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FROM LOCAL TO GLOBAL ONE-DIMENSIONAL UNSTABLE MANIFOLDS IN DELAY DIFFERENTIAL EQUATIONS

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A saddle-periodic orbit in a delay differential equation can be computed together with its unstable eigenfunctions using the latest version of the continuation package DDE-BIFTOOL. For the case of a single unstable Floquet multiplier we show how this information can be used to compute the one-dimensional unstable manifold of the associated fixed point of the Poincaré map.

In this paper we are concerned with a delay differential equation (DDE) with a single fixed delay $\tau \in \mathbb{R}^{>0}$, which takes the general form

$$\frac{dx(t)}{dt} = F(x(t), x(t-\tau), \lambda)\,. \tag{1}$$

Here $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is differentiable and $\lambda \in \mathbb{R}^p$ is a multi-parameter; see, for example, Refs. [1, 6] as general references.

The *phase space* of (1) is the space of continuous functions $\mathcal{C}$ defined on the interval $[-\tau, 0]$ with values in the *physical space* $\mathbb{R}^n$. A point $q \in \mathcal{C}$ in this infinite-dimensional phase space has a *headpoint* $q(0)$ and a *history* $q|_{[-\tau,0)} = \{q(t)\,|\,t \in [-\tau, 0)\}$. The dynamics of (1) is given by the *evolution*

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operator

$$\Phi^t : \mathcal{C} \to \mathcal{C}. \tag{2}$$

We are interested in a *periodic solution* $\Gamma(t)$ of (1), which has the property that $\Gamma(t) = \Gamma(t + \mathsf{T})$ for all t and for some (smallest) period $\mathsf{T} > 0$. A periodic orbit Γ is a closed curve in projection onto the physical space $\mathbb{R}^n$. We denote by $\mathcal{C}_\Sigma$ the space of points in $\mathcal{C}$ with headpoints in a prescribed section $\Sigma \subset \mathbb{R}^n$. Then the *Poincaré map* P is defined as

$$P :\ \mathcal{C}_\Sigma \to \mathcal{C}_\Sigma, \quad q \mapsto \Phi^{t_q}(q)\,, \tag{3}$$

where $t_q > 0$ is the return time to Σ. The periodic orbit Γ gives rise (locally) to the fixed point $q \in \mathcal{C}_\Sigma$ under P. The stability of Γ is given by its *Floquet multipliers*, which are the eigenvalues of the linearisation $DP(q)$ of the Poincaré map P at the respective fixed point $q \in \mathcal{C}_\Sigma$ (plus the trivial Floquet multiplier $+1$ of the direction tangent to Γ). For the case of a single fixed delay considered here, the spectrum of $DP(q)$ is isolated, except at the origin, and strictly inside some circle of radius $r > 0$ in the complex plane. In other words, the periodic orbit Γ has at most finitely many unstable eigendirections.

We consider the case that Γ has exactly one unstable eigendirection and all other Floquet multipliers, except the trivial one, are strictly inside the unit circle. In this situation, the corresponding fixed point $q \in \mathcal{C}_\Sigma$ has a one-dimensional unstable manifold $W^u(q)$ that is tangent at q to the one-dimensional linear unstable eigendirection $E^u(q)$.

The software package DDE-BIFTOOL[2] is able to find a saddle-periodic orbit Γ, together with its one-dimensional unstable eigendirection.[5] This is shown in Fig. 1 (a) and (b) for a periodic orbit of the DDE model of a semiconductor laser with phase-conjugate feedback (PCF), which we use as the illustrating example throughout; see the companion paper Ref. [3] for more details.

Our goal is to compute the one-dimensional unstable manifold $W^u(q)$. It is quite straightforward to extract the data of q from the data of Γ. However, as starting data for a manifold computation we also need two points q_δ^+ and q_δ^- on $E^u(q)$ at some small distance δ either side of q. Simply moving distance δ along $E^u(q)$ will not work, because the line field given by this unstable eigendirection is (generically) transverse to the section Σ; see Fig. 1 (c) and (d). This also means that the intersection of $E^u(q)$ with Σ is not a straight line.

The points $q_\delta^\pm$ can be extracted from the data in Fig. 1 (b) as follows. In the situation that the vector $E^u(q(0))$ points in the direction of the flow,

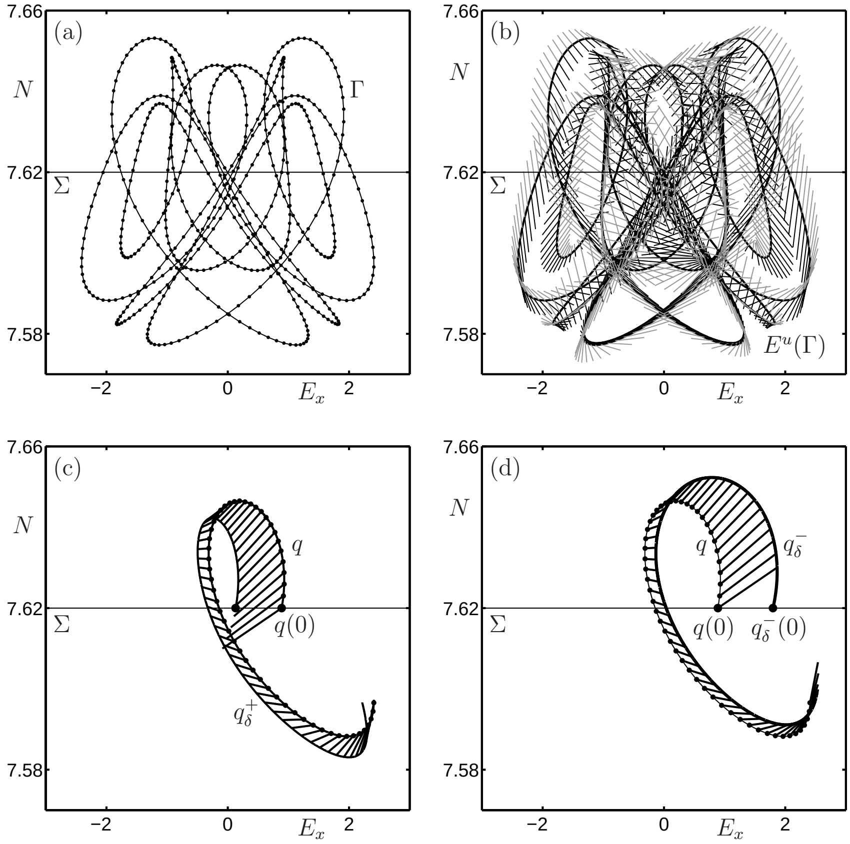

Figure 1. DDE-BIFTOOL data of a saddle periodic orbit Γ (a), and its one-dimensional unstable eigendirection $E^u(\Gamma)$ (b). Panels (c) and (d) show the starting data, the saddle-fixed point q and the two points $q_\delta^\pm$ on $E^u(q)$. We used the exaggerated value of $\delta = 3.0$ for illustration; this data is for the PCF laser[3] for $\kappa\tau = 2.49$.

as shown in Fig. 1 (c), the point, q_δ^+ in this case, can be found by moving backwards in t along the data of $\Gamma + \delta E^u(q)$ until a point in Σ is found. Then q_δ^+ is this headpoint together with the history of length τ. However, for q_δ^- in Fig. 1 (d) the vector $E^u(q(0))$ points against the direction of the flow, so that one must move forward in t along the data of $\Gamma - \delta E^u(q)$. The problem is that the data of $E^u(q)$ contains a discontinuity at $E^u(q(0))$, because over one period T any vector stretches by the Floquet multiplier. We take this scaling into account to extend the data of $\Gamma - \delta E^u(q)$ and then extract the desired point q_δ^-.

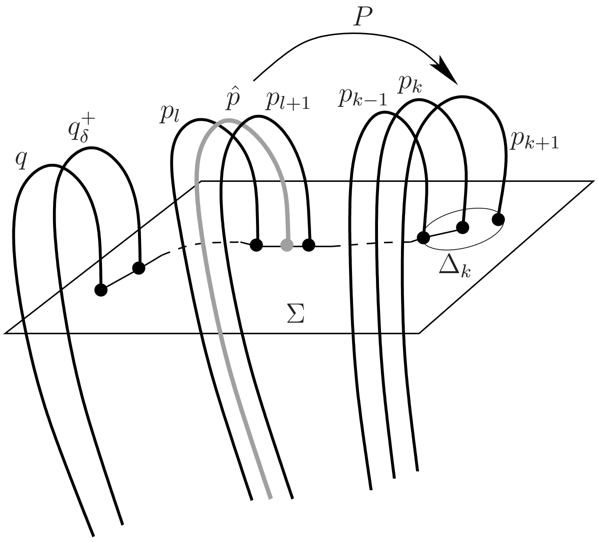

Figure 2. Sketch of the 1D manifold algorithm.

The procedure of finding q and $q_\delta^\pm$ (for a δ specified by the user) is now fully implemented in Matlab and can be used in conjunction with DDE-BIFTOOL routines; see Ref. [5] for details.

The next step is to grow the manifold away from q, which can be done with the algorithm in Ref. [7]. Each branch of the manifold is represented as a list of points $p_k \in \mathcal{C}_\Sigma$. Linear approximation is used between consecutive points of the list, and the distance between list points is determined by the curvature of the *trace* $W^u(q) \cap \Sigma$. The trace is a one-dimensional curve, but in contrast to a one-dimensional manifold of a (planar) map, it may have self-intersections and isolated points where it is not smooth. This is a result of the projection from an infinite-dimensional phase space.

The main step of the algorithm is sketched in Fig. 2. Suppose that the manifold has been computed up to the point p_k, so that we need to find the next list point, p_{k+1}, at a distance Δ_k from p_k. To this end, we find a point $\hat{p}$ on the part of the manifold that was already computed such that $P(\hat{p})$ lies on a circle of radius Δ_k around p_k. This typically involves interpolation between two list points, say, p_l and p_{l+1}. Provided that the angle between $p_{k-1}(0)$, $p_k(0)$ and $P(\hat{p})(0)$ (in Σ) is acceptable,[7] the point $P(\hat{p})$ is accepted as the new point p_{k+1}. If the angle is too sharp then Δ_k

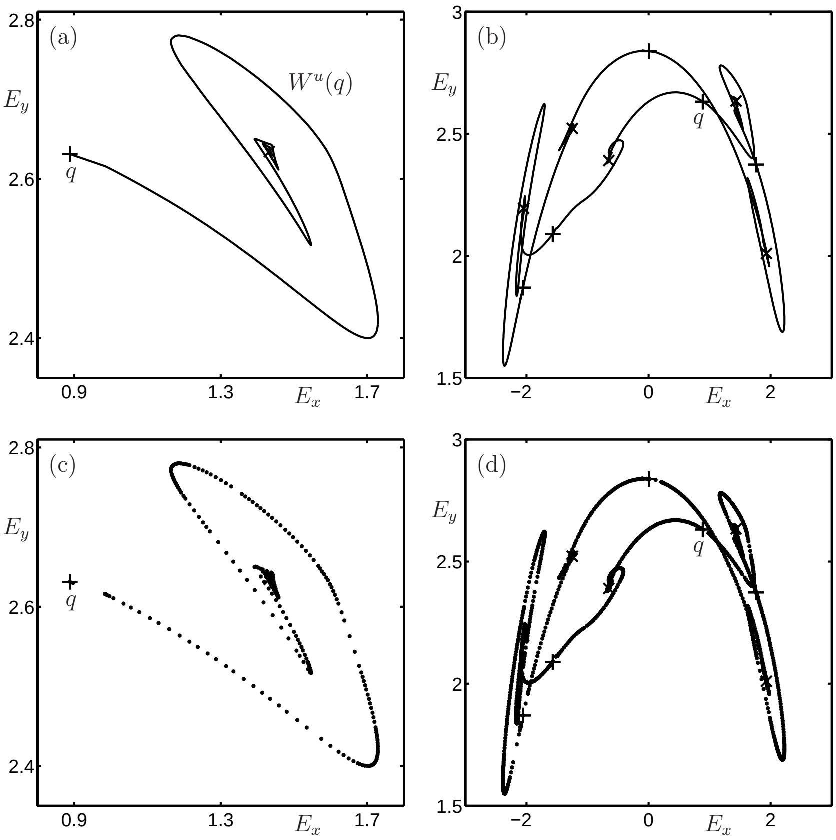

Figure 3. The trace of one branch of the saddle fixed point q (see Fig. 1) (a), and of all branches of a group of five saddle-fixed points (b). The points of the lists representing these manifolds are shown in panels (c) and (d), respectively. This data is for the PCF laser[3] for $\kappa\tau = 2.49$.

is decreased and a new attempt at finding p_{k+1} is made. A computation stops after a prescribed arclength of the trace has been computed, or when convergence to an attracting fixed point is detected.

The result of a computation is shown in Fig. 3, again for the example of the PCF laser.[3] Panel (a) shows one branch of the trace of the unstable manifold of the fixed point q from Fig. 1, and panel (b) shows all traces of the unstable manifolds of a group (including q) of five saddle-fixed points. All branches end up at an attracting period-five point (a fixed point of P^5), forming a continuous but non-smooth torus. Panels (c) and (d) show the actual points that were computed. It can clearly be seen how the density of points depends on the curvature of the trace.

In conclusion, we have demonstrated how the computation of 1D unstable manifolds of saddle fixed points of a Poincaré map of a DDE can be used in conjunction with the continuation software DDE-BIFTOOL. Examples of how this new tool can be applied can be found in Refs. [3, 4, 5].

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