Kowalczyk, P. S., & di Bernardo, M. (2000). On the existence of stable asymmetric limit cycles and chaos in unforced symmetric relay feedback systems.

On the existence of stable asymmetric limit cycles and chaos in unforced symmetric relay feedback systems

P. Kowalczyk, M.di Bernardo
Department of Engineering Mathematics
University of Bristol, Bristol BS8 1TR, U. K.
Fax: +44-117-9251154
*E-mail:*P.Kowalczyk,M.diBernardo@bristol.ac.uk
(October 13, 2000)

We study the occurrence of bifurcations leading to the formation of asymmetric orbits and chaos in unforced relay feedback systems with a symmetric relay characteristic. Specifically, we show that novel bifurcation mechanisms cause the formation of stable periodic solutions with segments of sliding motion. We then characterise the formation of asymmetric stable periodic solutions and detail the route to chaos exhibited by a representative third-order example.
Keywords: Complex Systems, Discrete Event and Hybrid Systems, Bifurcations and Chaos.

I. BACKGROUND

Relay Feedback Systems are widely used in applied science and enginering to model a wide variety of physical devices. Early examples come from mechanical and electromechanical systems [1–3], while recent attention has been motivated by variable-structure controllers [4], supervisory switched control [5], relay methods for tuning controllers in process industry [6], and delta–sigma converters in signal processing [7].

The class of relay systems of our interest are characterised by being unforced, symmetric and without hysteresis. Hence, they are described by the following state space representation:

$$\dot{x} = Ax + Bu \tag{1}$$
$$y = Cx \tag{2}$$
$$u = -\mathrm{sgn}\, y, \tag{3}$$

where $x \in \mathcal{R}^n$, $y \in \mathcal{R}$, $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times 1}$, $C \in \mathcal{R}^{1\times n}$ and $\mathrm{sign}\, y = 1$ if $y > 0$, $\mathrm{sign}\, y = -1$ if $y < 0$, and $\mathrm{sgn}\, y \in [-1, 1]$ if $y = 0$.

We define the *switching hyperplane* as the manifold:

$$H = \{x \in \mathcal{R}^n : Cx = 0\}. \tag{4}$$

Such hyperplane defines a boundary in phase space between the two regions, $G_1 = \{x \in \mathcal{R}^n : Cx > 0\}$ and $G_2 = \{x \in \mathcal{R}^n : Cx < 0\}$ associated with the two different system configurations (see Fig. 1). Note that for every initial condition outside H, the state trajectory will eventually cross H if the steady-state gain of the system transfer function, $G(s)$, is positive ($G(0) > 0$) and G is stable [8].

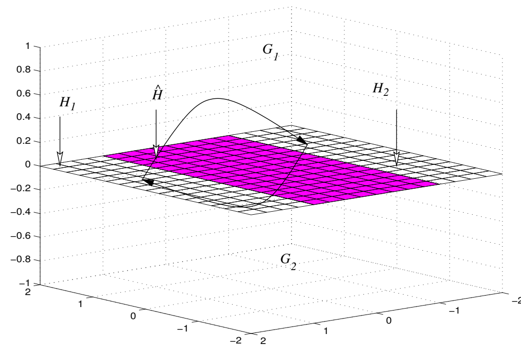


FIG. 1. Schematic representation of the phase space topology in the case of a third-order relay feedback system.

As shown in [9,4], systems like (1)-(3) can exhibit a peculiar type of solution termed *sliding* motion. This is characterised by lying within the switching hyperlane H and can be heuristically seen as associated to an infinite number of switchings between different system configurations. For this to occur, the system vector field in both G_1 and G_2 must point locally towards H. Hence, by studying the gradient of the system vector field in a neighborhood of H, we can identify a set $\widehat{H} \subset H$ where sliding is possible. For system (1)-(3) such *sliding region* can be defined as $\widehat{H} = \{x \in H : |CAx| < CB\}$ and only exists if $CB > 0$ (see [4] for the analytical details). In what follows we assume that this latter condition is indeed satisfied. The system dynamics within the sliding region can be studied by looking at an appropriate reduced order system. This can be obtained by applying Utkin's equivalent control method [4] or Filippov's convex method [9]. In particular, the state evolution under sliding motion is described by

$$\dot{x} = \widehat{A}x, \tag{5}$$

where $\widehat{A} = [I - (CB)^{-1}BC]A$ and I is the identity matrix.

According to the direction of the system vector field, one can also identify the two subsets $H_1 := \{x \in H :$

$CAx > CB\}$ and $H_2 := \{x \in H : CAx < -CB\}$ associated to trajectory leaving the switching hyperplane towards G_1 and G_2 respectively (see Fig. 1).

Although relay systems have been studied for more than a century, several problems remain unsolved. For instance, the global stability of periodic solutions is largely an open question [10,11,8,12]. Recently, the onset of bifurcations and chaos in these systems has also been studied. It is commonly assumed, though, that complex behaviour such as deterministic chaos is only present if the relay system under investigation is subject to an external forcing, say sinusoidal, and has either hysteresis or an asymmetric relay characteristic (see for example [13–15]). Similarly, a conjecture due to Tsypkin and reported in [3], p. 179, states that asymmetric solutions in relay feedback systems can only exist if the system is forced or has an intrinsic asymmetric relay characteristic.

In this paper, we show instead that symmetric unforced relay feedback systems can indeed exhibit asymmetric solutions and chaos. We present for the first time numerical evidence of the existence of *stable* asymmetric limit cycles whose bifurcations lead to stable chaotic dynamics. We find that sliding motion plays an important role in the formation of these solutions. In this sense, this paper expands and complete the work presented in [16], where the existence of stable asymmetric solutions and chaos in unforced, symmetric relay feedback systems was first conjectured.

For the sake of clarity, we will detail our presentation to the third-order relay feedback system studied in [16] where:

$$A = \begin{pmatrix} -(2\zeta\omega + \lambda) & 1 & 0 \\ -(2\zeta\omega\lambda + \omega^2) & 0 & 1 \\ -\lambda\omega^2 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} k \\ 2k\sigma\rho \\ k\rho^2 \end{pmatrix}, \tag{6}$$

$$C = (1 \quad 0 \quad 0), \tag{7}$$

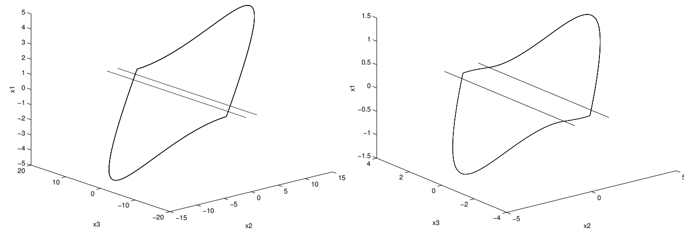
which corresponds to the transfer function

$$G(s) = k\frac{s^2 + 2\sigma\rho s + \rho^2}{(s^2 + 2\zeta\omega s + \omega^2)(s + \lambda)}.$$

It is relevant to point out that the analysis presented here can be applied, without major modifications, to other relay feedback configurations and switched dynamical systems where sliding solutions play an important role in organising the system dynamics [17].

II. PERIODIC ORBITS

Relay feedback systems, such as the one under investigation, tend to oscillate without external excitation and particular interest has been devoted to detect and study such self-oscillations, for example, by using describing function methods [3].



(a) (b)

FIG. 2. Phase space diagrams of periodic solutions for the third-order relay feedback system under investigation: non-sliding periodic solution for $\rho = 3$ (a), and sliding periodic solution for $\rho = 1$ (b). All the other parameters are set to $\omega = \zeta = \lambda = \kappa = -\sigma = 1$. The apparent change of the size of the sliding region, whose boundary is here represented by two solid lines, is only due to a change of scale.

Under certain conditions, it has been shown that periodic solutions of the system can connect the sliding set with itself, giving rise to so-called sliding orbits [8]. Some typical periodic solutions for the third order example considered in this paper, including a sliding orbit, are depicted in Fig. 2.

As shown in [16], to study the existence and stability of these periodic solutions one can introduce a set of appropriate Poincaré maps. For example in the case of orbits without sections of sliding motion (or *simple* orbits), one can construct the map, $\Pi : H \mapsto H$ from the switching plane back to itself by solving the system equations in each of the two phase space regions G_1 and G_2. In so doing, each of the system periodic solution will be associated to a corresponding fixed point, say x_0 of the Poincaré map Π. Notice that in the case of simple periodic orbits, Π is actually the composition of the two submappings $\Pi_1 : H_1 \mapsto H_2$ and $\Pi_2 : H_2 \mapsto H_1$ associated to motions in region G_1 and G_2 respectively.

Similarly, to analyze the system solutions characterized by sliding motion (sliding orbits), one can consider the additional *sliding* map $\Sigma : \widehat{H} \mapsto \partial\widehat{H}$, from a point $\hat{x} \in \widehat{H}$ to a point $\tilde{x} \in \partial\widehat{H}$, i.e. the map from the sliding region to its boundary. The sliding map Σ can be constructed by considering the equations of the reduced order system (5) describing the system evolution within the sliding region (see [16] for the analytical details).

By suitably composing the maps Π_1, Π_2 and Σ it is simple to obtain the map describing a periodic orbit with sliding (an example of this type of orbit is reported in Fig. 2-b). Notice that in the case of third-order relay feedback systems such composition give rise to a one-dimensional mapping from the boundary of the sliding region $\partial\widehat{H}$ back to itself.

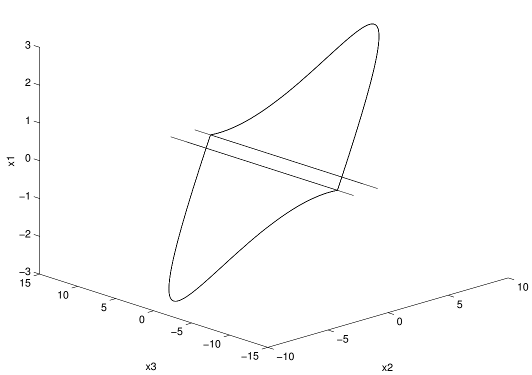
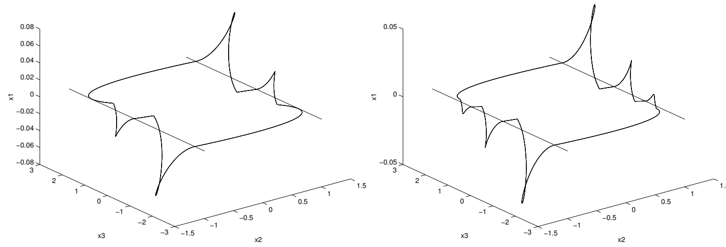


FIG. 3. Phase space diagrams of the simple limit cycle at the bifurcation point ($\rho \approx 2.1$). It can be clearly seen that the orbit intersects the switching plane on the boundary of the sliding strip



(a) (b)

FIG. 4. Multisliding bifurcation of a 2-sliding orbit (a) into a 3-sliding solution (b) as the parameter ω is varied between 9.5 and 10.5 while the other parameters are set to $\rho = \zeta = \lambda = \kappa = -\sigma = 1$.

III. LOCAL BIFURCATIONS

Both simple and sliding limit cycles were found to undergo several bifurcations as the systems parameters are varied. Standard local transitions such as the saddle-node, transcritical and period-doubling bifurcations are indeed possible and can be detected by studying the discrete time mappings described above. It is worth mentioning here that for a symmetric system such as the relay feedback system under investigation, the period-doubling of a symmetric solution corresponds to a symmetry-breaking point. Namely, as discussed in [16], at such a bifurcation point the transition occurs from a symmetric solution to a pair of conjugate asymmetric orbits.

In addition to the standard bifurcations, relay feedback systems (and more genereally switched dynamical systems) can exhibit a novel class of bifurcations involving sliding motion which were first reported in [16]. In what follows we briefly summarise the main characteristics of some of these novel transitions (see [18] for further details).

A. Sliding Bifurcation

This bifurcation describes the transition from a simple orbit to one with sliding motion. It occurs when, by varying the system parameters, a section of the simple orbit hits transversally the boundary of the sliding strip. As shown in Figs. 2 and 3 when this occurs, the fixed point of the switching map corresponding to the non-sliding orbit enters the sliding strip at the bifurcation point and sliding orbits are then generated. For further parameter variation, the newly formed sliding orbit is characterised by a longer and longer sliding section.

B. Multisliding Bifurcation

Sliding orbits can themselves undergo bifurcations to more complex solutions as the system parameters are varied. Namely, a novel bifurcation was found to take place when the sliding section of the orbit grazes tangentially the boundary of the sliding strip. In the simplest case, this is then followed by the formation of an orbit characterised by an additional sliding section as shown in Fig. 4; hence the name of *multisliding bifurcation*. Also, as reported later, a multisliding bifurcation can be associated to a saddle-node like scenario where orbits with a different number of sliding sections collide and disappear.

To avoid confusion, in what follows, we term a periodic solution characterised by N sections of sliding motion as an N-sliding orbit. Hence, at a multisliding bifurcation one can either observe the bifurcation of an N-sliding into a$(N+1)$-sliding solution [16] or the collision and disappearance of two orbits, unstable and stable ones having the same number N of sliding sections.

IV. STABLE ASYMMETRIC SELF-OSCILLATIONS AND CHAOS

It has been often assumed in the control literature that self-oscillations of symmetric, unforced relay feedback systems such as (1)-(2) are also symmetric. Specifically, as discussed in the introduction, it is usually conjectured that asymmetric periodic solutions in these systems can only exist either by means of some type of external forcing term acting on the system or because of an intrinsic asymmetric relay characteristic (see for example the conjecture by Tsypkin reported in [3], p. 179).

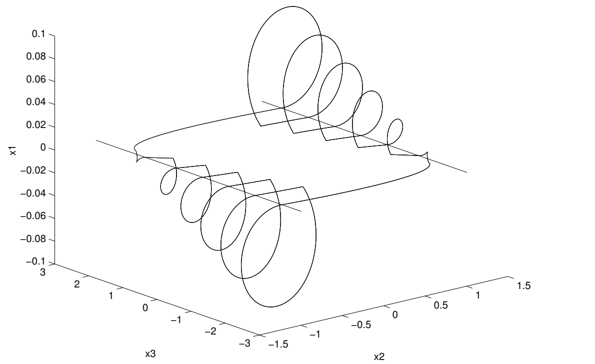


FIG. 5. Symmetric 12-sliding orbit observed when [include parameter values]. Note that the orbit is characterised by 6 lobes on each side.

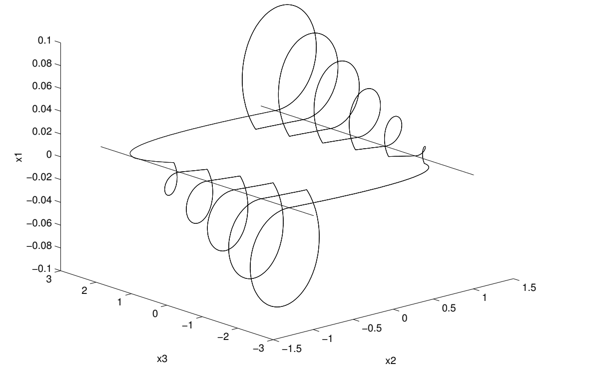


FIG. 7. Stable asymmetric 12-sliding orbit close to the multisliding bifurcation point. Note the near-tangency of one of its sliding sections. The orbit is characterised by 6 lobes on each side but one lobe on the right hand side is very small-due to broken symmetry.

In [16] it was shown via a third-order counterexample that, contrary to what is usually assumed, symmetric and unforced relay feedback systems can instead exhibit asymmetric periodic solutions. The asymmetric orbits reported in [16], though, were all unstable and the existence of stable asymmetric solutions and chaos was only conjectured. In what follows we report evidence of stable asymmetric solutions and give a detailed description of the bifurcation scenarios they are involved in. We will see that their existence is intrinsically related to the occurrence of sliding and multisliding bifurcations and leads to the formation of a chaotic attractor. We consider the bifurcation scenario obtained for decreasing values of the parameter $\zeta \in [-0.08, -0.06]$ while all the other parameters are assumed fixed to the values $k = 1, \lambda = 0.05, \omega = 10, \rho = 1, \sigma = -1$. This is a neighborhood of the parameter-space region where a seemingly chaotic solution was reported to exist in [16].

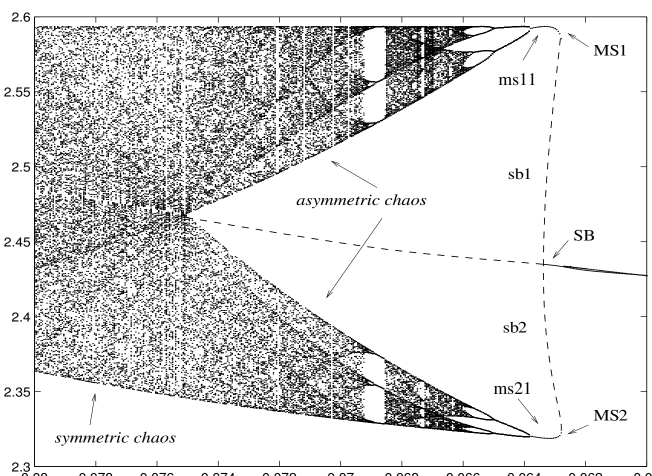


FIG. 6. Bifurcation diagram for the third-order relay feedback system under investigation, obtained by considering the variation of the parameter ζ in the interval $[-0.08, -0.06]$.

A. Symmetry Breaking

We begin by considering the 12-sliding stable symmetric orbit shown in Fig. 5 and represented by the solid branch to the right of the point SB in the bifurcation diagram reported in Fig. 6. Our numerical analysis shows that this solution undergoes a subcritical symmetry-breaking bifurcation at $\zeta \approx -0.0628$ (point SB in Fig. 6). Thus, the symmetric orbit involved in the bifurcation, becomes unstable giving rise to two branches of complex conjugate unstable 12-sliding orbits (branches sb_1 and sb_2 in Fig. 6).

This would appear to be a scenario similar to the one described in [16], since we only have a pair of asymmetric solutions which are unstable. In this case, though, following the two branches of asymmetric solutions, we find that they terminate into two multisliding bifurcation points located at $\zeta \approx -0.0623$ (points $MS1$ and $MS2$ in Fig. 6). Through these non-standard bifurcations, new branches of *stable* asymmetric solutions are then formed (branches $ms11$ and $ms21$ in Fig. 6). Thus, stable asymmetric solutions can indeed be found for $\zeta \in [-0.0638, -0.0623]$ in this entirely symmetric relay feedback system which is not subject to any external forcing.

More specifically, at the multisliding bifurcation point $MS1$, the stable 12-sliding orbit depicted in Fig. 7 ($ms1$) collides with the 12-sliding unstable asymmetric orbit ($sb1$), originating from the symmetry-breaking of the stable symmetric solution depicted in Fig. 5. Despite the apparent similarity of this transition with a saddle-node bifurcation, we emphasize that in this case this is due to a novel type of phenomenon caused by the tangency of a sliding section of the orbit with the boundary of the sliding region.

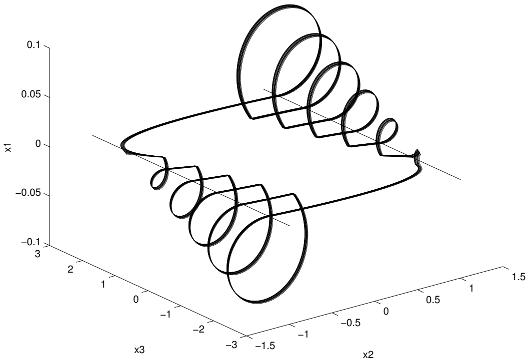


FIG. 8. One of the two stable asymmetric chaotic attractor exhibited by the system for $\zeta = -0.07$.

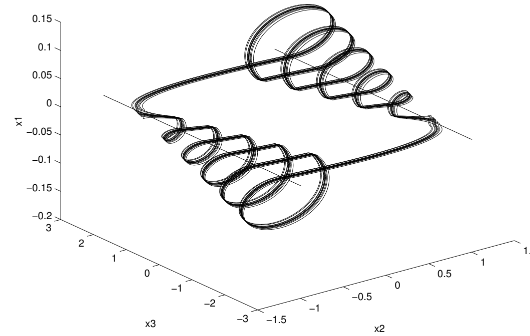


FIG. 9. Symmetric chaotic attractor exhibited by the system for $\zeta = -0.078$, formed by the merging of two bands of asymmetric chaos.

A similar scenario is observed at the bifurcation point $MS2$ for the symmetric conjugate of the orbit shown in Fig. 7 (which is characterised by 6 lobes on the left-hand side and 6 on the right-hand one (5 lobes can be seen the 6-th one is very small-due to broken symmetry). As expected both the stable asymmetric solutions are characterised by a small basin of attraction and are therefore extremely difficult to locate.

B. Deterministic Chaos

As we continue exploring this region of parameter space, we find that the newly formed stable asymmetric sliding orbits undergo a sequence of period-doublings which accumulate into two fully developed chaotic attractors. As shown in Fig. 8, these chaotic attractors are stable and organised by an underlying asymmetric multisliding orbit. Hence, they are both asymmetric.

For lower parameter values, these two bands of asymmetric chaos then merge into one (see Fig. 6) and the symmetric stable chaotic attractor depicted in Fig. 9 is therefore generated. Hence, stable chaos is also possible in these systems in the absence of any external forcing term or asymmetric relay characteristic. Moreover, the "route to chaos" is here characterised by an interesting combination of standard bifurcations such as period-doublings and symmetry-breaking together with novel transitions involving sliding.

As the system parameter, is further decreased we finally observe the sudden disappearance of this symmetric chaotic attractor at $\zeta \approx -0.08$ This is due to a global bifurcation phenomenon or "crisis" which is caused by the collision of an unstable coexisting periodic solution originated at a saddle-node point with the stable symmetric chaotic attractor.

V. CONCLUSIONS

We have discussed the occurrence of stable asymmetric solutions and chaos in relay feedback systems. In so doing, we confirmed the conjecture first reported in [16] that stable asymmetric orbit can exist in unforced and symmetric relay feedback configurations. In particular, we showed that standard bifurcations and novel transitions involving sliding give rise to an intricate bifurcation scenario. Namely, the occurrence of stable asymmetric orbits at a so-called multisliding bifurcation point is then linked with the existence of both stable asymmetric and symmetric chaotic attractors. Numerical evidence has been detailed to the case of a third-order representative example. We anticipate that similar results can be obtained in the case of other relay feedback systems with sliding.

Current work is addressing the derivation of an appropriate analytical explanation of the scenario observed through the derivation of appropriate normal form mappings in the neighborhood of the bifurcation points.

---

[1] I. Fluegge-Lotz. *Discontinuous Automatic Control.* Princeton University Press, 1953.

[2] A. A. Andronov, S. E. Khaikin, and A. A. Vitt. *Theory of Oscillators.* Pergamon Press, Oxford, 1965.

[3] Ya. Z. Tsypkin. *Relay Control Systems.* Cambridge University Press, Cambridge, U.K., 1984.

[4] V. I. Utkin. *Sliding Modes in Control Optimization.* Springer-Verlag, Berlin, 1992.

[5] A. S. Morse. Control using logic-based switching. In A. Isidori, editor, *Trends in Control. A European Perspective*, pages 69–113. Springer, 1995.

[6] K. J. Åström and T. Hägglund. *PID Controllers: Theory, Design, and Tuning.* Instrument Society of America, Research Triangle Park, NC, second edition, 1995.

[7] S. R. Norsworthy, R. Schreier, and G. C. Temes. *Delta-Sigma Data Converters—Theory, Design, and Simulation*. IEEE Press, New York, 1997.

[8] K. H. Johansson, A. Rantzer, and K. J. Åström. Fast switches in relay feedback systems. *Automatica*, 35(4), April 1999.

[9] A. F. Filippov. *Differential Equations with Discontinuous Righthand Sides*. Kluwer Academic Publishers, 1988.

[10] K. J. Åström. Oscillations in systems with relay feedback. In K. J. Åström, G. C. Goodwin, and P. R. Kumar, editors, *Adaptive Control, Filtering, and Signal Processing*, volume 74 of *IMA Volumes in Mathematics and its Applications*, pages 1–25. Springer-Verlag, 1995.

[11] A. Megretski. Global stability of oscillations induced by a relay feedback. In *Preprints IFAC 13th World Congress*, volume E, pages 49–54, San Francisco, CA, 1996.

[12] J. M. Gonçalves, A. Megretski, and M. A. Dahleh. Global stability of relay feedback systems. Technical Report Preprint LIDS-P-2458, Dept. of EECS, MIT, Cambridge, MA, 1999.

[13] P. A. Cook. Simple feedback systems with chaotic behaviour. *Systems & Control Letters*, 6:223–227, 1985.

[14] D. Amrani and D.P. Atherthon. Designing autonomous relay systems with chaotic motion. In *Proc. 28th IEEE Conf. on Decision and Control*, volume 1, pages 932–936, Tampa, FL, 1989.

[15] R. Genesio and A. Tesi. Chaos prediction in a third order relay system. Internal Report RT 29/90. Dipartimento di Sistemi ed Informatica, University of Florence, Italy., 1990.

[16] M. di Bernardo, K.H. Johansson, and F. Vasca. Self-oscillations in relay feedback systems: Symmetry and bifurcations. *International Journal of Bifurcations and Chaos*, (4), 2001. to appear.

[17] M. di Bernardo, A. R. Champneys, and C. J. Budd. Grazing, skipping and sliding: analysis of the nonsmooth dynamics of the DC/DC buck converter. *Nonlinearity*, 11:858–890, 1998.

[18] Work in progress by the authors and other collaborators is addressing the analysis of the normal forms associated to these novel bifurcations and their classification. These results will be soon published elsewhere. For further details check *http://www.enm.bris.ac.uk/anm/staff/enmdb/home.html*.