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The shape of two-dimensional space

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"There are indeed whole branches of mathematics dealing with the assessment of complexity, which no one has the ability or the imagination to make use of for grasping the biological situation... The difficulty about applying topological analysis to embryonic development, for instance, is that morphological form may change considerably, ... while its topological status remains unchanged." [1]

Genomics, so fashionable today, is only half of the secret of life. The other half of the secret is shape, form, morphogenesis and metamorphosis [1–4]. The gene may prescribe what is synthesized, but the proteins appear and operate in a pre-existing environment that they then change. The first step towards life is the appearance of a micelle, with a spherical membrane, a surface that separates the world into inside and outside.

We are here concerned with surfaces, with a particular subset of two-dimensional manifolds (a word that originally designated the intestines or bowels of an animal; an animal with its gut is toplogically a torus) embedded in three-dimensional Euclidean space, namely the non-self-intersecting, periodic minimal surfaces [5] of cubic symmetry, which separate the world into two regions as an infinite plane would do, but with much more complex topologies. Like the Platonic solids [6], these cubic surfaces are geometrical absolutes and have distinctive topologies but entail no arbitrary parameters (there are many more surfaces with other lower symmetries but these have one or more parameters on which the configuration depends; there will thus be critical parameters separating one topology from another and perhaps a complicated phase diagram). The objective is to enumerate at least some of these surfaces, for probably an infinite number answer to this description, to draw attention to their geometry and to point to some of their applications and occurrences on various scales between mega-engineering and nano-technology. These objects are solutions looking for problems.

Minimal surfaces have, at every point, except at certain singular 'flat' points (which in fact define the surfaces), two principal curvatures, k_1 and k_2, which are equal and opposite. This is equivalent to requiring that the divergence of the unit normal to the surface should be zero. Their product, k_1k_2, the Gaussian curvature, is thus negative (or zero). From this it follows that the metric on the surface is not that of a plane but is non-Euclidean, in that the perimeter of a small circle is more than $2\pi r$ and its area more than πr^2 [if we consider a sheet of cells, where each cell requires more than six neighbours then, as the cells multiply, more area than is appropriate for a plane sheet is produced and the result is a structure like seaweed

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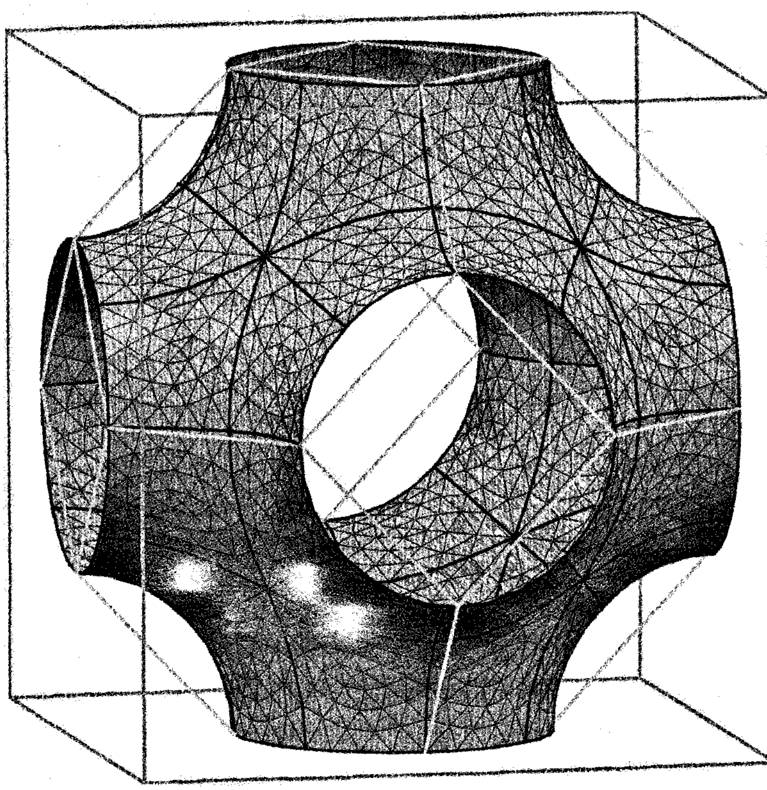
(e.g. *Fucus letuca*)]. This has the consequence that geodesic trajectories on the surface are usually chaotic and neighbouring geodesic trajectories may diverge with a certain Lyapunov coefficient. Each non-self-intersecting periodic minimal surface divides the world into two sub-spaces which may be either congruent (we refer to these as balanced) or different from each other. We may consider either the two sides of the surface or the two separated volumes as two parallel universes (cosmologists have also taken note of the properties of such surfaces).

To introduce the ideas we illustrate the P-surface (where P is for primitive; Figure 1), the simplest to apprehend, which was discovered by H.A. Schwarz in about 1865 [7]. A surface of zero mean curvature separates the world into two congruent domains. It is made up of elementary curved triangles (Figure 2), the angles of which are 90°, 45° and 30° (adding up to less than 180°), which are soap-film surfaces where the area of the film is minimized. The partial differential equation is simply $\mathrm{div}(n)=0$, where n is the unit normal. Exact algebraic description [8–11], starting from the Weierstrass integrals, leads to elliptical functions, but an approximate description is simply $\cos(x)+\cos(y)+\cos(z)=0$. The space group is $Pm\bar{3}m$ if the two sides of this triangle are different and $Im\bar{3}m$ if they are the same. The surface can also be regarded as being composed of catenoids between square frames.

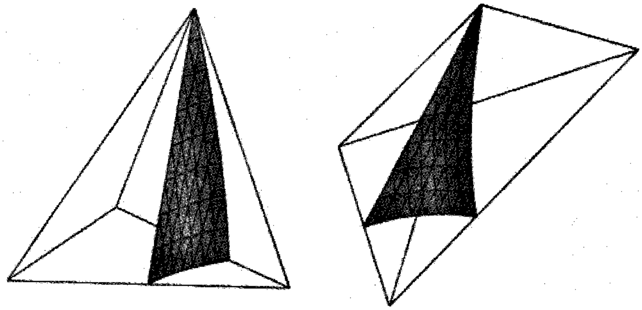
Figure 1



Parallel red and blue universes

The P-surface (due to H.A. Schwarz [7])

Figure 2



The asymmetrical unit of the P-surface, which is repeated by the kaleidoscopic cell

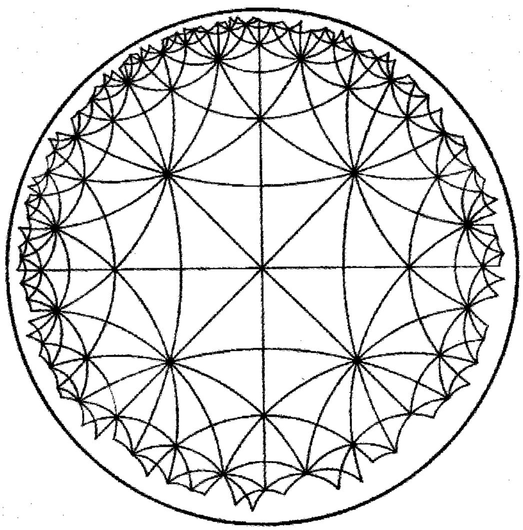
The angles are 90°, 45° and 30°, and the straight edge is a diad.

This surface is close to the surface of zero electrical potential in the structure of CsCl and we may recall Maxwell's equation $\mathrm{div}(E)=0$. Electrical equi-potentials in periodic arrangements of charges involve related surfaces.

The connectivity of the triangles can be plotted in a stereogram (Figure 3) in the hyperbolic space H^2, showing that the two-dimensional space is non-Euclidean, but in this projection the circuits which exist in R^3 are not apparent.

There are several main ways of regarding these periodic minimal surfaces (PMS). The first is to identify the geometric components from which they might be considered to be assembled by analogy with combinatorial chemistry. A catenoid is a tubular tunnel, a soap-film surface, hung between two circular or polygonal rings. These components are catenoids, branched catenoids, polyhedral

Figure 3



Hyperbolic stereogram of the P-D-G-surfaces (in which angles are preserved)

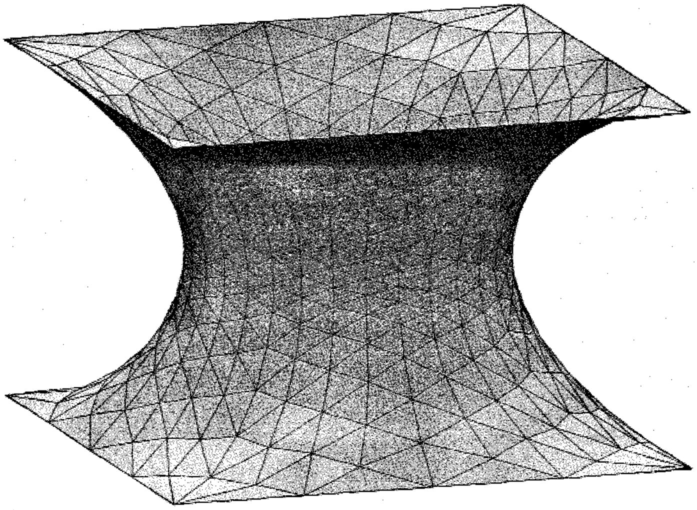
The stereogram shows the local connections of the unit triangles, which have angles of 90°, 45° and 30°.

joints and Scherk towers (intersecting planes sewn together with alternating catenoidal holes; see Figures 4, 5 and 6). An assembly of intersecting planes can be converted into what might be a minimal surface in this way by removing the self-intersections. This is geometrical combinatorics and a few of the possible combinations can be refined to become exact minimal surfaces. In some cases networks can be converted to PMS by the expansion of links into catenoids. The simplest example of this is the diamond network, bonds in which expand to give the D-surface (also discovered by Schwarz). The D-surface is built up of catenoids between equilateral triangles and the sides of these triangles are diad axes, which repeat the surface to fill space.

There are two classes of PMS, balanced and unbalanced, i.e. those where the two sub-spaces, and hence their networks of tunnels, are identical, and those where they are different. In space groups with diad axes, which frame the patches, giving balanced surfaces (probably 24 in number), the two sub-spaces must be congruent. In the case of Schoen's gyroid the diad axes are perpendicular to the surface and the two sub-spaces are inversion images of each other. The number in the balanced class is probably finite and the number in the unbalanced class appears indefinitely large.

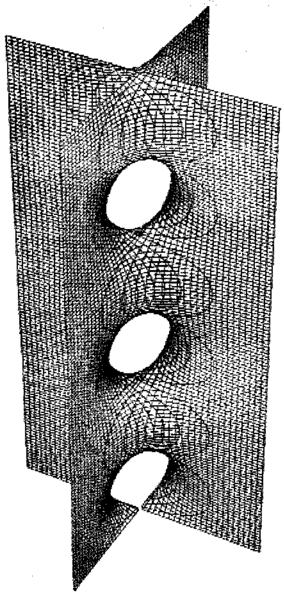
Since we are dealing with soap-film surfaces, which have a surface tension, then conditions for mechanical equilibrium apply. For example, in a tetrahedral asymmetrical region, bounded by four mirror planes, the lines of action of the forces that act normally to these planes must run through a common point; since the films intersect the mirrors normally, the forces, i.e. the lengths of the surface intersecting the mirror, must be of a magnitude proportional to the areas of the faces. The tetrahedron itself is the Maxwell polyhedron of forces. Although small regions of each surface are of minimal area, the structures as a whole are unstable and would collapse instantly unless there were a supporting framework.

Figure 4



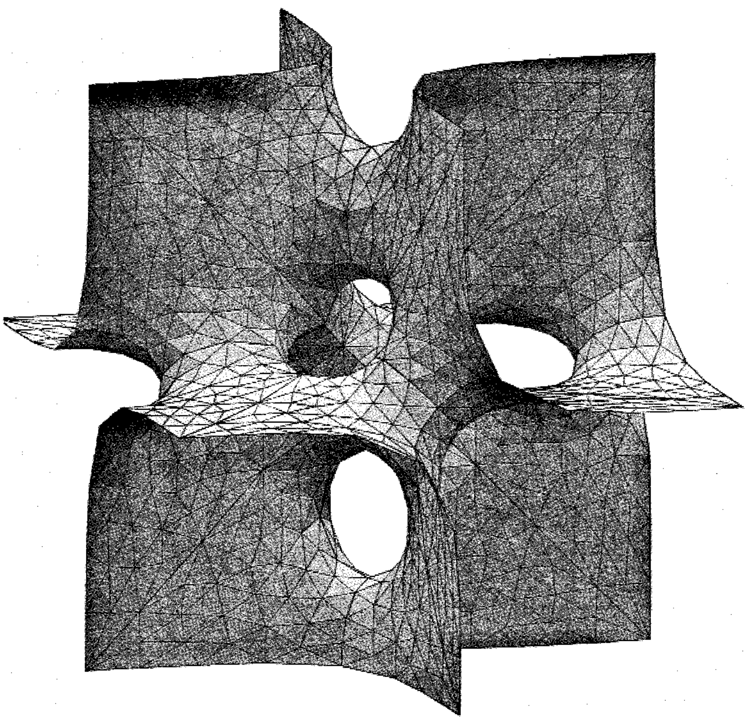
The square catenoid

Figure 5



A Scherk tower

Figure 6



A tetrahedral vertex

For the catenoid suspended between two rings (Figure 4) there is a critical distance of separation beyond which no minimal surface is possible. For a lesser separation there are two solutions, a stable one with a larger waist and an unstable one with a smaller waist. Each has $H=0$ everywhere. Willmore surfaces are those for which the integral of H^2 over the surface is a minimum, and PMS are a special case of this.

In connection with this we may ask how far a tetrahedral joint of soap-films intersecting the faces normally may be deformed. The tetrahedral network of sodalite is somewhat beyond this limit, so that, although it looks a promising structure, it is not an exact minimal surface.

The second way to regard PMS is to begin with the asymmetric region in a particular cubic space group and to hang a patch of surface across this region, usually a tetrahedron or a simple polyhedron, in conformity with the symmetry elements which repeat the region round to fill all space. In the simplest case the asymmetric region is a kaleidoscopic cell and all its faces are mirror planes, which an element of surface must intersect normally. Where diad axes occur the film must have straight lines and the surface may be called balanced. There may be two or perhaps more ways of choosing the diad axes to bound a soap film. If the P-surface is one way, then the second way is called the complement of P, abbreviated to C(P). Various catenoids and branched catenoids may then connect the patches bounded by straight lines in less obvious ways.

The third method is to consider the structure factor graphs for the various space groups [12]. These are contours for which a particular structure factor, the sum of symmetry-related sine waves of density, is strong or zero. This concept was introduced by W.L. Bragg and H. Lipson in 1936 [13] for solving crystal structures, but they could then compute only two-dimensional graphs. These contours have a direct connection with the placing of the scattering atoms. If all the atoms lie in such a surface then the corresponding structure factor will show up very strongly in the X-ray diffraction patterns. We have more recently been reminded of the structural implications of such surfaces by Brenner et al. [14]. It is now easy with the Mathematica program ContourPlot3D to plot $f(x,y,z)=0$, the zero-level contour of a function, but more difficult to plot a maximum.

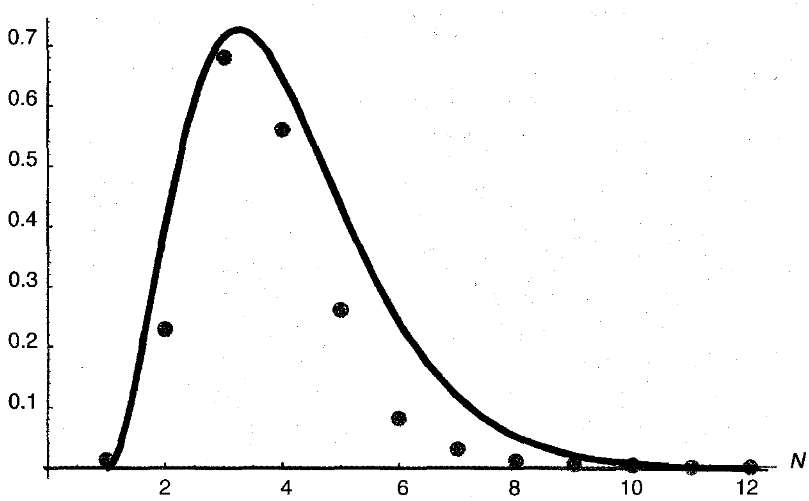
Some of these three-dimensional phase contours can be refined to become exact PMS but in any case they suggest surfaces of fascinating and intricate topologies. The adjustment can be expressed by adding further structure-factor terms to make up the nodal surface function $f(x,y,z)=0$ to approach with any required degree of accuracy the condition $H=0$.

Fourthly, K.A. Brakke has developed a method of adjoint or conjugate surfaces [15] to produce, with A.H. Schoen, several series of new surfaces, which may be seen on his website (http://www.susqu.edu/facstaff/b/brakke). Surfaces such as the P-, G- and D-surfaces are related to each other by the Bonnet transformation and this process can be applied more generally.

For a few surfaces it is possible to give an exact description in terms of the Weierstrass integrals, but in general it is necessary to use approximate methods, particularly the remarkable finite-element-analysis program Surface Evolver, developed by Brakke and made freely available (see [16] and http://www.susqu.edu/facstaff/b/brakke), which solves almost all problems to do with surface tension for both stable and unstable surfaces.

We are here engaged in combinatorics with strong limitations. An instructive analogy is provided by the Inorganic Crystal Structure Database (ICSD). How many different crystal structures can be formed with N different chemical elements (each atom in a proper place; solid solutions being excluded)? A naive expectation would be a rapid rise, such as factorial N, but the data (Figure 7) show that, after a maximum at three elements, the numbers drop off very rapidly. Complexity is thus severely limited. Examination of the unit cell dimensions of crystals shows similar limitations, and large unit cells are rare and usually hierarchical in structure (being simple assemblies of complex units). The plotted values fit reasonably to the calculated curve, which points in an interesting direction. The curve is, in fact, the distribution of energy against frequency for the modes of electromagnetic waves inside a black-body cavity. This was found by Max Planck $\{n^3/(\exp[n/t]-1)\}$ as a solution to the "ultraviolet catastrophe" problem implicit in the earlier theory of Rayleigh, and this solution involved the postulation of the quantization of the energies of modes of vibration. That is, this curve first led Planck to quantum theory, so that it is tempting to suggest that the frequency curve of the occurrence of spatial structures is a consequence of a corresponding spatial atomicity (a similar phenomenon is perhaps to be seen in the very limited number of circuits emerging in the randomly connected dynamic networks of Stuart Kauffmann; at certain connectivities they go periodic with surprisingly small periods [17]). Undoubtedly the requirement that a structure should exist in three-dimensional space is extremely stringent and we have a kind of Democritean Exclusion Principle: no two atoms can have the same x, y, z and t co-ordinates (t representing the time dimension). A ruler or a crystalline repeat period must be either N or $N+1$ atoms long. (E. Fredkin has suggested that space is grainy on the

Figure 7



Frequency distribution of inorganic crystal structures having *N* different elements

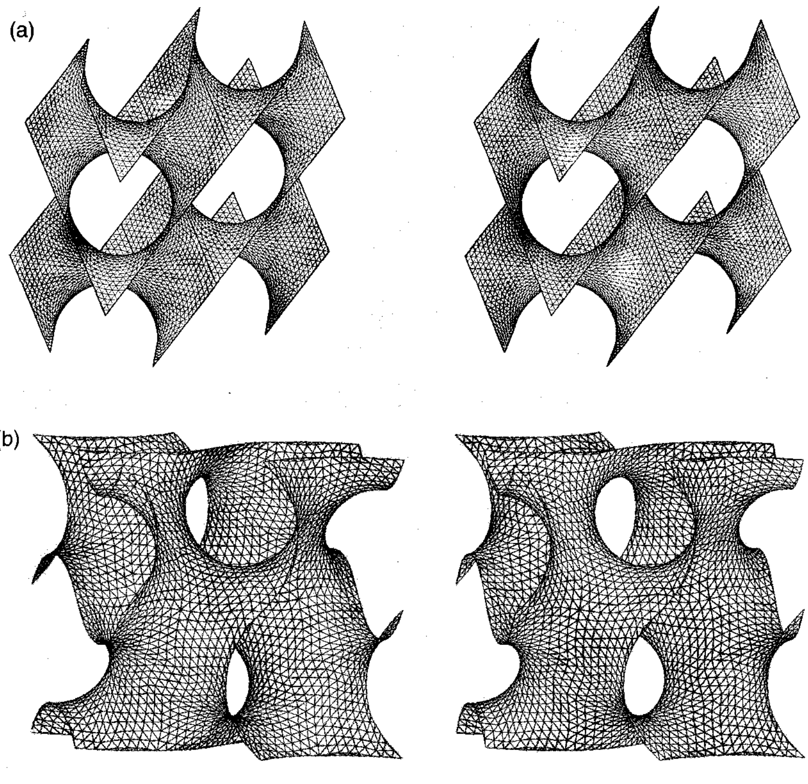
The curve of Planck's law of black-body emission versus frequency is shown. The maximum for N=3 is approx. 19000.

scale of the Planck length and that an infinite number of digits cannot be contained in an infinitesimal cube. Quasicrystal structures result when repeat periods are incommensurable. Fredkin implies that the square root of 2 must be in practice rational [17a].) The analogy is suggestive rather than rigorous.

As regards the various PMS we believe that the theoretical number is unlimited, but that when it comes to those actually realized by natural systems, the number is very small. Figure 8 shows a selection of the surfaces and Table 1 is a exhaustive list of the balanced surfaces. The P-, D- and G-surfaces are those most frequently observed and are the simplest. The rapidity with which the finite-element program converges in these shows why they are preferred by natural systems. Brakke's program [16] can show the eigenvalue structure of the refinement, which indicates the topography of the parameter space.

Methods have now been developed for characterizing networks in three dimensions by the Delaney–Dress symbol [18], which acts as a kind of inorganic gene, a linear symbol from which the network can be reconstructed. This concept has been applied by E.A. Lord to the more limited structure of a patch hung across the asymmetric unit of a cubic space group, often a tetrahedron, to give a unique name to each kind of surface. The nomenclature of the PMS is still fluid,

Figure 8



(contd.) ☞

Stereo pair views of representative surfaces

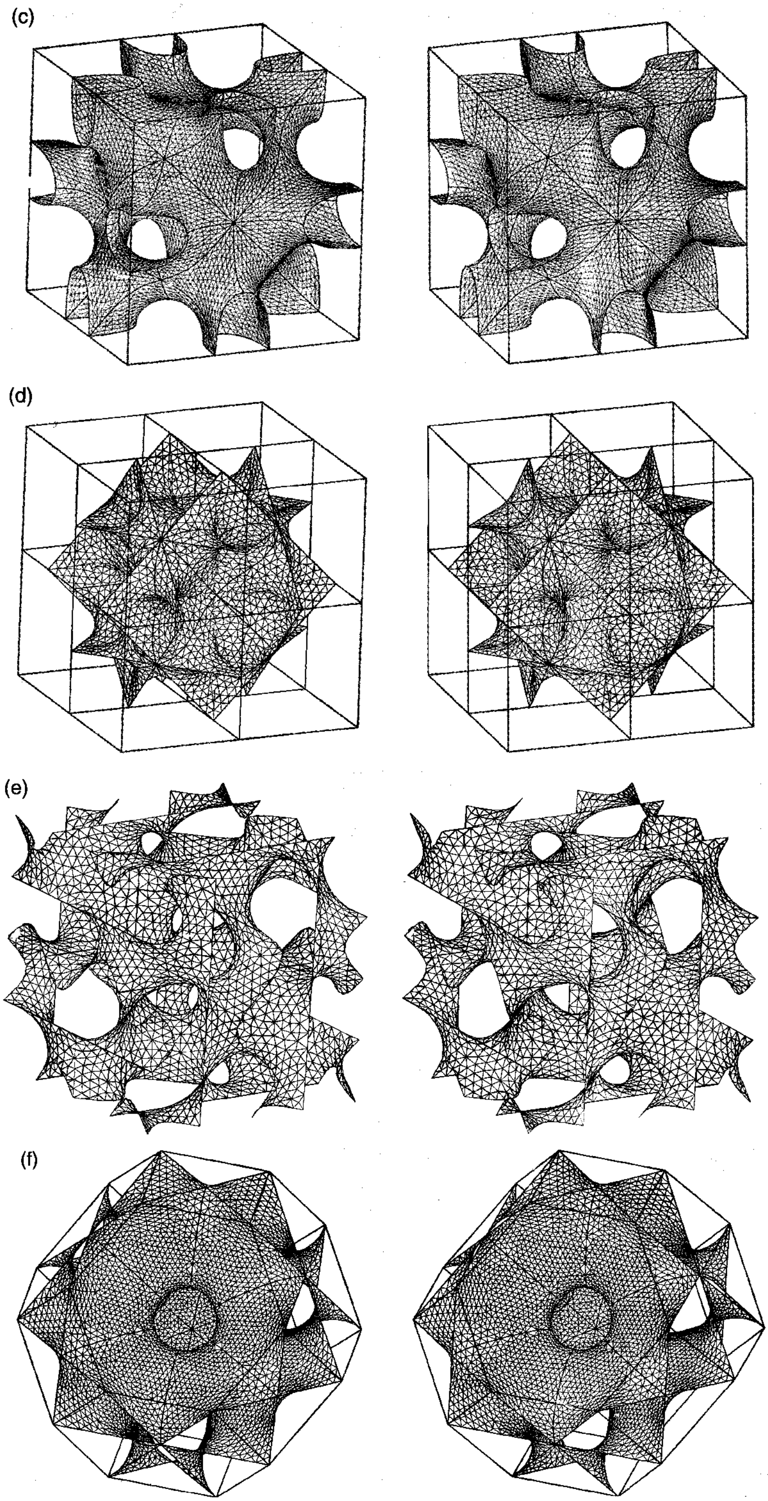
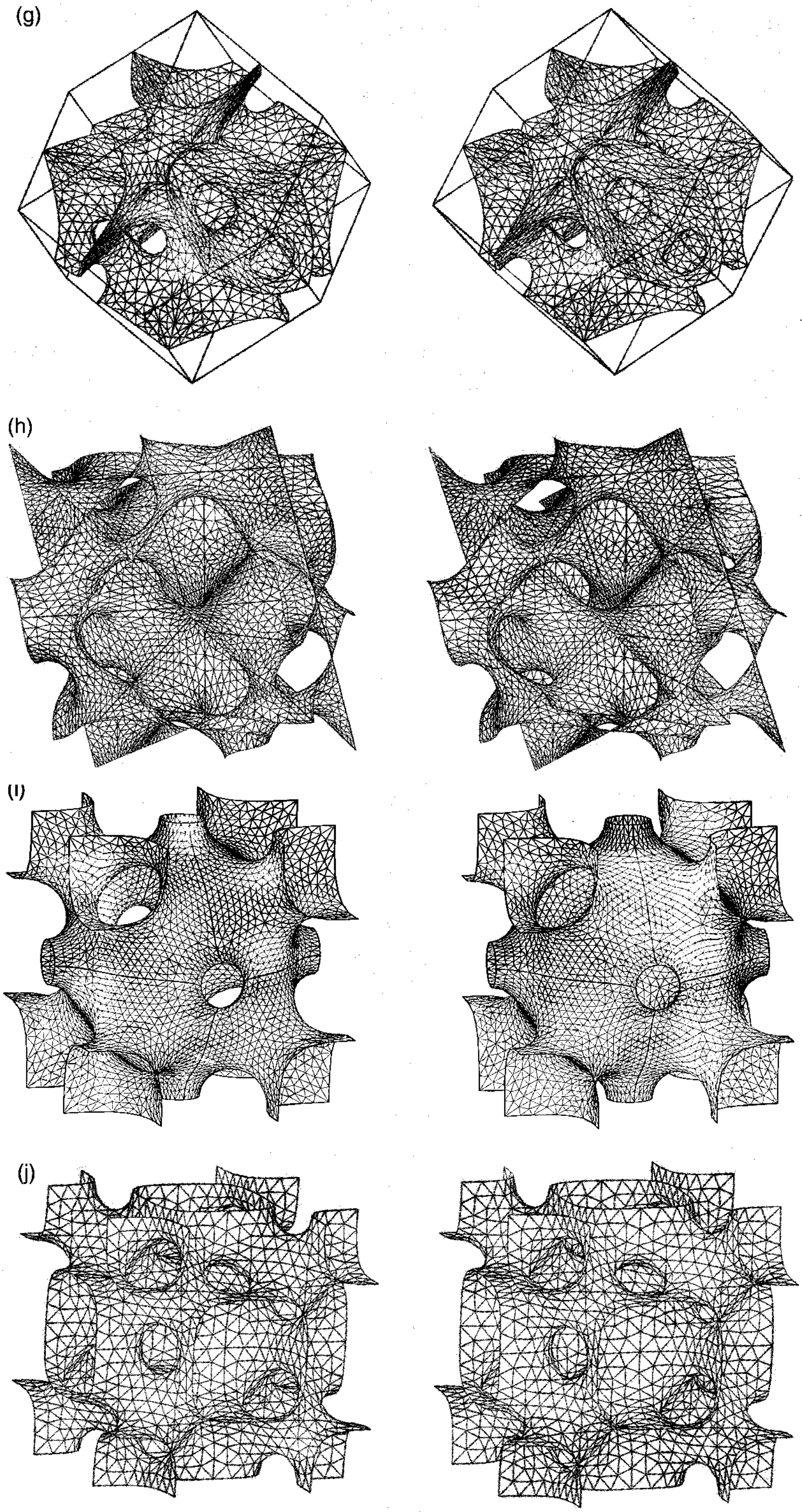


Figure 8 (contd.)

(contd.) ☞

Figure 8 (contd.)



(contd.) ☞

(k)

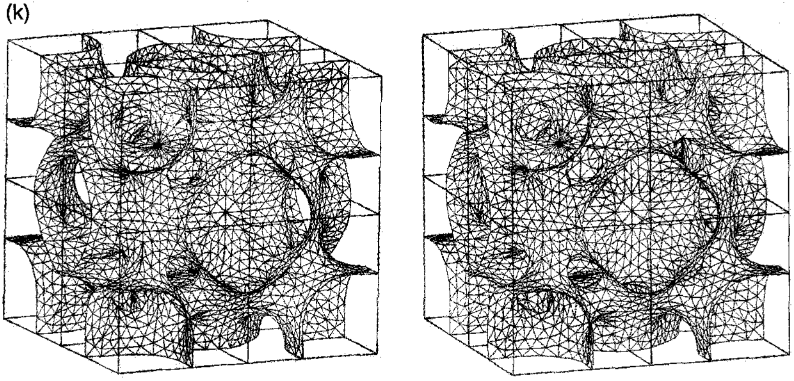


Figure 8 (contd.)

*(**a**) The D-surface (due to H.A. Schwarz; balanced). (**b**) The G-surface (A.H. Schoen's gyroid surface; mirror-image regions). (**c**) C(P), the complement of P (E.R. Neovius' surface; balanced). (**d**) P3a, a new surface due to E.A. Lord (balanced). (**e**) Yb, a new surface due to E.A. Lord (balanced). (**f**) C(D)c [the C(D)/H surface ofW. Fischer and E. Koch; balanced]. (**g**) Dpa, a new surface due to E.A. Lord (balanced). (**h**) C(Y) due to W. Fischer and E. Koch (balanced). (**i**) OCTO, an unbalanced surface due to A.H. Schoen. (**j**) FRD, an unbalanced surface due to A.H. Schoen. (**k**) A more complex surface, unbalanced, based on the Costa minimal surface, developed by E.A. Lord.*

but most of the surfaces actually observed have names, mostly those rather idiosyncratically given by A.H. Schoen [19] who systematically organized the situation in about 1965, and these have usually stuck. However, with more and more surfaces appearing some systematic organization is desirable.

Table 1 (E.A. Lord and A.L. Mackay, unpublished work) shows a list of the balanced surfaces. The data comprise, for each of the balanced surfaces: (i) the space-group pair, the space group of the surface if both sides are the same and the space group if the two sides are distinguishable; (ii) the arbitrary name of the surface; (iii) the genus for a unit cube — the numbers of faces, vertices and edges of the network of the polygonized surfaces are connected by $F+V-E=X$ and genus$=g=(2-X)/2$ and related numbers are provided by the Evolver program (the genus can also be defined as the number of cuts which can be made through the surface without separating it into disjunct pieces; the genus of a polyhedron is zero and that of a torus is one), but these need to be used with care — the definition of genus for a periodic polyhedron is somewhat ambiguous and is genus per unit cell, counting F, V and E per unit cell, thus depending on whether the primitive or the unit cube cell is chosen. The area is that of the surface within a unit cube, and the flat points designate umbilical points, where the principal curvatures are zero.

In the 1960s A.H. Schoen [19] revived interest in PMS and produced a study for NASA of about 18 such surfaces, most of which he had discovered himself, which might be useful for space structures. W. Fisher and E. Koch [20–26], starting from their intense study of the 230 space groups, systematically organized the topic from a crystallographic point of view, and discovered several new surfaces. Other mathematical authors have also greatly developed the subject

Table 1

Name, space group and reference	Genus	Area	Flat points Type I	Type II	Type III	Order
$Im\bar{3}m$–$Pm\bar{3}m$						
			c ($\frac{1}{4}$,$\frac{1}{4}$,$\frac{1}{4}$)			1
			d ($\frac{1}{2}$,$\frac{1}{4}$,$\frac{1}{4}$)			0
P [7]m	3	2.345		h (x,x,0) 0.325		0
C(P) [30]	9	3.520		e (x,0,0) 0.410		2
Pa (new)	25	5.022		f (x,x,x) 0.255		2
				e (x,0,0) 0.165		1
					i ($\frac{1}{2}-x$,$\frac{1}{2}$,x)	
Pb (new)	21	3.678		g ($\frac{1}{2}$,x,0) 0.173		0
				e (x,0,0) 0.408		2
					i ($\frac{1}{2}-x$,$\frac{1}{2}$,x)	
C(P)a [31] 'BFY'	19	4.964		h (x,x,0) 0.139		0
				f (x,x,x) 0.106		1
					j (x,y,0)	
C(P)b [20–26] 'C(P)/H'	15	3.607		g ($\frac{1}{2}$,x,0) 0.023		0
					k (x,y,y)	
$Pm\bar{3}n$–$Pm\bar{3}$						
		0	c ($\frac{1}{2}$,$\frac{1}{4}$,0)			
		0	d ($\frac{1}{4}$,$\frac{1}{2}$,0)			
		1	e ($\frac{1}{4}$,$\frac{1}{4}$,$-\frac{1}{4}$)			

Triply periodic minimal balanced surfaces with cubic symmetry (generated by finite patches with straight-edged boundaries) (contd.) ☞

Table 1 (contd.)

Name, space group and reference	Genus	Area	Flat points Type I	Type II	Type III	Order
P2b (new)	15	3.615		f $(\frac{1}{2},\frac{1}{2},x)$ 0.0612		2
				h $(x,\frac{1}{2},0)$ 0.3395		0
$Pn\bar{3}m$–$P\bar{4}3m$						
			b $(\frac{1}{4},\frac{1}{4},\frac{1}{4})$			1
			f $(\frac{1}{2},\frac{1}{4},0)$			0
P3a (new)	21	4.629		e $(x,x,-x)$ 0.132		1
				g $(x,0,0)$ 0.250		2
$Pn\bar{3}m$–$Fd\bar{3}m$						
			b $(\frac{1}{4},\frac{1}{4},\frac{1}{4})$			1
			d $(\frac{1}{2},\frac{1}{2},0)$			0
			f $(\frac{1}{2},\frac{1}{4},0)$			0
D [7]	3	1.920				
C(D) [19]	19	3.960		e $(x,x,-x)$ 0.200		4
D′a (new)	47	4.817		e $(x,0,0)$ 0.407		2
				g (x,x,x) 0.168		1
				g $(x,x,-x)$ 0.131		1
C(D)a (new)	31	4.723		e $(x,0,0)$ 0.141		2
				g (x,x,x) 0.693		1
Dc (new)	43	4.110		h $(\frac{1}{4}+x,\frac{1}{4}+x,x-\frac{1}{4})$		1
				j $(\frac{1}{4}+x,\frac{1}{4},x-\frac{1}{4})$ 0.19		0
				g $(x,x,-x)$ 0.21		1

Triply periodic minimal balanced surfaces with cubic symmetry (generated by finite patches with straight-edged boundaries) (contd.) ☞

Table 1 (contd.)

Name, space group and reference	Genus	Area	Flat points: Type I	Type II	Type III	Order
C(D)c [20–26] 'C(D)/H'	27			j ($\frac{1}{4}+x,\frac{1}{4},x-\frac{1}{4}$)		0
					k (x,x,z)	
					k ($x,y,-y$)	
D3a (new)	31	3.877		e (x,0,0) 0.205		2
				g ($x,x,-x$) 0.1024		1
D2a (new)	35	4.698		g (x,x,x) 0.156		1
				g ($x,x,-x$) 0.110		1
$P\bar{4}_2 32$–$F\bar{4}_1 32$						
			b ($\frac{1}{4},\frac{1}{4},\frac{1}{4}$)			1
			d ($\frac{1}{2},\frac{1}{2}$,0)			0
			e ($\frac{1}{2},\frac{1}{4}$,0)			0
			f ($\frac{1}{4},\frac{1}{2}$,0)			0
D2′c (new)	17	4.05		g (x,x,x) 0.02		1
				l ($\frac{1}{4}$ $\frac{1}{4}+x,x-\frac{1}{4}$) 0.2		0
$P\bar{4}3m$–$F\bar{4}\bar{3}m$						
			c ($\frac{1}{2},\frac{1}{2}$,0)			0
			d ($\frac{1}{2}$,0,0)	a		0
Db (new)	17	2.594		e (x,x,x) 0.10		1
				g ($\frac{1}{2},\frac{1}{2},x$) 0.21		0
$Fd\bar{3}m$–$F4\bar{3}m$						
			c ($\frac{1}{8},\frac{1}{8},\frac{1}{8}$)			

Triply periodic minimal balanced surfaces with cubic symmetry (generated by finite patches with straight-edged boundaries) (contd.) ☞

Table 1 (contd.)

Name, space group and reference	Genus	Area	Flat points: Type I	Flat points: Type II	Flat points: Type III	Order
Dpa (new)	13	5.985		e (x,x,x) 0.054		1
				e $(x,x,-x)$ 0.066		1
Pda (new)	11	5.882		e (x,x,x) 0.045		1
				f $(x,0,0)$ 0.091		0
$I4_132$–$P4_332$						
			a $(1/8,1/8,1/8)$			1
			b $(1/8,0,1/4)$			1
			d$(-3/8,0,1/4)$			1
C(Y) [20]	13	4.43		e (x,x,x) 0.08		1
				f $(x,0,1/4)$		
Yb (new)	25	4.57		g $(x/4-1/4,-1/8,x/4-1/8)$		0
C(Y)b [20–26] 'C(Y)/H'	21	4.58		g $(x/4-1/4,-1/8,x/4-1/8)$		0
$Ia\bar{3}d$–$I\bar{4}3d$						
			a $(0,0,0)$			1
			b $(1/8,1/8,1/8)$			1
S [20–26]	11	5.44		g $(1/8,x,1/4-x)$ 0.25		1

Triply periodic minimal balanced surfaces with cubic symmetry (generated by finite patches with straight-edged boundaries)

with the appearance of the computer, and computer graphics have now even reached mathematicians.

Materials on all scales with regular arrays of pores have become important. The zeolites, with tunnels in the 10 Å range, are silicate cages used for the cracking and reforming of oil molecules. Their annual commercial production is on the megaton scale. Larger tunnels can be produced by electron-beam etching or mechanical drilling and, for large-scale structures, by the joining of tiles. On all these scales the passage of particles and of electromagnetic waves and of acoustic waves is of interest.

A number of surfaces have been observed in liquid-crystal phases, in polymer interfaces and in biological structures such as aetioplasts and the lipids found in the lungs of the newborn infants. There is a large literature on the subject; see [27,28].

In principle the use of this PMS geometry for electronic circuit boards is attractive, although clearly manufacture would be prohibitive. Plastic models of PMS with repeats of about 5 cm have been made by stereo-lithography, a technique that is still developing rapidly. The necessary *.STL files can be generated using Mathematica.

There are obvious applications of PMS in heat exchangers, or two liquid phases could be kept separate until the interface collapsed, leaving them intimately mixed. It seems that the flow of a fluid across a surface of negative Gaussian curvature cannot be laminar but must become turbulent. It might be noted too that with a uniform current of a gas passing through the P-surface an element of the gas must expand and contract by a factor of four, Bernouilli's theorem requiring corresponding local changes in velocity. It is possible that ram-jet effects or 'acoustic engine' properties might be generated with localized inputs of heat, like the negative drag radiator of the Spitfire aircraft.

There are also interesting mechanical properties that have yet to be investigated, but methods of manufacture are a great problem. Design and art applications are less stringent [29].

In other contexts my collaborator Eric Lord, now at the Indian Institute of Science in Bangalore, India, has appeared as co-author and this work would have been almost impossible without him. We are both indebted to the Indian Institute of Science for their warm hospitality. The computer programs Surface Evolver by K.A. Brakke and Mathematica 3.0, due to Stephen Wolfram, are the indispensible tools that we have used here. During this programme we have received financial help from the Royal Society and from the Université de Paris-Sud.

Notes and references

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