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CONTINUATION OF BIFURCATIONS IN PERIODIC DELAY DIFFERENTIAL EQUATIONS USING CHARACTERISTIC MATRICES

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Abstract. In this paper we describe a method for continuing periodic solution bifurcations in periodic delay-differential equations. First, the notion of characteristic matrices of periodic orbits is introduced and equivalence with the monodromy operator is proved. An alternative formulation of the characteristic matrix is given, which can efficiently be computed. Defining systems of bifurcations are constructed in a standard way including the characteristic matrix and its derivatives. For following bifurcation curves in two parameters, the pseudo-arclength method is used combined with Newton iteration. As a test example, an interrupted machining model is analyzed.

Key words. delay-differential equations, periodic solutions, bifurcations, continuation

AMS subject classifications. 37M20, 34K13, 34K18

1. Introduction. The aim of this paper is to give a method for bifurcation analysis of periodic orbits in periodic delay-differential equations (DDE). There are several models in engineering and science, which involve delay and parametric forcing, but as far as we are aware, there is no general method to analyze bifurcations in these systems. For time-invariant problems, DDE-BIFTOOL [12] is a widely used continuation software, which is capable of continuing periodic orbits and detecting their bifurcations using Floquet multipliers, but it cannot continue these bifurcations. However, there are time-periodic systems which possess an S^1 group symmetry, so that a periodic solution can be considered as the fixed point of a transformed system (see e.g. Haegeman et al. [17]). In such cases, one could use DDE-BIFTOOL to continue period doubling or Neimark-Sacker bifurcations. Unfortunately, fold bifurcations cannot be continued this way. The reason is that the trivial zero characteristic root related to the periodic solution cannot be distinguished from the additional zero root related to the fold bifurcation when one applies codimension-1 continuation algorithms. The above S^1 group symmetry simplification can also be achieved in systems, where the equation can be rewritten into polar coordinate system, separating out the angle variable into a separate equation. In this case the remaining equations, including the amplitude, can be considered as an autonomous problem (see e.g. Green and Krauskopf [16]).

It is well known [19, 6] that linear autonomous delay equations admit the construction of a characteristic matrix. Thus, the task of finding the spectra of the infinite dimensional problem can be reduced to finding the roots of a nonlinear function, which is just the determinant of the characteristic matrix. Also, Jordan chains [6] of the characteristic matrix at a spectral point can be used to obtain (generalized) eigenvectors of the infinitesimal generator. In the case of codimension-1 bifurcations of fixed points, the characteristic matrix at the critical spectral point has a one dimensional kernel subspace, which can be used to define test functionals by bordered systems and also to construct the defining systems of bifurcations. DDE-BIFTOOL uses this approach to continue Hopf and fold bifurcation curves of fixed points.

For periodic delay equations, there is no such general characteristic matrix. In [18], there is a first order scalar example for integer delays where by using Floquet's theorem, one

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can obtain the characteristic function to investigate stability. In [23], this method was used to compute stability of a 2D periodic system and to obtain a simple implicit function for period doubling stability curves. In [19], another n D example can be found for systems with integer delays. This one does not imply a generally applicable method as the one in [23] does, because it excludes equations having non-invertible coefficient matrices. However, it shows an equivalence between the characteristic matrix and the inverse of the monodromy operator. In this paper, we show that for all periodic delay equations there is a characteristic matrix which is equivalent to the monodromy operator in the same sense as it can be found in [19].

Continuation is a very effective technique to follow invariant structures in dynamical systems under parameter variation. This technique is also useful to follow special solutions such as those at bifurcation points. The preferred continuation technique is the pseudo-arclength continuation, which was used in several continuation software packages including AUTO [8], CONTENT [22], DDE-BIFTOOL [12]. We will use it, too.

The paper is organized as follows. In section 2, we introduce characteristic matrix for periodic orbits. We derive its equivalence with the monodromy operator and show how its computational cost can be reduced to that of the solution of one system of linear equations. In section 3, the method of continuation of periodic orbits is discussed using the pseudo-arclength technique. Continuation of bifurcations is discussed in section 4. In section 5, we give implementation details of the algorithm. In section 6, an example of interrupted machining model is analyzed.

2. Characteristic matrices. In this section, we generalize the results of Kaashoek and Lunel [21] to periodic delay equations. Let x be a continuous function $x : I \subset \mathbb{R} \rightarrow \mathbb{R}^n$ and let $h > 0$ be a real number. Define $x_t \in C([-h, 0], \mathbb{R}^n)$ by $x_t(\theta) = x(t + \theta)$. Consider the following delay differential equation

$$x'(t) = f(t, x_t), \quad (2.1)$$

where $f : \mathbb{R}^n \times C([-h, 0], \mathbb{R}^n) \rightarrow \mathbb{R}^n$ is of class C^N , $N \geq 2$ in the second variable, and the prime denotes time-derivative. Further, assume that f is periodic in the first variable, i.e., $f(t, \cdot) = f(t + \omega, \cdot)$ with $\omega \geq h$. Consider a ω -periodic solution $v(t)$ of (2.1). Since f is smooth, we can linearize (2.1) at v in the form

$$x'(t) = \int_0^h d_\theta \eta^v(t, \theta) x(t - \theta), \quad (2.2)$$

where η^v is a matrix valued function that is of bounded variation in θ , and it is associated with the periodic solution v . We can uniquely represent η if we take the space NBV , which consists of functions of bounded variations normalized such that $\eta(\theta) = 0$ for $\theta \in (-\infty, 0]$, right continuous on $(0, h)$ and constant on $[h, \infty)$. According to the basic theory of delay equations, (2.2) has a unique (forward) solution if we specify an initial function with $x_s = \phi$, $s \in \mathbb{R}$ fixed. The linearized equation can be written equivalently

$$x'(t) = \int_0^{t-s} d_\theta \eta^v(t, \theta) x(t - \theta) + \int_{t-s}^\infty d_\theta \eta^v(t, \theta) \phi(t - s - \theta), \quad (2.3)$$

where the first term includes merely the unknown function, and the second term includes the initial function. Since the initial function is in the equation now, we can solve it uniquely by specifying an initial value $x(s) = \phi(0)$ only.

In what follows, we will consider a bigger space $X = L^\infty([- \omega, 0], \mathbb{R}^n)$, where $C([- \omega, 0], \mathbb{R}^n)$ is a closed subspace in X . Note that X is chosen to maintain the compatibility with the duality framework for delay equations developed by Clement *et al.* [2, 3] (see also Diekmann *et al.* [6]). Let us define the operator

$$(\mathcal{A}(v, s)\varphi)(\theta) = \varphi'(\theta) - \int_0^{\omega+\theta} d_\vartheta \eta^v(s + \theta, \vartheta) \varphi(\theta - \vartheta) \quad (2.4)$$

with domain of definition $D(\mathcal{A}(v, s)) = \text{Lip}([- \omega, 0], \mathbb{R}^n)$ and

$$(\mathcal{B}(v, s)\varphi)(\theta) = \int_{\omega+\theta}^{\omega} d_\vartheta \eta^v(s + \theta, \vartheta) \varphi(\omega + \theta - \vartheta). \quad (2.5)$$

on X . Note that $\overline{D(\mathcal{A}(v, s))} = C([- \omega, 0], \mathbb{R}^n)$ and $\mathcal{A}(v, s)$ has an n dimensional kernel since, as we mentioned, we can solve $0 = \mathcal{A}(v, s)\varphi$ by specifying $\varphi(-\omega) = c$, for any $c \in \mathbb{R}^n$. To obtain the monodromy operator, we solve the equation $\mathcal{A}(v, s)\psi = \mathcal{B}(v, s)\varphi$ (which is just a reformulation of (2.3)) with the boundary condition $\psi(-\omega) = \varphi(0)$. Thus the monodromy operator becomes

$$U(s + \omega, s)\varphi = \psi. \quad (2.6)$$

The boundary condition can be eliminated by introducing the (even) bigger space $\hat{X} = \mathbb{R}^n \times X$, which includes the initial function ϕ and the initial value $\phi(0)$, as well. For ease of notation we use the following boundary value operators

$$L : X \rightarrow \mathbb{R}^n, \quad L\varphi = \varphi(-\omega),$$

$$M : X \rightarrow \mathbb{R}^n, \quad M\varphi = \varphi(0).$$

These definitions, indeed, make sense since we consider only functions in the domain of $\mathcal{A}(v, s)$. Let

$$\hat{\mathcal{A}}(v, s) = \begin{pmatrix} 0 & L \\ 0 & \mathcal{A}(v, s) \end{pmatrix}$$

and

$$D(\hat{\mathcal{A}}(v, s)) = \{(c, \varphi) \in \hat{X} \mid \varphi \in D(\mathcal{A}(v, s)), c = M\varphi\}.$$

With this definition, it is easy to check that $\hat{\mathcal{A}}(v, s)$ is one-to-one and onto. $\hat{\mathcal{B}}(v, s)$ is defined similarly

$$\hat{\mathcal{B}}(v, s) = \begin{pmatrix} I & 0 \\ 0 & \mathcal{B}(v, s) \end{pmatrix}.$$

The extended monodromy operator becomes $\hat{U}(s + \omega, s) = \hat{\mathcal{A}}^{-1}(v, s)\hat{\mathcal{B}}(v, s)$. Note that

$$\hat{\mathcal{A}}(v, s) \begin{pmatrix} \psi(0) \\ \psi \end{pmatrix} = \hat{\mathcal{B}}(v, s) \begin{pmatrix} \varphi(0) \\ \varphi \end{pmatrix} \Leftrightarrow \psi(-\omega) = \varphi(0), \mathcal{A}\psi = \mathcal{B}\varphi$$

gives (2.6).

It can easily be proved that the eigenvalues of $\hat{U}(s + \omega, s)$ and $U(s + \omega, s)$ are the same (see [21, Proposition 1.1.]). Therefore we can equivalently analyze the spectrum

$$\sigma(\hat{U}(s + \omega, s)) := \mathbb{C} \setminus \{\lambda \in \mathbb{C} \mid (\lambda I - \hat{U}(s + \omega, s))^{-1} \text{ is a bounded linear operator}\}.$$

In case of delay equations, $\hat{U}(s + \omega, s)$ is compact, thus its spectrum consists of eigenvalues which have a cluster point at 0. Since we are not interested in 0, we can define $\mu = 1/\lambda$ and analyze the equivalent problem

$$\lambda I - \hat{U}(s + \omega, s) = \frac{1}{\mu} \hat{\mathcal{A}}^{-1}(v, s) (\hat{\mathcal{A}}(v, s) - \mu \hat{\mathcal{B}}(v, s))$$

or, for the sake of simplicity, $\hat{\mathcal{A}}(v, s) - \mu \hat{\mathcal{B}}(v, s)$. In this latter case,

$$\sigma(\hat{U}(s + \omega, s)) = \mathbb{C} \setminus \left\{ \frac{1}{\mu} \in \mathbb{C} \mid (\hat{\mathcal{A}}(v, s) - \mu \hat{\mathcal{B}}(v, s))^{-1} \text{ is a bounded linear operator} \right\}.$$

From now on, we suppress the initial time s , the periodic solution v or both in the notation of $\mathcal{A}(v, s)$, $\hat{\mathcal{A}}(v, s)$, $\mathcal{B}(v, s)$ and $\hat{\mathcal{B}}(v, s)$.

2.1. Equivalence. Kaashoek and Lunel [21] derived equivalence of the infinitesimal generator of some evolutionary systems with a characteristic matrix $\Delta(z)$, with some spectral parameter z . Here, we do the same for periodic DDEs. Clearly, our operator \mathcal{A} has the following properties

1. $\mathcal{N} := \ker(\mathcal{A})$ is finite dimensional, $\dim(\mathcal{N}) = n$ and $\mathcal{N} \neq \{0\}$
2. the operator \mathcal{A} has a restriction $\mathcal{A}_0 : X \rightarrow X$ such that
 - (a) $D(\mathcal{A}) = \mathcal{N} \oplus D(\mathcal{A}_0)$,
 - (b) $\Omega := \{\mu \in \mathbb{C} \mid (\mu \mathcal{B} - \mathcal{A}_0)^{-1} \text{ is a bounded operator on } X\} \neq \emptyset$.

Also, denote the isomorphism from \mathbb{R}^n onto \mathcal{N} by j . In what follows, we will use the graph norm $\|x\|_{\hat{\mathcal{A}}} = \|x\| + \|\hat{\mathcal{A}}x\|$ on $D(\hat{\mathcal{A}})$. Using this norm, $\hat{\mathcal{A}}$ becomes bounded and $\hat{X}_A := D(\hat{\mathcal{A}})$ becomes a closed space (see Hille and Phillips [20]).

THEOREM 2.1. *If the above conditions hold, we have the equivalence*

$$F(\mu)(\hat{\mathcal{A}} - \mu \hat{\mathcal{B}})E(\mu) = \begin{pmatrix} \Delta(\mu) & 0 \\ 0 & I_X \end{pmatrix}, \quad \mu \in \Omega \quad (2.7)$$

where $E : \Omega \rightarrow \mathcal{L}(\hat{X}, \hat{X}_A)$ and $F : \Omega \rightarrow \mathcal{L}(\hat{X}_A, \hat{X})$ are holomorphic bijective operator valued functions and

$$\Delta(\mu) = (\mu M - L)(\mu \mathcal{B} - \mathcal{A}_0)^{-1} \mu \mathcal{B} j \quad \mu \in \Omega. \quad (2.8)$$

Furthermore

$$\begin{aligned} E(\mu) \begin{pmatrix} c \\ \varphi \end{pmatrix} &= \begin{pmatrix} M(Q(\mu)jc + (\mu \mathcal{B} - \mathcal{A}_0)^{-1}\varphi) \\ Q(\mu)jc + (\mu \mathcal{B} - \mathcal{A}_0)^{-1}\varphi \end{pmatrix}, \\ E^{-1}(\mu) \begin{pmatrix} M\psi \\ \psi \end{pmatrix} &= \begin{pmatrix} j^{-1}Q(0)\psi \\ (\mu \mathcal{B} - \mathcal{A})\psi \end{pmatrix}, \\ F(\mu) \begin{pmatrix} c \\ \varphi \end{pmatrix} &= \begin{pmatrix} c - (\mu M - L)(\mu \mathcal{B} - \mathcal{A}_0)^{-1}\varphi \\ \varphi \end{pmatrix}, \\ F^{-1}(\mu) \begin{pmatrix} c \\ \varphi \end{pmatrix} &= \begin{pmatrix} c + (\mu M - L)(\mu \mathcal{B} - \mathcal{A}_0)^{-1}\varphi \\ \varphi \end{pmatrix}, \end{aligned}$$

where

$$Q(\mu)\varphi = \varphi - (\mu\mathcal{B} - \mathcal{A}_0)^{-1}(\mu\mathcal{B} - \mathcal{A})\varphi$$

Proof. Is similar to the one in [21, Theorem 3.1.], therefore is omitted. \square

Remark. The proof is based on the observation that for all $\mu \in \Omega$, $(\mu\mathcal{B} - \mathcal{A})$ has an n dimensional kernel and that the projection onto $\ker(\mu\mathcal{B} - \mathcal{A})$ is $Q(\mu)$. Since $\ker Q(\mu)$ is independent of μ , $Q(\mu_1) = Q(\mu_1)Q(\mu_2)$ and therefore $Q(\mu)j$ is an isomorphism from \mathbb{R}^n to $\ker(\mu\mathcal{B} - \mathcal{A})$. Applying $(MQ(\mu), Q(\mu))^T$ to $\mu\tilde{\mathcal{B}} - \tilde{\mathcal{A}}$, we find that the first equation is the characteristic matrix and the second equation is identically zero.

PROPOSITION 2.2. *The operator $(\mu\mathcal{B} - \mathcal{A}_0)$ has an inverse for all $\mu \in \mathbb{C}$, i.e., $\Omega \equiv \mathbb{C}$.*

Proof. Solve $(\mu\mathcal{B} - \mathcal{A}_0)\varphi = \psi$. To do so we expand it into

$$\varphi'(\theta) = \int_0^{\omega+\theta} d_\vartheta \eta(s + \theta, \vartheta)\varphi(\theta - \vartheta) + \mu \int_{\omega+\theta}^{\omega} d_\vartheta \eta(s + \theta, \vartheta)\varphi(\omega + \theta - \vartheta) - \psi(\theta)$$

by using (2.4) and (2.5). This is very similar to a renewal equation, but it cannot be solved by means of resolvent matrices. After integration by parts and changing integration limits, we get

$$\begin{aligned} \varphi'(\theta) &= \eta(s + \theta, \omega + \theta)\varphi(-\omega) + \int_{-\omega}^{\theta} \eta(s + \theta, \theta - \vartheta)\varphi'(\vartheta)d\vartheta + \\ &\mu \left(\eta(s + \theta, \omega)\varphi(\theta) - \eta(s + \theta, \omega + \theta)\varphi(0) + \int_{\theta}^0 \eta(s + \theta, \omega + \theta - \vartheta)\varphi'(\vartheta)d\vartheta - \psi(\theta) \right), \end{aligned}$$

where we used that $D(\mathcal{A}_0) = \{\varphi \in D(\mathcal{A}) \mid \varphi(-\omega) = 0\}$. Denote $\phi(\theta) = \varphi'(\theta)$, then the equation becomes

$$\phi(\theta) = \int_{-\omega}^{\theta} h_{s,\mu}(\theta, \vartheta)\phi(\vartheta)d\vartheta + \int_{\theta}^0 k_{s,\mu}(\theta, \vartheta)\phi(\vartheta)d\vartheta - \psi(\theta),$$

where

$$\begin{aligned} h_{s,\mu}(\theta, \vartheta) &= \eta(s + \theta, \theta - \vartheta) + \mu\eta(s + \theta, \omega) - \mu\eta(s + \theta, \omega + \theta) \\ k_{s,\mu}(\theta, \vartheta) &= \mu\eta(s + \theta, \omega + \theta - \vartheta) + \mu\eta(s + \theta, \omega + \theta). \end{aligned}$$

Define $\tilde{h}_{s,\mu}(t, \theta) = h_{s,\mu}(t, \theta)e^{-\gamma(\omega+\theta-\vartheta)}$, $\tilde{k}_{s,\mu}(t, \theta) = k_{s,\mu}(t, \theta)e^{-\gamma(\omega+\theta-\vartheta)}$ and $\tilde{\phi}(t) = \phi e^{-\gamma(\omega+\theta)}$. Then the original problem is equivalent to

$$\tilde{\phi}(\theta) = \int_{-\omega}^{\theta} \tilde{h}_{s,\mu}(\theta, \vartheta)\tilde{\phi}(\vartheta)d\vartheta + \int_{\theta}^0 \tilde{k}_{s,\mu}(\theta, \vartheta)\tilde{\phi}(\vartheta)d\vartheta - \psi(\theta)e^{-\gamma(\omega+\theta)}, \quad (2.9)$$

Choose γ such that

$$\sup_{\theta \in [-\omega, 0]} \left[\int_{-\omega}^{\theta} |h_{s,\mu}(t, \theta)|e^{-\gamma(\omega+\theta-\vartheta)}d\vartheta + \int_{\theta}^0 |k_{s,\mu}(t, \theta)|e^{-\gamma(\omega+\theta-\vartheta)}d\vartheta \right] < 1,$$

which is clearly possible, then (2.9) is a contraction mapping, which has a unique fixed point ϕ . After integrating ϕ , we get the solution. Then it follows that $(\mu\mathcal{B} - \mathcal{A}_0)$ is one-to-one and onto. \square

In numerical applications, computing (2.8) is not efficient. Instead, we use the isomorphism $R(\mu) : \mathbb{R}^n \rightarrow \ker(\mu\mathcal{B} - \mathcal{A})$,

$$R(\mu)c = \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} 0 & -L \\ 0 & \mu\mathcal{B} - \mathcal{A} \end{pmatrix}^{-1} \begin{pmatrix} c \\ 0 \end{pmatrix},$$

which is well defined if

$$D \left(\begin{pmatrix} 0 & -L \\ 0 & \mu\mathcal{B} - \mathcal{A} \end{pmatrix} \right) = D(\hat{\mathcal{A}})$$

in light of proposition 2.2. In this case, the equivalent characteristic matrix becomes

$$\tilde{\Delta}(\mu) = (\mu M - L)R(\mu).$$

3. Continuation of periodic orbits. Define the following nonlinear operator

$$G : X \times \mathbb{R}^p \rightarrow X, \quad G(\varphi, \lambda)(\theta) = x'(\theta) - f(\theta, \iota_\theta(\varphi), \lambda),$$

where $t \in [-T, 0]$, $(\iota_t(\varphi))(\theta) = \varphi((t + \theta) \bmod \omega)$. The domain of G is $D(G(\cdot, \lambda)) = \{\varphi \in \text{Lip}([- \omega, 0], \mathbb{R}^n) \mid \varphi(-\omega) = \varphi(0)\}$. We can extend this operator in the same way as in the previous section:

$$\hat{G}((c, \varphi), \lambda) = \begin{pmatrix} L\varphi - c \\ G(\varphi, \lambda) \end{pmatrix}, \quad (c, \varphi) \in D(\hat{G}(\cdot, \lambda)) = D(\hat{\mathcal{A}}).$$

Now we have the nonlinear equation for the periodic solutions of (2.1) in the form

$$\hat{G}((c, \varphi), \lambda) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (3.1)$$

Let $(c_0, \varphi_0) = u_0$ be a solution of (3.1) at λ_0 . Then the Frechet derivative of (3.1) is $\hat{\mathcal{A}}(\varphi_0) - \hat{\mathcal{B}}(\varphi_0)$. By the equivalence (2.7) we infer that $\hat{\mathcal{A}}(\varphi_0) - \hat{\mathcal{B}}(\varphi_0)$ is one-to-one and onto at a regular solution, i.e., when there is no fold bifurcation. In this case, the solution u_0 can be continued according to the Implicit Function Theorem.

3.1. Pseudo-arclength continuation. In our computations, we use the pseudo-arclength method to continue branches beyond limit points [9]. Here, we will use the following bilinear form

$$\langle \cdot, \cdot \rangle : \hat{X}_A \times \hat{X}_A \rightarrow \mathbb{R}, \quad \langle u, v \rangle = (Mv)^* Mu + \frac{1}{\omega} \int_{-\omega}^0 u^*(\theta)v(\theta)d\theta,$$

where the $*$ denotes the (conjugate) transpose. Let us denote a branch of solutions by $(u, \lambda) : I \subset \mathbb{R} \rightarrow \hat{X} \times \mathbb{R}$ parametrized with the arclength γ w.r.t. the norm $\|(u, \lambda)\| = \sqrt{\langle u, u \rangle} + |\lambda|$. Consider a solution $(u_0, \lambda_0) = (u(0), \lambda(0))$, which we want to continue in λ . Since u_0 is a regular solution, we can obtain the tangents $\dot{u}_0, \dot{\lambda}_0$ from the equation

$$(\hat{\mathcal{A}}(u_0) - \hat{\mathcal{B}}(u_0))\dot{u}_0 + D_\lambda \hat{G}(u_0, \lambda)\dot{\lambda}_0 = 0.$$

Further, we normalize the tangents such that $\|(\dot{u}_0, \dot{\lambda}_0)\| = 1$. With these tangents we predict the next solution point on the branch of solutions $u_1^{(0)} = d\gamma \dot{u}_0, \lambda_1^{(0)} = d\gamma \dot{\lambda}_0$, where $d\gamma$ is

the requested step-size. Now the Newton iteration of the correction step proceeds by solving

$$\begin{aligned} & \begin{pmatrix} \hat{A}(u_1^{(\nu)}) - \hat{B}(u_1^{(\nu)}) & D_\lambda \hat{G}(u_1^{(\nu)}, \lambda_1^{(\nu)}) \\ \langle \dot{u}_0, \cdot \rangle & \lambda_0 \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta \lambda \end{pmatrix} \\ &= \begin{pmatrix} -\hat{G}(u_1^{(\nu)}, \lambda_1^{(\nu)}) \\ d\gamma - \langle \dot{u}_0, u_1^{(\nu)} - u_0 \rangle - (\lambda_1^{(\nu)} - \lambda_0)\dot{\lambda}_0 \end{pmatrix} \end{aligned} \quad (3.2)$$

and updating $u_1^{(\nu+1)} = u_1^{(\nu)} + \Delta u$, $\lambda_1^{(\nu+1)} = \lambda_1^{(\nu)} + \Delta \lambda$. According to bordering theorems (see e.g. [9]), the coefficient operator in (3.2) is also one-to-one and onto, since $\hat{A}(u_0) - \hat{B}(u_0)$ is one-to-one and onto and $\lambda_0 \neq 0$ at a regular solution. The same is true in the case of a quadratic fold when $D_\lambda \hat{G}(u_0, \lambda_0) \notin \mathcal{R}(\hat{A}(u_0) - \hat{B}(u_0))$.

4. Continuation of bifurcations. In this section we reformulate classical continuation theorems of codimension-1 bifurcations for periodic DDEs. Since we have the equivalence with the finite dimensional characteristic matrix, these classical theorems can be applied almost directly. In order to compute the Jacobians, we will need derivatives of the characteristic matrix with respect to u , λ and the spectral parameter μ . For the sake of simplicity (and computational efficiency) we use $\tilde{\Delta}(\mu)$. In general, the derivative of the inverse of an operator valued function A is

$$(D_x A^{-1}(x))(y) = -A^{-1}(x)(D_x A(x))(A^{-1}(x), y). \quad (4.1)$$

Using this relation

$$\begin{aligned} D_u \tilde{\Delta}(\mu)c &= -(\mu M - L) \begin{pmatrix} 0 & -L \\ 0 & \mu \mathcal{B} - \mathcal{A} \end{pmatrix}^{-1} \times \\ &\times D_u \begin{pmatrix} 0 & -L \\ 0 & \mu \mathcal{B}(u) - \mathcal{A}(u) \end{pmatrix} \left(\begin{pmatrix} 0 & -L \\ 0 & \mu \mathcal{B} - \mathcal{A} \end{pmatrix}^{-1} \begin{pmatrix} c \\ 0 \end{pmatrix}, \cdot \right), \end{aligned}$$

$$\begin{aligned} D_\lambda \tilde{\Delta}(\mu)c &= -(\mu M - L) \begin{pmatrix} 0 & -L \\ 0 & \mu \mathcal{B} - \mathcal{A} \end{pmatrix}^{-1} \times \\ &\times D_\lambda \begin{pmatrix} 0 & -L \\ 0 & \mu \mathcal{B}(u) - \mathcal{A}(u) \end{pmatrix} \begin{pmatrix} 0 & -L \\ 0 & \mu \mathcal{B} - \mathcal{A} \end{pmatrix}^{-1} \begin{pmatrix} c \\ 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} D_\mu \tilde{\Delta}(\mu)c &= M R(\mu)c - (\mu M - L) \begin{pmatrix} 0 & -L \\ 0 & \mu \mathcal{B} - \mathcal{A} \end{pmatrix}^{-1} \times \\ &\times \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{B}(u) \end{pmatrix} \begin{pmatrix} 0 & -L \\ 0 & \mu \mathcal{B} - \mathcal{A} \end{pmatrix}^{-1} \begin{pmatrix} c \\ 0 \end{pmatrix}. \end{aligned} \quad (4.2)$$

Note, that in the above formulae, it is always the same operator that is inverted.

4.1. Fold bifurcation. In case of a fold bifurcation there is a +1 eigenvalue of the monodromy operator, which condition is equivalent to $\det \Delta(1) = 0$. This implies that we have unit vectors p and q , ($p^* p = 1$, $q^* q = 1$) such that $0 = p^* \Delta(1)$ and $0 = \Delta(1)q$. The

kernel of $\hat{\mathcal{A}} - \hat{\mathcal{B}}$ and $(\hat{\mathcal{A}} - \hat{\mathcal{B}})^*$ is spanned by $\varphi = E(1)(q, 0)^T$ and $\psi = F^*(1)(p, 0)^T$, respectively. The defining system is (see e.g. Beyn et al. [1])

$$\begin{aligned}\hat{G}(u, \lambda) &= 0, \\ \tilde{\Delta}(1)q &= 0, \\ q_0^*q - 1 &= 0.\end{aligned}$$

The Jacobian with respect to (u, q, λ) is

$$\begin{pmatrix} \hat{\mathcal{A}}(u_0) - \hat{\mathcal{B}}(u_0) & 0 & D_\lambda \hat{G}(u_0, \lambda_0) \\ D_u \tilde{\Delta}(1)q_0 & \tilde{\Delta}(1) & D_\lambda \tilde{\Delta}(1)q_0 \\ 0 & q_0^* & 0 \end{pmatrix}. \quad (4.3)$$

PROPOSITION 4.1. *The Jacobian in (4.3) is one-to-one and onto at a quadratic fold.*

Proof. Same as for algebraic systems, therefore omitted. (see Doedel et al. [9, Section 2.3.]). \square

4.2. Period doubling. Similarly, the defining system is

$$\begin{aligned}\hat{G}(u, \lambda) &= 0, \\ \tilde{\Delta}(-1)q &= 0, \\ q_0^*q - 1 &= 0.\end{aligned}$$

and the Jacobian with respect to (u, q, λ) yields

$$\begin{pmatrix} \hat{\mathcal{A}}(u_0) - \hat{\mathcal{B}}(u_0) & 0 & D_\lambda \hat{G}(u_0, \lambda_0) \\ D_u \tilde{\Delta}(-1)q_0 & \tilde{\Delta}(-1) & D_\lambda \tilde{\Delta}(-1)q_0 \\ 0 & q_0^* & 0 \end{pmatrix}. \quad (4.4)$$

4.3. Neimark-Sacker bifurcation. The defining system

$$\begin{aligned}\hat{G}(u, \lambda) &= 0, \\ \tilde{\Delta}(e^{i\alpha})q &= 0, \\ q_0^*q - 1 &= 0\end{aligned}$$

and the Jacobian with respect to (u, q, α, λ)

$$\begin{pmatrix} \hat{\mathcal{A}}(u_0) - \hat{\mathcal{B}}(u_0) & 0 & 0 & D_\lambda \hat{G}(u_0, \lambda_0) \\ D_u \tilde{\Delta}(e^{i\alpha_0})q_0 & \tilde{\Delta}(e^{i\alpha_0}) & D_\alpha \tilde{\Delta}(e^{i\alpha_0})q_0 & D_\lambda \tilde{\Delta}(e^{i\alpha_0})q_0 \\ 0 & q_0^* & 0 & 0 \end{pmatrix}. \quad (4.5)$$

Here we use complex values in the last two rows of the Jacobian, which can be expanded to reals, and then use the same argument to prove the regularity as in [9, Section 2.3.].

4.4. Remark on the autonomous case. The presented method can easily be extended for autonomous equations by adding a phase condition to the defining systems and the period to the list of unknown variables [11]. In the case of a fold bifurcation in autonomous systems, the characteristic multiplier $+1$ has algebraic multiplicity 2, therefore, it is not sufficient to use $\Delta(1)q$, but by the equivalence of Jordan chains [21],

$$\tilde{\Delta}(1)q_1 = 0,$$

$$D_\mu \tilde{\Delta}(1)q_1 + \tilde{\Delta}(1)q_2 = 0, \quad (4.6)$$

$$q_1^*q_2 = 0. \quad (4.7)$$

The size of this system of equations can be reduced by computing q_1 and $D_\mu \Delta(1)q_1$. The eigenfunction ϕ_1 of the trivial multiplier 1 is always the derivative of the solution, which equals the right hand side of the equation $\phi_1(\theta) = f(i_\theta(u_0), \lambda)$. Further, from the analysis above, we infer that $q_1 = M\phi_1 = f(i_{(0)}(u_0), \lambda)$. On the other hand, $D_\mu \Delta(1)q_1$ can be computed according to (4.2) as

$$D_\mu \tilde{\Delta}(\mu)q_1 = f(i_{(0)}(u), \lambda) - (M - L) \begin{pmatrix} 0 & -L \\ 0 & \mathcal{B} - \mathcal{A} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \mathcal{B}f(i_{(\cdot)}(u), \lambda) \end{pmatrix}.$$

By using bordering, (4.6) reduces to

$$\begin{pmatrix} \tilde{\Delta}(\mu) & D_\mu \tilde{\Delta}(\mu)q_1 \\ q_1^* & 0 \end{pmatrix} \tilde{q}_2 = 0,$$

where \tilde{q}_2 is a new variable, which contains q_2 in its first n coordinates. We remark that in [7], test functionals for folds in ODEs are constructed in a similar way.

5. Implementation. In this section, we summarize the numerical methods used in our continuation software that uses collocation [5] to discretize the operators \mathcal{A} and \mathcal{B} of equations with discrete fixed delays. Thus, equation (2.1) can be written as

$$x'(\tilde{t}) = \tilde{f}(\tilde{t}, x(\tilde{t} - \tilde{\tau}_1), \dots, x(\tilde{t} - \tilde{\tau}_m)),$$

where $\tilde{f} : \mathbb{T}^1 \times \mathbb{R}^n \times \dots \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $0 \leq \tilde{\tau}_1 < \tilde{\tau}_2 < \dots < \tilde{\tau}_m \leq \omega$. By rescaling the time with $\tilde{t} = \omega t$ and defining $f(t, x_1, \dots, x_m) = \tilde{f}(\omega t, x_1, \dots, x_m)$, our equation becomes

$$x'(t) = \omega f(t, x(t - \tau_1), \dots, x(t - \tau_m)).$$

Its variational system at v is

$$x'(t) = \sum_{i=1}^m D_{x_i} f(t, v(t - \tau_1), \dots, v(t - \tau_m))x(t - \tau_i).$$

Also, assume that the initial time $s = 0$. Let $A_i, B_i : X \times [-1, 0] \rightarrow \mathbb{R}^{n \times n}$ defined by

$$A_i(v, \theta) = \omega H(1 + \theta - \tau_i) D_{x_i} f(\theta, v((\theta - \tau_1) \bmod 1), \dots, v((\theta - \tau_m) \bmod 1)),$$

$$B_i(v, \theta) = \omega H(\tau_i - 1 - \theta) D_{x_i} f(\theta, v((\theta - \tau_1) \bmod 1), \dots, v((\theta - \tau_m) \bmod 1)),$$

where mod maps to $[-1, 0]$ and H is the Heaviside function. Then (2.4) and (2.5) become

$$(\mathcal{A}(v)\varphi)(\theta) = \varphi'(\theta) - \sum_{i=1}^m A_i(\theta)\varphi(\theta - \tau_i)$$

$$(\mathcal{B}(v)\varphi)(\theta) = \sum_{i=1}^m B_i(\theta)\varphi(1 + \theta - \tau_i).$$

5.1. Collocation. Here, we summarize the main steps of the method, its detailed discussion can be found in [11]. We approximate the solution φ on n intervals by degree m polynomials. The method requires the space X to be $C^m([-1, 0], \mathbb{R}^n)$ and consequently $D(\mathcal{A}(v)) = C^{m+1}([-1, 0], \mathbb{R}^n)$ and f also C^m -smooth in t . The mesh is represented by the intervals $[\theta_i, \theta_{i+1}]$, $i = 0, \dots, n-1$ with $-1 = \theta_0 < \theta_1 < \dots < \theta_n = 0$. Also denote the length of an interval by $h_i := \theta_{i+1} - \theta_i$. The approximate solution can be written on the i -th subinterval as

$$\tilde{\varphi}(\theta) = \sum_{j=0}^m \varphi(\theta_{i+\frac{j}{m}}) P_{i,j}(\theta),$$

where

$$P_{i,j}(\theta) = \prod_{r=0, r \neq j}^m \frac{\theta - \theta_{i+\frac{r}{m}}}{\theta_{i+\frac{j}{m}} - \theta_{i+\frac{r}{m}}}$$

and $\theta_{i+\frac{j}{m}} := \theta_i + j \frac{h_i}{m}$. Let c_i , $i = 1, \dots, m-1$ be the roots of the degree m Legendre polynomial scaled from $[-1, 1]$ to $[0, 1]$. Then define the collocation points by $c_{j,i} = \theta_j + h_j c_i$. The discretized version of operator \mathcal{A} becomes

$$(\mathcal{A}_d(v)\tilde{\varphi})(c_{j,i}) = \tilde{\varphi}'(c_{j,i}) - \sum_{i=1}^m A_i(c_{j,i}) \tilde{\varphi}(c_{j,i} - \tau_i). \quad (5.1)$$

Operator \mathcal{B} is discretized similarly. Note that (5.1) is an operator from $\mathbb{R}^{(m \times n - 1) \times n}$ to $\mathbb{R}^{m \times n \times n}$, so it has (at least) an n dimensional kernel just as its abstract counterpart (2.4) has. The discretized version of $\hat{\mathcal{A}}$ acting on $(M\varphi, \varphi)$ can be represented by a square matrix by placing the $-L\varphi$ vector into the first n rows, which is of full rank at a regular solution v according to [10, Theorem 3.1] when $|h| = \max_{i=0}^{n-1} h_i$ is sufficiently small. Moreover, if we choose the space $X = C^m([-1, 0], \mathbb{R}^n)$ as described above then there exists a $\delta > 0$ and $C > 0$ such that the solutions of $\hat{\mathcal{A}}\psi = \hat{\mathcal{B}}\varphi$ and $\hat{\mathcal{A}}_d\tilde{\psi} = \hat{\mathcal{B}}_d\tilde{\varphi}$ satisfy

$$\max_{\theta \in [-1, 0]} |\psi(\theta) - \tilde{\psi}(\theta)| \leq C|h|^m \quad (5.2)$$

if $|h| \in (0, \delta]$ (see Engelborghs and Doedel [10, Theorem 3.3]). Although superconvergence is lost for delay equations, some improvement can be achieved over (5.2) by using modified collocation schemes (see [10, Section 4] and [11]).

5.2. Solving bordered linear systems. Equations (3.2), (4.3), (4.4) and (4.5) contain the operator $\hat{\mathcal{A}} - \hat{\mathcal{B}}$, functionals like $\langle \dot{u}_0, \cdot \rangle$ on \hat{X} , vectors like $D_\lambda \hat{G}(u_0, \lambda_0)$ in \hat{X} , and also the finite dimensional characteristic matrix. From a computational point of view, it is better to factorize the discretized counterparts of these components separately, rather than copying them into one big matrix. Usually, the discretized infinite dimensional part has a special sparsity structure, which can be exploited in computations. Other reasons for bordering methods might be the reuse of code and avoiding the frequent copying of large data structures. In our code, we used methods developed by Govaerts [13, 14] and Govaerts and Pryce [15]. These methods have the advantage over classical bordering methods [9, 1] that they do not require the calculation of the kernel of singular matrices for the computation and can be used with arbitrary stable solvers. This feature makes it possible to use the same routine for sparse (e.g., $\hat{\mathcal{A}}_d - \hat{\mathcal{B}}_d$ with discrete delays) and dense (e.g., $\hat{\mathcal{A}}_d - \hat{\mathcal{B}}_d$ with distributed delays) matrices, leaving space for future improvements.

The method of Mixed Block Elimination (BEM) is used in factorizing the discretized counterpart of the Jacobian in Eqn. (3.2). For solving

$$\begin{pmatrix} A & b \\ c^* & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

type of equations, the method requires a solution with A and with A^* . Note, that for this solution A is factorized only once. For the bigger problem

$$\begin{pmatrix} A & 0 & b \\ C & D & e \\ f^* & g^* & h \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix},$$

which appears in Eqns. (4.3) and (4.4), we apply the Generalized Mixed Block Elimination (GMBE) technique, which requires one solution with D^T and two BEM steps with A and D , respectively. In case of systems augmented with wider borders, BEMW is used, which is a straighten out form of the recursive version of BEM. These wider borders occur in Eqns. (4.5), (4.3), and also in (4.4) when it is further augmented with derivatives and tangent vectors in the two-parameter continuation context similarly to Eqn. (3.2). These kinds of equations can be solved using GMBE but BEM replaced with BEMW.

Factorization of sparse matrices was carried out using UMFPACK¹, which implements the Unsymmetric Multifrontal method of direct LU factorization (see Davies [4]). For (nearly) optimal performance we construct the matrices $\hat{A}_d - \hat{B}_d$ directly in compressed row form instead of the default compressed column form. After factorization, we interchange the solution methods for the matrix and its transpose to get the correct results. For dense matrices, we use the standard LAPACK routine GESVX.

6. Example. In this section we analyse bifurcations of period-2 orbits in an interrupted machining model. To this end, we consider an extended version of the equation of motion in [23] in the form

$$\begin{aligned} x''(t) + 2\zeta x'(t) + x(t) &= wg(t)H(1 + x(t - 2T) - x(t - T))F_c(1 + x(t - T) - x(t)) \\ &+ wg(t)H(1 + x(t - T) - x(t - 2T))F_c(2 + x(t - 2T) - x(t)), \end{aligned} \quad (6.1)$$

where H is the Heaviside function,

$$g(t) = \begin{cases} 0 & \text{if } \exists k : kT \leq t < (k + 1 - \rho)T \\ 1 & \text{if } \exists k : (k - \rho)T \leq t < kT \end{cases}, \quad k \in \mathbb{Z}$$

and

$$F_c(x) = H(x) \left(\frac{120}{59}x - \frac{48}{59}x^2 + \frac{35}{177}x^3 \right).$$

This cutting force function F_c is a polynomial approximation of the common three-quarter rule $F_c(x) \approx 4/3 x^{3/4}$ (see [24]). Also, $\zeta = 0.015$ is the relative damping, $\rho = 0.1$ is the ratio between the period length T and the time spent with cutting. Equation (6.1) has a unique T -periodic solution, which is described by the ODE

$$x''(t) + 2\zeta x'(t) + x(t) = wg(t)F_c(1).$$

¹We remark that this package is incorporated into MATLAB as well.

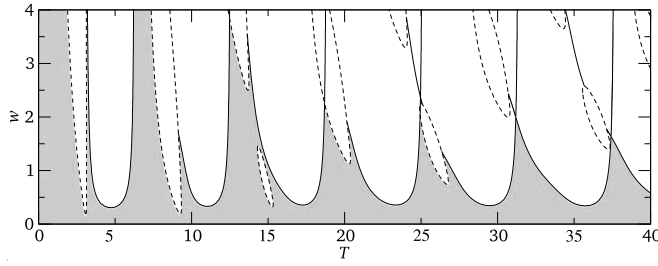


FIGURE 6.1. Stability chart showing period doubling boundaries (dashed lines) and Neimark-Sacker boundaries (continuous lines). In the gray region the T periodic solution is stable. ($\zeta = 0.015$)

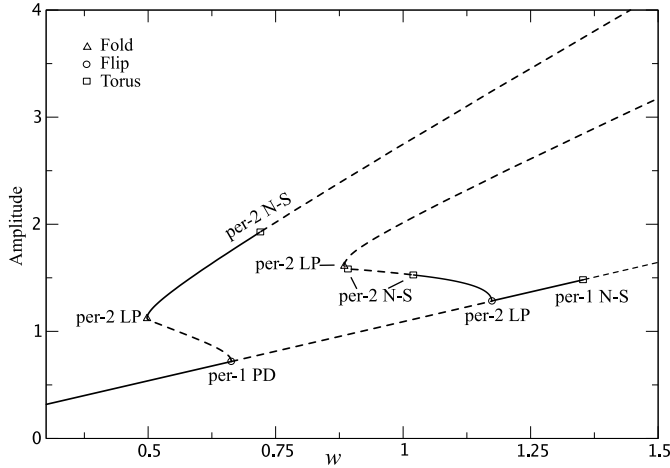


FIGURE 6.2. This bifurcation diagram shows the period-2 orbits emanating from the period-1 solution at the stability boundary in the vertical cross-section of a lens at $T = 14.75$ in Fig. 6.1. Continuous lines denote stable solutions while dashed lines correspond to unstable solutions.

In a quite large neighborhood of this periodic solution, (6.1) simplifies to

$$x''(t) + 2\zeta x'(t) + x(t) = wg(t)F_c(1 + x(t-T) - x(t)), \quad (6.2)$$

while the second line of (6.1) is turned on if the tool did not cut the workpiece in the previous period. Its variational equation around the T -periodic solution is

$$x''(t) + 2\zeta x'(t) + x(t) = wg(t)(x(t-T) - x(t)).$$

The stability of the above equation was analyzed analytically in [23] in great detail. The stability chart is shown in Fig. 6.1. We analyze the lowest lens at $T \approx 15$. The bifurcation diagram at $T = 14.75$ can be seen in Fig. 6.2. The first period doubling bifurcation at $w \approx 0.6629$ is subcritical. The arising unstable period two orbit folds back and then loses its stability through a torus bifurcation. The other period doubling bifurcation is supercritical. On the stable branch there is a torus bifurcation window. This branch folds back, too and becomes unstable.

The bifurcations in Fig. 6.2 were continued in two parameters using our method. First, the stability boundaries of (6.2) are obtained, which coincide the semi-analytical results of [23]. The period T orbits are also checked analytically. In order to avoid difficulties

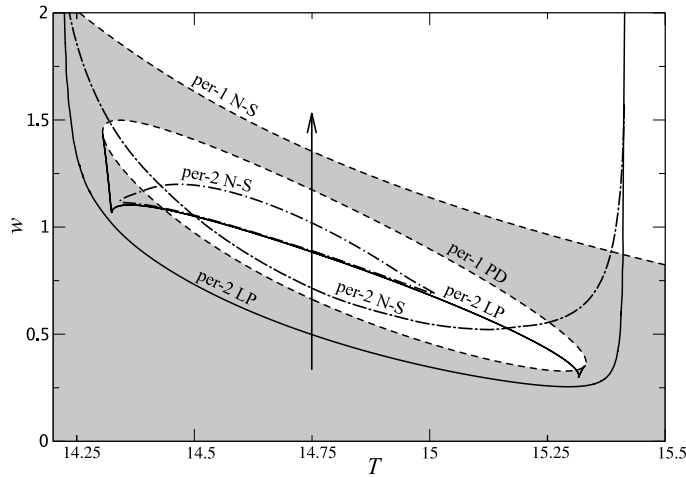


FIGURE 6.3. Bifurcations of period-2 orbits. Dashed lines correspond to the stability of period-1 solutions (see Fig. 6.1). Continuous lines are fold bifurcation curves, dash-dotted lines are Neimark-Sacker stability curves of period-2 solutions.

arising from the non-smoothness of (6.1), we substitute the Heaviside function $H(x)$ by $(1 + \tanh(Cx))/2$ with $C = 30$. Choosing bigger values for C is not reasonable because it only gives results within line thickness but makes the computation more difficult due to derivatives tending to infinity. The period-2 orbits arising at the lens in the space of two parameters extended with the solution amplitude form a highly distorted tube-like object. Figure 6.3 shows the computed bifurcation curves. The lower continuous curve corresponds to the fold bifurcation of orbits arising at the lower part of the lens. The other continuous curve shows the fold of the upper solution branch in Fig. 6.2. It can be seen that there are two cusp points on the bifurcation curve, which in addition, connect to the period doubling boundary of period-1 solutions separating sub- and supercritical period doubling bifurcations. The dashed-dotted lens near to the upper fold curve is an unstable island on the supercritically bifurcated family of period-2 solutions, while the other dash-dotted line corresponds to the stability loss of motion on the other branch of orbits.

7. Conclusions. Periodic delay differential equations frequently arise in applications describing engineering, biological, etc. phenomena. Usually, the first question about these parameter dependent models is the bifurcation curves of periodic orbits. In this paper we constructed a characteristic matrix for general periodic DDEs in order to use it in the continuation of bifurcations. We also presented the equivalence of the characteristic matrix with the monodromy operator, which makes regularity proofs of defining systems of equations similar to the proofs in the case of finite dimensional algebraic systems. We implemented the algorithm in a software written in C++, which uses collocation to discretize the infinite dimensional operators. Bordering techniques were also used in the solution of linear systems extended with the characteristic matrix, some derivatives, and tangent vectors. Further, an example of an interrupted machining model was analyzed.

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