Sieber, J., & Krauskopf, B. (2003). Dynamics of an inverted pendulum subject to delayed control.

Early version, also known as pre-print

DYNAMICS OF AN INVERTED PENDULUM SUBJECT TO DELAYED CONTROL

J. SIEBER* AND B. KRAUSKOPF

Bristol Centre for Applied Nonlinear Mathematics, Department of Engineering Mathematics, Queen's Building, University of Bristol, BS8 1TR, U.K.
E-mail: Jan.Sieber@bristol.ac.uk
E-mail: B.Krauskopf@bristol.ac.uk

We investigate an inverted pendulum on a cart subject to a delayed feedback control force which tries to balance the pendulum. This is modelled by a two-dimensional system of delay-differential equations and can be considered as a prototype system for control problems arising in mechanical engineering. The linear stability analysis shows that there is only a bounded region of linear stability of the origin (corresponding to successful balancing), and identifies a singularity of codimension three as the organizing center for all dynamics of small amplitude.

Here we present the numerical bifurcation analysis of the ordinary differential equation governing the dynamics on the three-dimensional center manifold. This is compared directly with a bifurcation study of the full delay system in the vicinity of the singularity.

1. Introduction of the mathematical model

We consider the classic control problem[2,7] of an inverted planar pendulum on a motorized cart on a track (see Ref. [6] for further references and details of the analysis). The goal of the feedback controller is to control the pendulum in the upright position, where the control action is the movement of the cart.

The dynamics of the inverted pendulum are given by the non-dimensionalized second-order differential equation for the angle θ of its deviation from the upright position:

$$\left(1-\frac{3\varepsilon}{4}\cos^2\theta\right)\ddot{\theta}+\frac{3\varepsilon}{8}\dot{\theta}^2\sin(2\theta)-(\sin\theta+D\cos\theta)=0. \qquad (1)$$

*The research of J.S. is supported by EPSRC Grant GR/R72020/01.

Here ε is the relative mass of the uniform pendulum with respect to the mass of the cart, and D is the horizontal driving force applied to the cart. The dynamics of the displacement δ of the cart depends on θ by

$$\ddot{\delta} = L\frac{\frac{\varepsilon}{2}\sin\theta\dot{\theta}^2 + \frac{2}{3}D - \frac{\varepsilon}{4}\sin(2\theta)}{1 - \frac{3\varepsilon}{4}\cos^2\theta} \tag{2}$$

where L is the length of the pendulum. The force D is applied as a feedback control depending on the state of the system with the goal of stabilizing the pendulum at its upright position, $\theta = 0$. Due to inherent delays, the feedback control force D is a function of the state of the system at some fixed delay time τ ago. We consider the situation that this delay is not negligible and are interested in its influence on the dynamics of the overall system. To this end, we study the case of a linear control force

$$D(t) = -a\theta(t-\tau) - b\dot{\theta}(t-\tau) \tag{3}$$

with the control gains a and b. Note that increasing the delay time τ is equivalent to decreasing the length L of the pendulum.

After rescaling the delay time to 1, equation (1) can be written as a delay differential equation (DDE) of the form

$$\dot{x}(t) = f(x(t), x(t-1), \lambda) \tag{4}$$

which relates to (1) and (3) by setting $x_1 = \theta$, $x_2 = \dot{\theta}$ and $\lambda = (a, b, \tau)$. The right-hand-side $f : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^2$ has the form

$$\begin{aligned} f_1(x,y,\lambda) &= x_2 \\ f_2(x,y,\lambda) &= \frac{-\frac{3}{8}\varepsilon\sin(2x_1)x_2^2 + \tau^2\sin x_1 - \cos x_1(\tau^2 a y_1 + \tau b y_2)}{1 - \frac{3}{4}\varepsilon\cos^2 x_1}. \end{aligned} \tag{5}$$

The phase space of (4)–(5) is the space of continuous functions over the delay interval $[-1, 0]$ with values in $\mathbb{R}^2$.

System (4)–(5) has $\mathbb{Z}_2$-symmetry, because

$$f(-x, -y, \lambda) = -f(x, y, \lambda). \tag{6}$$

Consequently, the origin 0 is always an equilibrium, and any solution of (4)–(5) is either symmetric under this symmetry or has a counterpart under reflection at the origin. A stable symmetric attractor close to 0 corresponds to successful balancing of the pendulum whereas $\delta(t)$ is always unbounded for $t \to \infty$ for any non-symmetric attractor.

2. Linear stability analysis of the origin

The origin undergoes a pitchfork bifurcation for $a = 1$ (two non-symmetric equilibria emerge for $a > 1$) and it has a pair of purely imaginary eigenvalues $\pm i\omega\tau$ ($\omega \in \mathbb{R}$) if

$$a = \cos(\omega\tau)\left[\omega^2\left(1 - \frac{3}{4}\varepsilon\right) + 1\right], \quad b = \frac{\sin(\omega\tau)}{\omega}\left[\omega^2\left(1 - \frac{3}{4}\varepsilon\right) + 1\right] \quad (7)$$

which generically implies a Hopf bifurcation. These two bifurcations form curves in the (a, b)-plane as is sketched in Fig. 1. The Hopf curve emanates

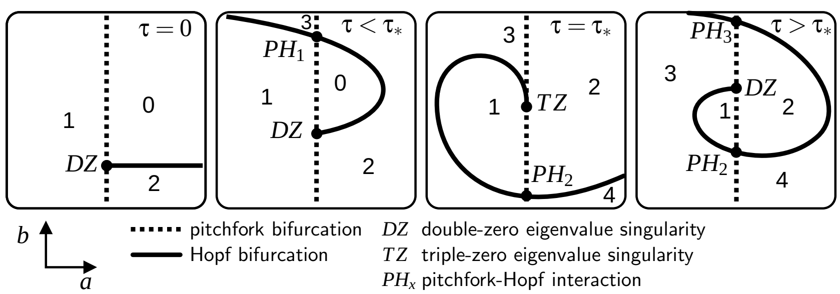

Figure 1. Sketches of local bifurcation curves in the (a, b)-plane for different values of τ. The number of unstable eigenvalues of the origin is indicated in each region.

from a double-zero eigenvalue singularity, DZ, where $(a, b) = (1, \tau)$ and spirals outward in counterclockwise direction for $\tau > 0$. A bounded parameter region of linear stability exists for delays $\tau \in (0, \tau_*)$ where $\tau_* := \frac{1}{2}\sqrt{8 - 6\varepsilon}$ is the critical delay[7]. At $\lambda_* = (a, b, \tau) = (1, \tau_*, \tau_*)$ the double-zero eigenvalue singularity DZ and the first pitchfork-Hopf interaction PH_1 coincide and the region of linear stability shrinks to a point TZ which is a triple-zero eigenvalue singularity of the origin. For $\tau > \tau_*$ there can be no stable small-amplitude dynamics as the origin has at least one strongly unstable eigendirection. The triple-zero eigenvalue bifurcation TZ at λ_* organizes the stable small-amplitude dynamics of system (4)–(5). The unfolding of the generic triple-zero eigenvalue bifurcation was studied in Ref. [4]; the full dynamics of the normal form are not well understood. A special case with the same symmetry (6) was recently found in Chua's circuit[1].

3. Dynamics on the center manifold near the singularity

Near the triple-zero singularity λ_*, system (4)–(5) has a three-dimensional center manifold at the origin. We compute the basis B for the center invariant subspace of the origin for its linearization, split $x = Bu + z$ into a center part $u \in \mathbb{R}^3$ and a hyerbolic part z, zoom into the neighborhood of the origin by using the rescaling

$$(u_1, u_2, u_3) \to (r^3 u_1, r^5 u_2, r^7 u_3), \quad z \to r^3 z, \quad t \to r^2 t,$$
$$a = 1 + \frac{1}{3} r^6 \alpha, \quad b = \tau_* + \tau_* \frac{1}{3} r^2 \beta, \quad \tau = b + \tau_* \frac{1}{3} r^4 \gamma, \tag{8}$$

and expand the flow on the center manifold in terms of the small scaling parameter r. This procedure[6] results in the following theorem:

Theorem 3.1. *The flow of system* (4)–(5) *on the center manifold of the origin for parameter values λ near λ_* is governed by the system of ordinary differential equations*

$$\dot{u} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha & \gamma & \beta \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ u_1^3 \end{bmatrix} + r^2 R(u, \alpha, \beta, \gamma, r) \tag{9}$$

where R is a smooth function with respect to all arguments.

We call system (9) for $r = 0$ the *truncated system*. As it has cone structure we can restrict our bifurcation analysis to the parameter sphere $\alpha^2 + \beta^2 + \gamma^2 = 1$, which we parametrize by

$$\alpha = \sin \frac{\pi}{2}\varphi, \quad \beta = \cos \frac{\pi}{2}\varphi \ \cos 2\pi\psi, \quad \gamma = \cos \frac{\pi}{2}\varphi \ \sin 2\pi\psi.$$

Any phenomenon found in the truncated system that is robust with respect to regular perturbations exists in (9) and, thus, in (4)–(5) as well. Fig. 2(a) shows the bifurcation diagram of the truncated system in the (φ, ψ)-plane as computed with AUTO[3]. The most prominent feature of the diagram is region II of stable symmetric periodic motion next to region I of linear stability of the origin. The two-parameter family of symmetric periodic orbits undergoes infinitely many pitchfork bifurcations and is bounded by a Hopf bifurcation, a curve of heteroclinic connections between the non-symmetric saddle equlibria, and a curve of figure-eight homoclinic connections to the origin. The curves of heteroclinic and homoclinic connections spiral toward the point HC where a multiple heteroclinic chain exists.

For comparison Fig. 2(b) shows a part of the bifurcation diagram, as computed with DDE-BIFTOOL[5], for the full DDE system (4)–(5) for the parameter sphere around TZ of radius $r = 0.5$ (see scaling (8)) and $\varepsilon = 0$. Figures 2(a) and 2(b) agree not only qualitatively but also quantitatively which illustrates that the triple-zero singularity indeed acts as an organizing center for the small-amplitude dynamics of the DDE system (4)–(5) for a large parameter range.

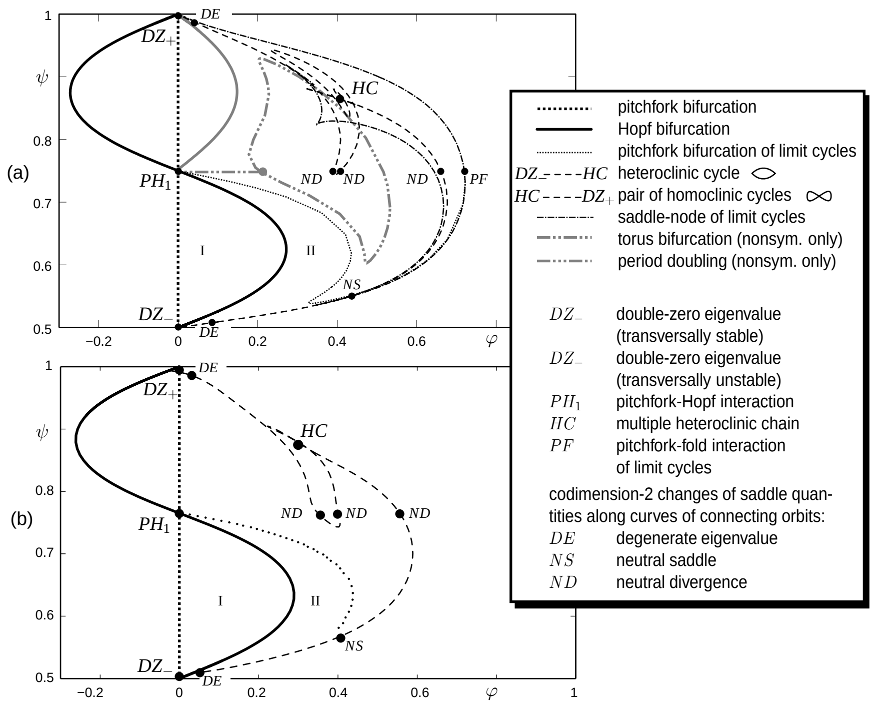

Figure 2. Comparison between (a) the bifurcation diagram for the truncated system (9), and (b) a partial bifurcation diagram for the DDE system (4)–(5) in the (φ, ψ)-plane. (Gray curves in panel (a) are of bifurcations of non-symmetric equilibria and periodic orbits, which are not shown in panel (b).)

4. Conclusions

We have studied a triple-zero eigenvalue singularity on its center manifold in a $\mathbb{Z}_2$-symmetric DDE system by center manifold reduction and a numerical bifurcation analysis of its partial unfolding. In this way, we have found a region of stable small-amplitude motions for the pendulum with delayed feedback control even outside the region of linear stability.

We finish with some challenges for future work. First, it remains an open problem to find the complete unfolding of the triple-zero eigenvalue singularity. Second, it would be interesting to know whether it is possible to find controllers that stabilize small-amplitude motion for values of the delay τ above the critical delay τ_*. Finally, we plan to extend our analysis to more realistic balancing models that include physical effects such as backlash, and this leads to discontinuities in the right-hand-side.

References

1. A. Algaba, M. Merino, E. Freire, E. Gamero, and A. Rodriguez-Luis. Some results on Chua's equation near a triple-zero linear degeneracy. *International Journal of Bifurcation and Chaos*, 13(3):583–608, 2003.
2. J. L. Cabrera and J. G. Milton. On-off intermittency in a human balancing task. *Physical Review Letters*, 89(158702), 2002.
3. E. J. Doedel, A. R. Champneys, T. F. Fairgrieve, Y. A. Kuznetsov, B. Sandstede, and X. Wang. *AUTO97, Continuation and bifurcation software for ordinary differential equations*, 1998.
4. F. Dumortier, S. Ibañez, and H. Kokubu. New aspects in the unfolding of the nilpotent singularity of codimension three. *Dynamical Systems*, 16(1):63–95, 2001.
5. K. Engelborghs, T. Luzyanina, and G. Samaey. DDE-BIFTOOL v.2.00: a Matlab package for bifurcation analysis of delay differential equations. Report TW 330, Katholieke Universiteit Leuven, 2001.
6. J. Sieber and B. Krauskopf. Bifurcation analysis of an inverted pendulum with delayed feedback control near a triple-zero eigenvalue. Preprint 2003.7, Bristol Centre for Applied Nonlinear Mathematics, University of Bristol, 2003.
7. G. Stepán. *Retarded Dynamical Systems: Stability and Characteristic Functions*. Longman Scientific and Technical, 1989.