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Bifurcations to Chaos in a Semiconductor Laser with Phase-Conjugate Feedback

Bernd Krauskopf and Kirk Green

Department of Engineering Mathematics, University of Bristol, Bristol BS8 1TR, UK,
B.Krauskopf@bristol.ac.uk, TEL: +44 117 928 7957, FAX: +44 117 925 1154

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Due to possible applications, such as mode locking, phase locking, and frequency control, there has been considerable interest recently in the nonlinear dynamics of a semiconductor laser subject to phase-conjugate feedback (PCF). It is now known that the PCF laser exhibits a wealth of dynamics, including stable periodic operation, quasiperiodic motion and chaos[1,2]. Transitions to chaos were recently studied with a combination of bifurcation diagrams and phase plots[2]. However, the PCF laser is a delay differential system with an infinite dimensional phase space[2,3], and this makes it difficult to study, both analytically and with numerical tools that go beyond mere simulation.

We present a detailed study of bifurcations to chaos involving tori in the system. This is made possible by the new numerical technique of computing what are called unstable manifolds of periodic orbits of saddle type, which we developed for this purpose. These manifolds are one-dimensional curves in a suitable Poincaré section, and knowing them is tentamount to knowing the dynamics of the system.

A single-mode semiconductor laser subject to weak (instantaneous) PCF is described by the rate equations[2]

$$\frac{dE}{dt} = \frac{1}{2}\left[-i\alpha G_N(N(t)-N_{sol}) + (G-\frac{1}{\tau_p})\right]E(t) + \ \kappa E^*(t-\tau)\exp(2i\delta(t-\tau/2)) \tag{1}$$

$$\frac{dN}{dt} = \frac{I}{q} - \frac{N(t)}{\tau_e} - G\,|E(t)|^2 \tag{2}$$

for the complex electric field $E(t)$ and for the inversion $N(t)$. The main parameter we study is the dimensionless product $\kappa\tau$ of the feedback rate κ and the external-cavity round trip time τ, where we set all other parameters to realistic values as in Ref. [2].

Notice that (1) and (2) are a three-dimensional delay-differential system that decribes how the history, defined on the interval $[-\tau, 0)$ with values in (E, N)-space, evolves in time. Because one needs to specify values of E and N over the entire interval $[-\tau, 0]$ as 'initial condition', the system actually has an infinite dimensional phase space. As is common, we consider the time-evolution in the three-dimensional (E, N)-space, but it is important to keep in mind that this is a projection of the infinite-dimensional dynamics. An important property of Eqs. (1) and (2) is their symmetry with respect to the transformation $E \to -E$, which is physically a phase shift by π. As a consequence, any attractor is either symmetric or it has a symmetric counterpart[3].

As $\kappa\tau$ is changed the dynamics of the PCF laser is organised in regions with stable periodic output with 'bubbles' in between. These bubbles correspond to more complicated dynamics and one finds period-doublings and torus bifurcations[2]. However, a detailed understanding of routes to chaos in this infinite-dimensional system is still missing.

Here we consider in detail a transition to chaos at the beginning of one such bubble, shown in Fig. 1(a). For each $\kappa\tau$ we allowed the system to settle down to an attractor and then plotted a normalised value of the inversion $\hat{N}$ when the trajectory crosses the value of avarage power in the positive direction. In other words, Fig. 1(a) was created by numerically integrating Eqs. (1) and (2). For $\kappa\tau < 2.307$ there is a single point in this bifurcation diagram, corresponding to a stable periodic orbit. At $\kappa\tau \approx 2.307$ one notices the birth of a torus, on which the dynamics is initially quasiperiodic. At $\kappa\tau \approx 2.440$ the dynamics on the torus becomes locked to a stable periodic orbit. A feature not found earlier in this system is the fact that the new stable periodic orbit undergoes a torus bifurcation itself at $\kappa\tau \approx 2.556$. Then this torus suddenly disappears and the dynamics is chaotic for $\kappa\tau > 2.571$.

In order to understand the details of this transition to chaos it is not sufficient to use mere numerical simulation, because for $2.440 < \kappa\tau < 2.555$ one will only get an image of the stable periodic orbit, and not of the torus on which it lies. This is why we computed the unstable manifold of the saddle-periodic orbit on the torus in a suitable Poincaré map. This is shown in Fig. 1(b), where the whole torus for $\kappa\tau = 2.480$ can be seen in cross section with the Poincaré section $\{N = N_{\text{ave}} = 7.620 \times 10^8\}$. The crosses mark the five intersection points of a saddle-type periodic orbit on the torus with the section. It was computed by rotating the stable orbit around the orbit by a suitable angle and, starting from this initial condition, 'following the stable manifold' towards the periodic orbit of saddle type. From each cross there emanate two branches of the unstable manifold, which converge to two neighboring stable points (intersection points of the stable periodic orbit) in a spiralling fashion. This physically corresponds to damped modulations of the periodic laser output. The unstable manifold was computed by iterating suitable initial conditions near the saddle-type periodic orbit (approximately along the direction of the unstable eigenspace). Our computations show that the torus (which was smooth just after the torus bifurcation at $\kappa\tau \approx 2.307$) has lost its smoothness by starting to 'curl up' along the stable periodic orbit. Nevertheless, the torus is still present as a continuous object, as evidenced by the continuous (but not smooth) closed loop in Fig. 1(b). Notice that the different branches of the unstable manifold intersect each other, which is allowed because we are again looking at a two-dimensional projection of an infinite-dimensional system.

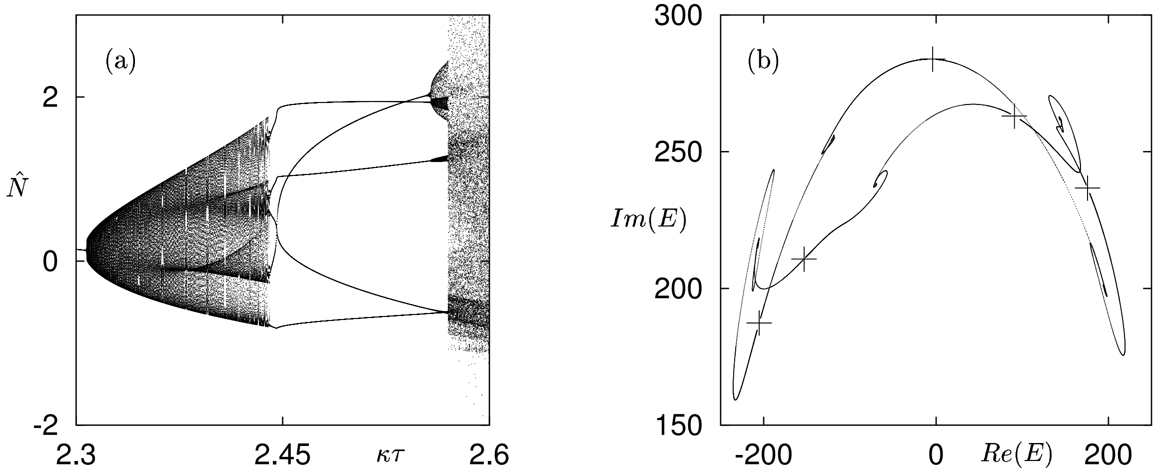

Fig. 1. Detail of the bifurcation diagram (a) and the torus with locked dynamics for $\kappa\tau = 2.480$ (b).

Our new numerical technique revealed the following bifurcation scenario as $\kappa\tau$ is varied from 2.3 to 2.6 as in Fig. 1(a). The stable periodic orbit undergoes a torus bifurcation at $\kappa\tau \approx 2.307$, and then the dynamics on the torus locks at $\kappa\tau \approx 2.440$, but the torus is still smooth. Smoothness of the torus is lost at $\kappa\tau \approx 2.445$, after which there is spiralling around the stable, locked solution; see Fig. 1(b). At $\kappa\tau \approx 2.556$ this new stable solution undergoes a torus bifurcation itself, which leads to a new smooth torus that looks much like a closed long piece of garden hose. The unstable manifold of the saddle-type periodic orbit goes to and spirals around this new torus. In what appears to be an attractor crisis, the new attracting torus disappears and chaotic dynamics is born at $\kappa\tau \approx 2.571$. The shape of the ensuing chaotic attractor is that of the unstable manifold of the saddle-type periodic orbit just before the crisis.

In summary, we have demonstrated how our new method for computing one-dimensional unstable manifolds in delay differential systems can be used to understand complicated transitions to chaos in the PCF laser. The example we used is a particularily interesting sequence to chaos involving several tori, which cannot be studied by mere simulation of the rate equations. Other routes to chaos, involving sudden transitions of folded and high-dimensional tori will be studied with this new method in the future. We expect that this will reveal important information of the types of chaos in the PCF laser. This is relevant for possible applications of chaotic signals, such as communication schemes using a chaotic carrier.

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3. B. Krauskopf, G.H.M. van Tartwijk, and G.R. Gray, "Symmetry properties of lasers subject to optical injection", *Opt. Comm.* **177**(1-6) (2000) 347.