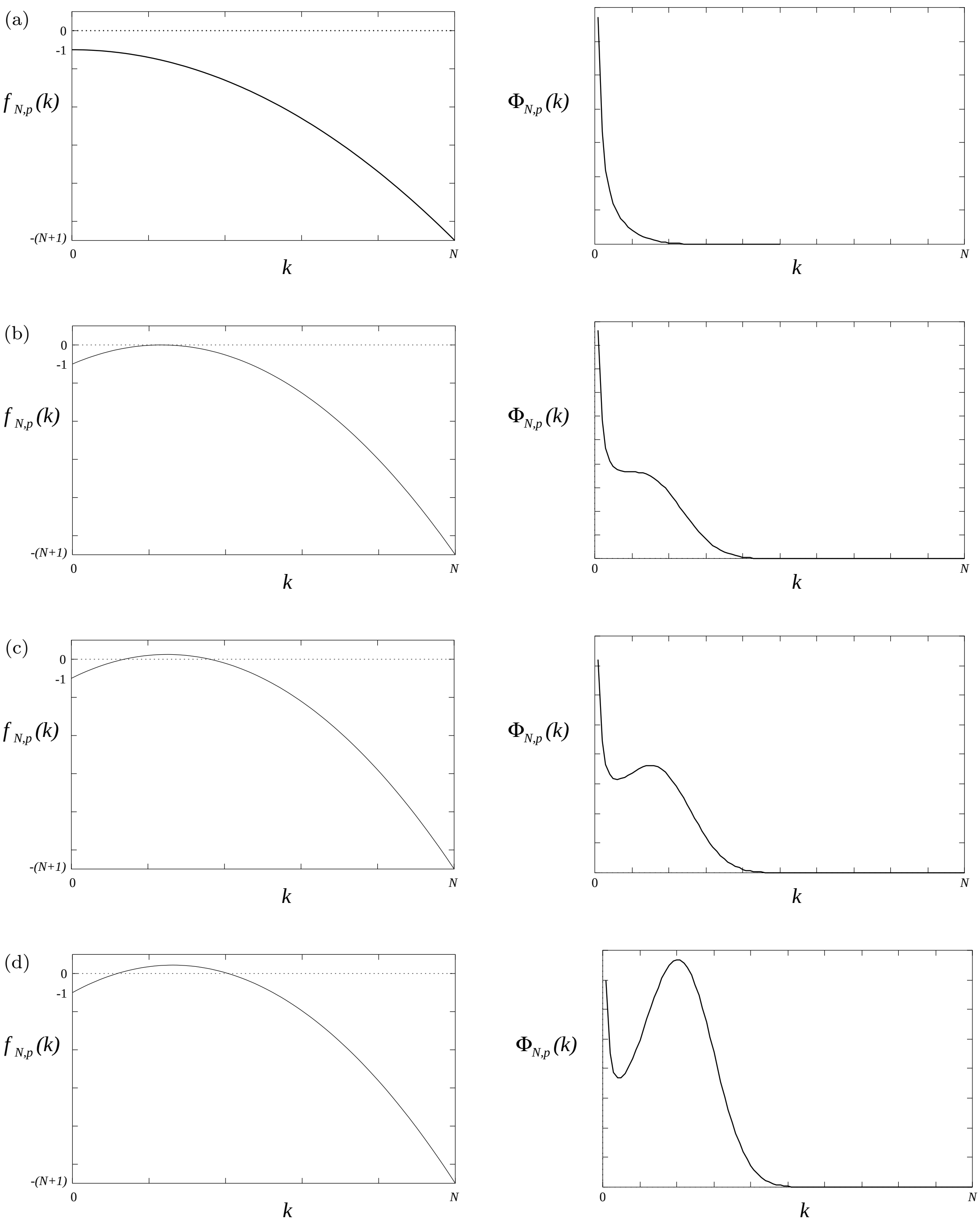


Figure 6: A mechanism for the sudden jump in most likely number of impacting pipes, with impact probability increasing from top to bottom

and the random forcing has just one component

$$F_j(t) = \alpha_{1,j} \cos \Omega t + \beta_{1,j} \cos \Omega t \tag{18}$$

where $\alpha_{1,j}$ and $\beta_{1,j}$ are identically distributed independent normal random variables, with zero mean and variance σ^2, and we choose $\Omega = \sqrt{2}$. We use the impact map to simulate the system of equations (17) with 100 pipes (taking the two end walls to be indistinguishable from the pipes), and record the number of pipes undergoing impacts after a suitable transient, solutions with no impacts after the transient are discarded; for further details see [29]. Figures 7 and 8 show the simulation data: plots of the most likely and expected number of impacting pipes versus forcing σ. We see that for small forcing there are very few impacts, but as the forcing is increased past a

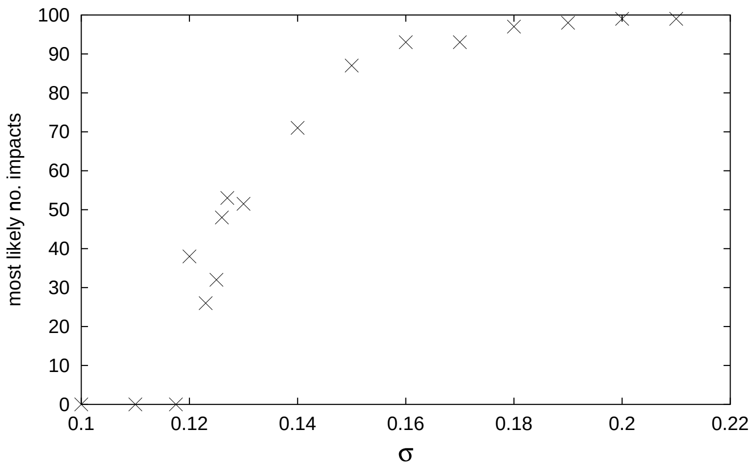


Figure 7: Numerical simulation data: the most likely number of pipes undergoing impact versus forcing (σ) for numerical simulation of the heat exchanger model

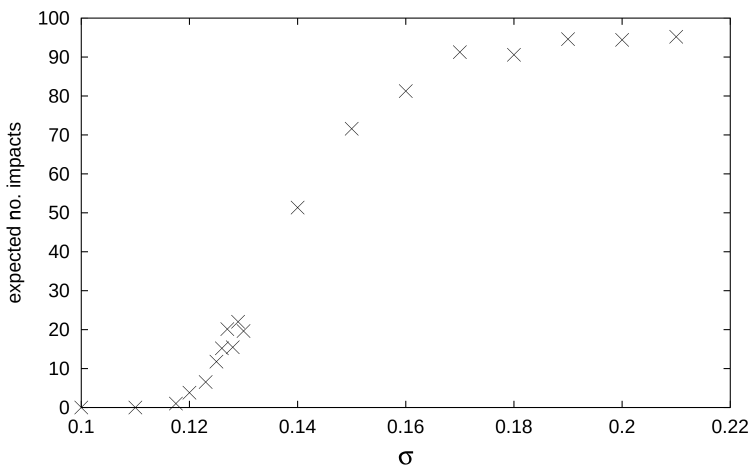


Figure 8: Numerical simulation data: the expected number of pipes undergoing impact versus forcing (σ) for numerical simulation of the heat exchanger model

threshold ($\sigma \approx 0.12$) the number of impacts increases very rapidly. Figure 7 shows a sudden jump in the most likely number of impacts, as predicted by our theory, and observed in other simulations and in physical experiments [42]. Before we can make any quantitative comparisons, however, we must find a way to compute p from the parameters in the physical model: equilibrium pipe-pipe spacing, frequency and amplitude of the forcing, natural frequency of oscillation of the pipes and coefficient of restitution, for example. We describe a possible candidate for this relationship below.

Recall that the differential equation governing the motion of pipe j is

$$\ddot{x}_j + \omega^2(x_j - jd) = F_j(t) \tag{19}$$

where d is the equilibrium pipe spacing, and the forcing F_j is given (in general) by

$$F_j(t) = \sum_{i=1}^{\infty} \alpha_{i,j} \cos \Omega_i t + \beta_{i,j} \sin \Omega_i t \tag{20}$$

with coefficients $\alpha_{i,j}$ and $\beta_{i,j}$ independent random variables. Since these coefficients are independent of time, we may solve equation (19) to show that the separation of pipes j and $j+1$, $\Delta_j(t) = x_{j+1}(t) - x_j(t)$, is given by

$$\begin{aligned}\Delta_j(t) = d + (\alpha_{0,j+1} - \alpha_{0,j}) \cos \omega t + (\beta_{0,j+1} - \beta_{0,j}) \sin \omega t \\ + \sum_{i=1}^{\infty} \frac{\alpha_{i,j+1} - \alpha_{i,j}}{\omega^2 - \Omega_i^2} \cos \Omega_i t + \frac{\beta_{i,j+1} - \beta_{i,j}}{\omega^2 - \Omega_i^2} \sin \Omega_i t\end{aligned} \tag{21}$$

The quantities $\alpha_{0,j}$ and $\beta_{0,j}$ are related to the initial conditions of pipe j; we shall assume these are also random variables (and we shall integrate over all possible initial conditions).

Now the probability of not having an impact, $1-p$, is just the probability that $\Delta_j > 0$ for all $t > 0$, that is $P(\inf_{t>0} \Delta_j > 0)$, which gives

$$1 - p = P\left(Y_0 + \sum_{i=1}^{\infty} \frac{Y_i}{|\omega^2 - \Omega_i^2|} < d\right) \tag{22}$$

(assuming the summation converges) where Y_i is defined to be

$$Y_i = \sqrt{(\alpha_{i,j+1} - \alpha_{i,j})^2 + (\beta_{i,j+1} - \beta_{i,j})^2}. \tag{23}$$

Equation (22) may be written in terms of the density functions of all the random variables $\alpha_{i,j}$ and $\beta_{i,j}$ with the aid of standard identities from probability theory [23]. For example, if all the random coefficients $\alpha_{i,j}$ and $\beta_{i,j}$ above are normally distributed with zero mean and variance σ_i^2, then the probability density function of Y_0 is given by

$$f_{Y_0}(x) = \frac{x}{2\sigma_0^2} \exp\left(-\frac{x^2}{4\sigma_0^2}\right) H(x) \tag{24}$$

and that for $Y_i/|\omega^2 - \Omega_i^2|$ by

$$f_{Y_i/|\omega^2-\Omega_i^2|}(x) = \frac{(\omega^2 - \Omega_i^2)^2 x}{2\sigma_i^2} \exp\left(-\frac{(\omega^2 - \Omega_i^2)^2 x^2}{4\sigma_i^2}\right) H(x) \tag{25}$$

where $H(x)$ is the Heaviside function. Hence it is possible to obtain an equation for p in terms of the parameters in the system, $\{\sigma_i\}$, ω, $\{\Omega_i\}$ and d. In our numerical simulation, we choose the simplest possible random forcing, with only one component (so $Y_i = 0$ for all $i \geqslant 2$), and so equation (22) reduces to

$$1 - p = \int_{-\infty}^{d} \int_{-\infty}^{\infty} f_{Y_0}(t) f_{Y_1/|\omega^2-\Omega_1^2|}(x-t)\, dt\, dx \tag{26}$$

that is

$$p = 1 - \frac{(\omega^2-\Omega_1^2)^2}{4\sigma_0^2\sigma_1^2} \int_0^d \int_0^x t(x-t) \exp\left(-\frac{t^2}{4\sigma_0^2}\right) \exp\left(-\frac{(\omega^2-\Omega_1^2)^2(x-t)^2}{4\sigma_1^2}\right) dt\, dx \tag{27}$$

We show in figure 9 the curve $p = p(\sigma)$ given by equation (27), showing how the impact probability p varies with the standard deviation of the forcing and initial conditions σ for the parameter values used in the numerical simulation, namely $\sigma = \sigma_1 = \sigma_2$, $\omega = 1$, $\Omega_1 = \sqrt{2}$ and $d = 1$.

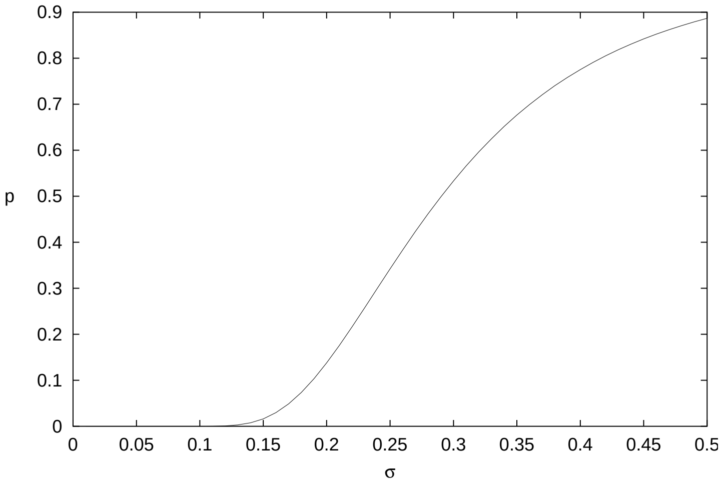


Figure 9: The relationship between the standard deviation of pipe forcing (σ) and impact probability (p).

We are now able to make a quantitative comparison of the predictions of the probabilistic graph theory method and the results of our numerical simulations, with the aid of the above relationship. Figure 10 shows a plot of the expected number of impacts plotted against impact probability, for both the predictions of the graph theory method and the results of our numerical simulations. We plot *expected* number of impacts (rather than the most likely number), since the mean is a much easier statistic to compute reliably. For a system with 100 pipes, all having random initial conditions and forcing, we would have to perform hundreds of thousands of separate runs for each value of σ accurately to predict the most likely number of impacts, particularly if the distribution is not strongly unimodal.

It is clear from figure 10 that we have a qualitative agreement between theory and experiment: both graphs have a similar shape, particularly for larger values of p. The numerics seem to predict impacts for extremely small p, while the theory does not. The value $p(\sigma)$ is extremely small for $\sigma < 0.15$, and we suspect our method of estimating p is poor here. The similarity is encouraging,

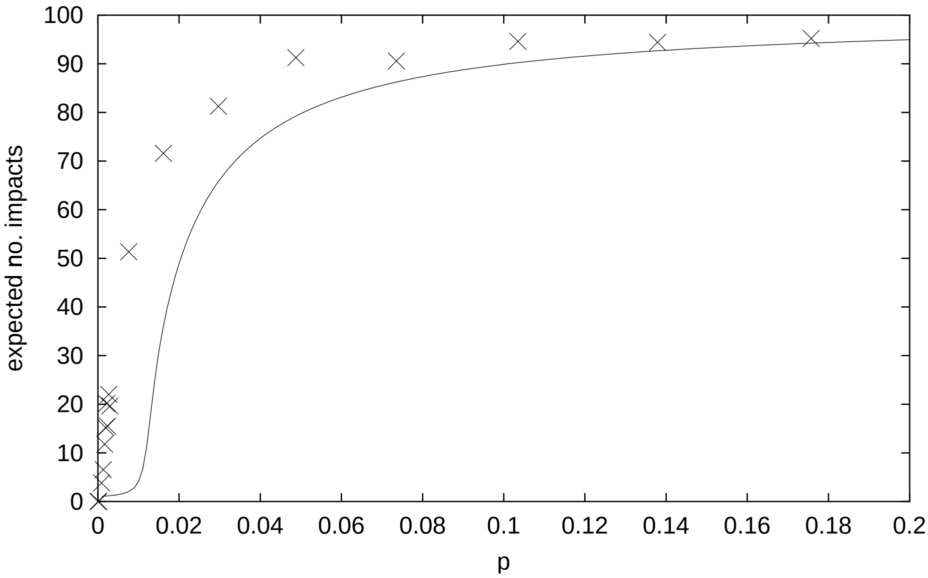


Figure 10: Comparison of expected number of impacting pipes for graph theory predictions (solid line) and numerical simulations ($\times$).

however, since so many factors were neglected to use the probabilistic graph theory method: non-independence of impacts, the existence of aperiodic and possibly chaotic solutions, not to mention higher order periodic solutions. We shall now seek to relax this last assumption, in the hope that we can improve the fit.

8 Higher order circuits

One important restriction of the graph theory method is that we have considered only simple circuits; that is periodic orbits in which each pipe undergoes at most one impact per period. Clearly this is not a realistic assumption; we would expect more complicated behaviour to dominate, particularly as the forcing is increased. We shall now consider the possibility of including higher order circuits in our method.

Once again we shall exploit the fact that the heat exchanger graph is fully connected to count higher order circuits. First we consider the class of circuits in which exactly one vertex is visited twice, all other vertices being visited no more than once; let the number of such circuits of length k in a graph with N vertices be $\phi_N^1(k)$ (so $2 \leqslant k \leqslant N+1$); each of these has $k-1$ different repeated pipe-pipe impacts.

Initially let vertex i be visited twice. Then we seek circuits which start at vertex i, visit γ_1 distinct vertices (via a path of length $\gamma_1 + 1$), vertex i again, then γ_2 more distinct vertices (a path length $\gamma_2 + 1$), and finally return to vertex i; for a circuit of length k, we have the constraint

$$(\gamma_1 + 1) + (\gamma_2 + 1) = k. \qquad (28)$$

and so the number of such circuits is

$$\begin{aligned}
&(N-1)(N-2)\dots(N-\gamma_1)\cdot 1\cdot(N-(\gamma_1+1))(N-(\gamma_1+2))\dots(N-(\gamma_1+\gamma_2)) \\
&=(N-1)(N-2)\dots(N-(\gamma_1+\gamma_2)) &(29)\\
&=(N-1)(N-2)\dots(N-(k-2)) &(30)\\
&=\frac{(N-1)!}{(N-(k-1))!} &(31)
\end{aligned}$$

Thus the total number of circuits of length k, passing through vertex i twice, and no other vertex more than once is

$$\phi_N^1(k)=\frac{1}{2}\sum\sum_{\substack{\gamma_1,\gamma_2\in\mathbb{N}\\ \gamma_1+\gamma_2=k-2}}\frac{(N-1)!}{(N-(k-1))!}=\frac{\gamma(k,2)}{2}\frac{(N-1)!}{(N-(k-1))!} \tag{32}$$

The factor of $\frac{1}{2}$ arises because each circuit is counted twice; $\gamma(n,j)$ is the number of solutions of the equation

$$\gamma_1+\gamma_2+\cdots+\gamma_j=n \tag{33}$$

where $\gamma_i\in\mathbb{N}^+$ for all i. This is a standard quantity in combinatoric theory [60]:

$$\gamma(n,j)=\binom{n-1}{j-1}=\frac{(n-1)!}{(j-1)!(n-j)!} \tag{34}$$

Thus

$$\phi_N^1(k)=\frac{1}{2}(k-1)\frac{N!}{(N-(k-1))!} \tag{35}$$

This expression is valid for $k>2$: for $k=2$ there is no repeated counting, and so

$$\phi_N^1(2)=N \tag{36}$$

By a similar method, we may count other classes of higher order circuits. We label the number of circuits of length k in a fully connected graph with N vertices where exactly δ_j vertices are visited $j+1$ times, all other vertices visited at most once (omitting leading zeros in the exponent) as

$$\phi_N^{\dots\delta_4,\delta_3,\delta_2,\delta_1}(k) \tag{37}$$

For example the number of distinct circuits of length $k>3$ where one vertex is visited three times, and no other more than once, $\phi_N^{1,0}(k)$, is

$$\phi_N^{1,0}(k)=\frac{1}{6}\frac{(k-1)!}{(k-3)!}\frac{N!}{(N-(k-2))!}, \tag{38}$$

the number of distinct circuits of length $k>4$ where one vertex is visited four times, and no other more than once, $\phi_N^{1,0,0}(k)$, is

$$\phi_N^{1,0,0}(k)=\frac{1}{24}\frac{(k-1)!}{(k-4)!}\frac{N!}{(N-(k-3))!}, \tag{39}$$

the number of distinct circuits of length $k>4$ where two vertices are visited twice, and no other more than once, $\phi_N^2(k)$, is

$$\phi_N^2(k)=\frac{1}{8}\frac{(k-1)!}{(k-4)!}\frac{N!}{(N-(k-2))!} \tag{40}$$

It is now easy to form distribution functions for the number of pipes undergoing impact, where the sample space is extended to include higher order circuits. Once again we assume impacts occur independently, thus the probability of a circuit length k is simply p^k multiplied by the number of such circuits, up to a normalising factor. We choose to consider a sample space made up of the five classes of circuit we have counted, that is the simple circuits, and the four higher order circuits above. So the number of circuits with k different pipe-pipe impacts per period in this case is

$$\phi_N(k) + \phi_N^1(k+1) + \phi_N^{1,0}(k+2) + \phi_N^{1,0,0}(k+3) + \phi_N^2(k+2) \tag{41}$$

Figure 11 shows plots of the distribution of the number of impacting pipes, and their contributions from the five terms of the sum, for various values of p. These diagrams show an extremely

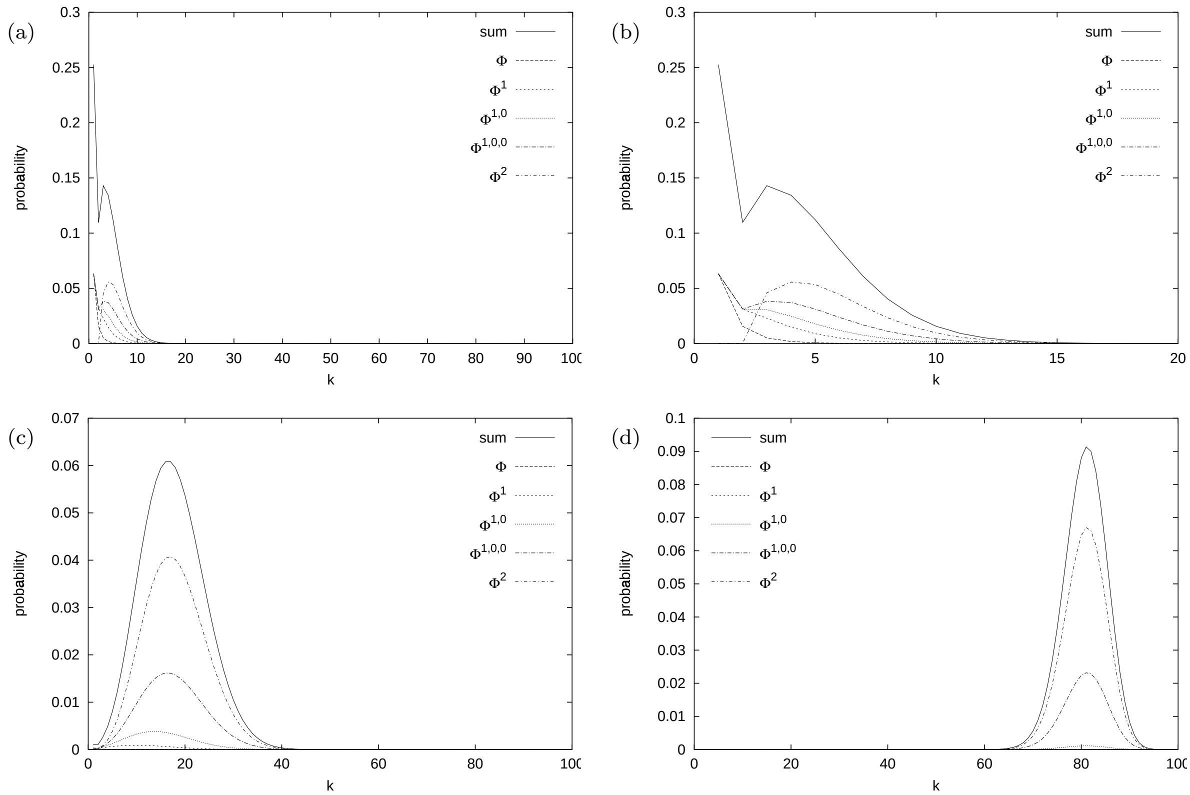
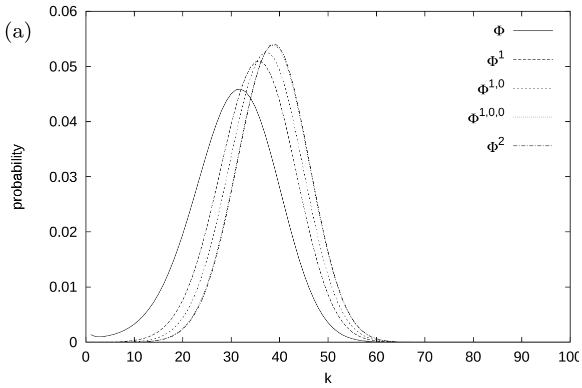


Figure 11: Distributions of the number of impacting pipes (k) with sample space containing higher order circuits (solid lines) for a heat exchanger with 100 pipes, and (a) $p = 0.005$*, (b)* $p = 0.005$*, detail of small k region, (c)* $p = 0.01$*, (d)* $p = 0.05$*. Also shown are the probabilities of observing the various classes of circuits.*

interesting effect. Even for very small p, higher order circuits appear to be significant, and as p increases, the higher order circuits dominate, and the probability of observing a simple circuit becomes insignificant. Despite this, however, the shape of each distribution becomes essentially identical as p increases, as shown in figure 12, and thus the expectation of the simple circuit distribution alone is a very good approximation to that of the sum. This explains why we have good agreement between theory and experiment for large p; adding higher order circuits does not significantly alter the expected circuit length. For small p, however, the expected circuit length



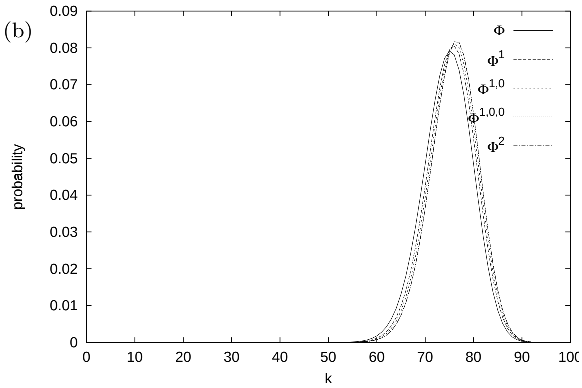


Figure 12: Convergence of simple and higher order circuit distributions as p increases; (a) $p = 0.015$, (b) $p = 0.04$.

is measurably different, perhaps going some way to account for the poor correspondence between the simple theory and numerics in this regime. The behaviour of the most likely circuit length will change also: we expect a sudden jump as before, but at a different critical probability. Figure 13 shows the change in the expectation of the circuit length, together with our numerical simulations. We do indeed see the most change for small p, and improved agreement.

9 Conclusions

In this paper we have described an application of the graph theory method of [27], to enable the prediction of large scale properties of a simple model of a impact dynamics in a heat exchanger as local parameters are varied.

Motivated by ideas from percolation theory and random graphs, we have introduced a hybrid method linking graph theory and probability theory, in which we associate with each edge in the graph a weight p, which can be thought of as the probability of an impact. This has led to distribution functions for the simple circuits. We can then examine how the distributions change as we vary p. We then plot the mode of the distribution, which corresponds to the most likely circuit length $k^\star$, against p, and prove that as p is increased from zero there is a discontinuity where $k^\star$ jumps from zero, in qualitative agreement with experimental observations and numerical simulation.

We then go on to describe a possible mapping from parameters in the system to p, and hence compare expected circuit length predicted by the graph theory to the average number of impacting pipes observed in our numerical simulations, and show good agreement.

In an attempt to improve the agreement between theory and numerics, we extend the method to take account of higher order periodic orbits. We derive distribution functions for higher order circuits, and then form a sum of the simple and higher order circuit distributions. This demonstrates an improved fit to numerical simulations for small p, and explains why simple circuits alone provide good agreement to simulations for large p.

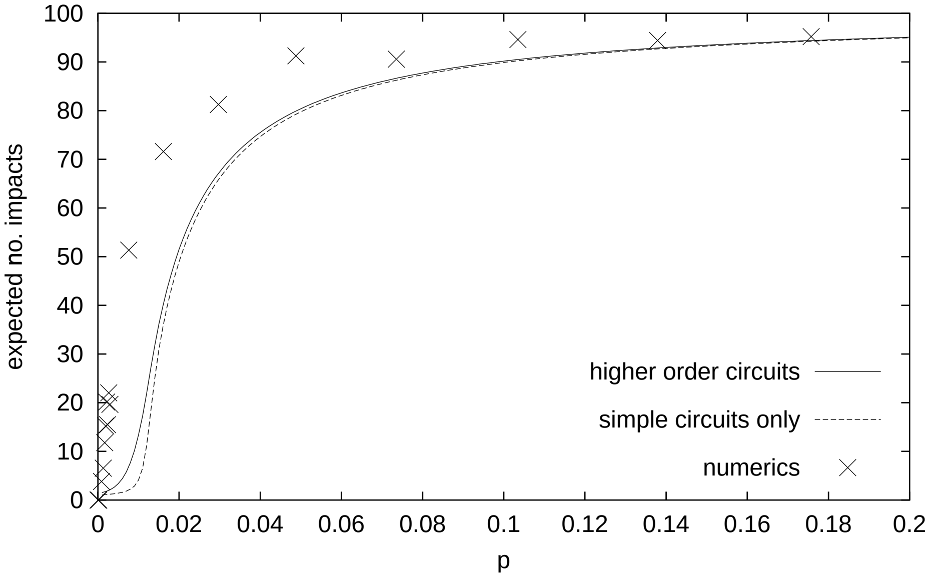


Figure 13: Comparison of expected number of impacting pipes for graph theory predictions including higher order circuits (solid line) and simple circuits only (dotted line), and numerical simulations (×).

There are many possible refinements and improvements to our method. We are keen to relax some of the many assumptions; allowing non-independence of impacts, or having different values of p for different edges in the graph, which is crucial to the possibility of applying the method to a two dimensional array of impacting pipes. There are undoubtedly better methods of computing p; it might be interesting to use a more realistic model of pipe vibration, and then compute p purely numerically. Most of all we are extremely keen to make a direct comparison with experimental data [42]. We hope this will help refine the graph theory method and lead to exciting improvements.

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