The particle number in Galilean holography

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Abstract

Recently, gravity duals for certain Galilean-invariant conformal field theories have been constructed. In this paper, we point out that the spectrum of the particle number operator in the examples found so far is not a necessary consequence of the existence of a gravity dual. We record some progress towards more realistic spectra. In particular, we construct bulk systems with asymptotic Schrödinger symmetry and only one extra dimension. In examples, we find solutions which describe these Schrödinger-symmetric systems at finite density. A lift to M-theory is used to resolve a curvature singularity. As a happy byproduct of this analysis, we realize a state which could be called a holographic Mott insulator.

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1 Introduction

Particle production is a dramatic, necessary consequence of relativistic field theory. There is no particle production in Galilean-invariant field theories, which therefore have an extra conserved quantity. This quantity is often thought of as the particle number, but (since we can and will work in units of the mass of one particle) it is equivalent to the total rest mass.

In systems with multiple species, to be discussed more below, the latter definition is the more useful one.

Recently, candidate gravity dual descriptions of certain Galilean-invariant, scale-invariant field theories were proposed [1, 2]. More specifically, these theories are non-relativistic conformal field theories (NRCFTs): their symmetry group, the Schrödinger group, contains a special conformal generator. This same group is respected by the dynamics of fermions with unitarity-limited two-body interactions [3, 4], which arise by tuning ultracold fermionic atoms to a Feshbach resonance. These gravity solutions have since been embedded in string theory and put at finite temperature and density [5, 6, 7, 8, 9].[1]

In the known examples, the symmetry associated with the conserved rest mass is realized geometrically in the gravity dual as the isometry of a circle, whose coordinate we call ξ. Compactifying on a circle with circumference L_ξ produces a spectrum of possible values of the rest mass of states in the theory of the form

$$\{\text{masses}\} = \frac{1}{2\pi L_\xi} \, \mathbb{Z}_+ \quad . \tag{1.1}$$

The main purpose of this (somewhat polemical) article is to point out that this particular spectrum is not a necessary consequence of the existence of a gravity dual.

The form of the spectrum (1.1) seems to be responsible for the strange thermodynamics found in [5, 6, 7, 8]:

$$F \sim -\frac{T^4}{\mu^2}, \ \mu < 0. \tag{1.2}$$

This is quite different from the behavior of unitary fermions, where in particular the chemical potential is positive, and the free energy scales like a positive power of μ. These theories are closely related to relativistic field theories, via (modifications of) the discrete lightcone quantization procedure (DLCQ). This fact is made particularly vivid in the calculation of the free energy (1.2) from a free relativistic field theory in DLCQ by [20]. The modifications of DLCQ in [21] (associated with "β-deformation") simplify the theory by removing most of the troublesome [22] lightcone zeromodes, but do not change the spectrum of the lightcone momentum operator, $i\partial_\xi$.

In this work, our goal is to learn how to construct gravity duals for NRCFTs with

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[1]Related earlier work on geometric realizations of the Schrödinger group includes [12, 13, 14]. Earlier work on holography for spaces with degerate boundaries includes [15, 16]. Subsequent work, including examples of Schrodinger-invariant supergravity solutions which preserve some supersymmetry, includes [17, 18].

other (ideally, more realistic) spectra. We demonstrate that it is not necessary to realize the Schrödinger algebra in a gravity dual entirely via isometries of the bulk metric. It was natural to try to realize the full algebra by isometries, since the obviously-geometric momentum and boost generators commute to the particle number operator $\hat{N}$,

$$[K_i, P^j] = iN\delta_j^i. \tag{1.3}$$

However, here we show that this algebra can be realized without the introduction of a ξ dimension, if the boost generator acts by gauge transformations on fields charged under an additional abelian gauge symmetry in the bulk. The construction is quite similar to the way that these symmetries are realized on states of a quantum system [23]: a Galilean boost by velocity $\vec{v}$ acts on the phase of the wavefunction (in the Schrödinger representation) of a particle of mass m by

$$\psi(x, t) \mapsto e^{im\left(\frac{1}{2}v^2t + \vec{v}\cdot\vec{x}\right)}\psi(x - vt, t) \; ; \tag{1.4}$$

from this expression one can show that (1.3) is satisfied.

Using this idea we construct solutions describing $d = 2$, $z = 2$ NRCFT without the additional circle. For most of this paper we will employ a practical approach to holography advocated in *e.g.* [24, 25]: we do not yet know the constraints that quantum gravity imposes on effective field theories of gravity coupled to matter (known examples [26, 27, 28] are not very forceful, and it is clear that our grasp on the space of less-supersymmetric string vacua is poor), and so we will employ the simplest gravity models with which we can approach the physics of interest. We will, however, find it useful in §4 to lift one of our solutions to 11-dimensional supergravity in order to resolve a curvature singularity. That solution describes a system (at finite density and at zero temperature) with a gap for the charged excitations; it appears to provide a holographic description of a Mott insulator. This is an improvement over a previous holographic realization of an insulating state [29], which had zero density. We also succeed in constructing some examples where there are several species of particles, so that the spectrum of the number operator is not just integer multiples of a single mass; this is described in §5. Finally, in §6, we discuss a conjugate issue, namely whether the NRCFTs described by the constructions of [1, 2] have superfluid groundstates (at low temperature and finite density), and what such a state would look like from the point of view of the gravity dual. We relegate to an appendix a curious black hole solution with the new realization of asymptotic Schrödinger symmetry.

2 Getting rid of the ξ direction

In this section, we study the dimensional reduction on the particle-number circle of the systems discussed previously in [1, 2]. We are doing this because it provides a proof of principle that there can be gravity theories with Schrödinger symmetry which don't have this annoying extra dimension. Our real goal is to find new solutions where the spectrum of the mass operator can be different (*i.e.* not the KK tower of momentum modes on a circle), and where the thermodynamics may therefore be more like that of unitary fermions. Our immediate goal is to understand how the symmetries are realized.

A concern which remains even after dimensionally reducing to replace the role of the ξ-dimension with a gauge field in a lower-dimensional description is charge-conjugation invariance. In a relativistic QFT with charge-conjugation invariance (like the one living in the bulk here), the spectrum of a $U(1)$ symmetry must include both positive and negative charges. Below, we explicitly break this symmetry by imposing boundary conditions which introduce a background electric field.

2.1 Review of gravity duals with Schrödinger symmetry

In this paper, we are interested in non-relativistic conformal field theories which are governed by the Schrödinger symmetry algebra. The Schrödinger group includes Galilean invariance, scale invariance and one special conformal transformation. Geometries which realize these symmetries as isometries were constructed in [1, 2] and the metric is

$$ds^2 = -\alpha^2 \frac{dt^2}{r^4} + \frac{2d\xi dt + d\vec{x}^2 + dr^2}{r^2} L_{\text{AdS}}^2 \tag{2.1}$$

where $\vec{x}$ is a vector of d spatial dimensions, and we will work in units with $L_{\text{AdS}} = 1$. This metric solves the equations of motion of Einstein gravity coupled to a massive gauge field and a negative cosmological constant. In the above metric, ξ is a compact direction and the particle number symmetry is realized as translation symmetry along this circle. When α is zero, the metric is just AdS in light cone coordinates with one of the null directions compactified [30, 31].

This solution can be embedded in string theory [5, 6, 7] and the dual field theory is a modified DLCQ of $\mathcal{N} = 4$ SYM theory (or of another quiver gauge theory dual to type IIB on AdS_5 times a Sasaki-Einstein manifold). A black hole solution asymptotic to the

metric (2.1) was found in [5, 6, 7]; this describes the dual NRCFT at finite density and finite temperature. In the black hole solution, ξ is not null everywhere because $g_{\xi\xi}$ is not identically zero as in the vacuum solution. This implies that the radius of the circle is non-zero in the bulk and the supergravity approximation can be trusted in regions where the radius is large compared to the string length scale. Thus the nonzero $g_{\xi\xi}$ component acts like a regulator and this fact will be used here to construct alternate holographic descriptions of Schrödinger algebra.

Dimensional reduction of this solution along ξ yields a lower-dimensional system with asymptotic Schrödinger symmetries. The matter content of the lower-dimensional gravity theory consists of a massive vector field, $U(1)$ gauge field and two scalars (higher-dimensional dilaton and the radion). We would like to have a simpler system that can aid us in understanding the lower dimensional realization of Schrödinger symmetry.

In §3.4 of [7], we studied a scaling limit of this black hole solution (finite μ, T) which had zero temperature, but had a non-trivial $g_{\xi\xi}$ component (let's call this solution $Sch_{\Omega\neq0,T=0}$). This solution is singular in the IR ($r \to \infty$) and should not be taken too seriously. We will use it here as a helpful device to learn about possible bulk realizations of the Schrödinger algebra.

2.2   Dimensional Reduction of $Sch_{\Omega\neq0,T=0}$

The geometry of $Sch_{\Omega\neq0,T=0}$ is described by the following line element

$$ds_5^2 = \frac{1}{r^2\kappa^{2/3}}\left(\frac{-dt^2}{r^2} + 2dtd\xi + (\kappa-1)r^2d\xi^2\right) + \kappa^{1/3}\left(\frac{d\vec{x}^2 + dr^2}{r^2}\right) \tag{2.2}$$

where $\kappa = 1+\Omega^2r^2$, for some constant Ω which determines the density. This can be obtained as a classical solution of the following action[2]:

$$S_5 = C_0 \int d^5x\sqrt{-g_5}\left[R_5 - \frac{4}{3}(\nabla\Phi)^2 + V(\Phi) - \frac{1}{4}e^{-8\Phi/3}F_5^2 - \frac{m^2}{2}A_5^2\right] \tag{2.3}$$

with $\Phi = -\frac{1}{2}\log\kappa$, $A_5 = \frac{1}{2}r^{-2}\kappa^{-1}(dt + (\kappa-1)r^2d\xi)$ and $F_5 = dA_5$. In this subsection, we will perform a series of manipulations using this solution to identify a 4-dimensional system that admits asymptotic solutions which respect the Schrödinger group in two space dimensions.

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[2] The potential $V(\Phi)$ is [5, 6] $V(\Phi) = 4e^{2\Phi/3}\left(e^{2\Phi} - 4\right)$ but it will disappear soon.

Dimensional reduction of the above action along ξ direction yields a lower-dimensional system with Schrödinger symmetry. In this system, the particle number symmetry is realized as a bulk gauge symmetry. The $g_{\xi\xi}$ component of the higher-dimensional metric appears as a scalar field (the radion field $e^{2\sigma}$) in the lower-dimensional system. With the metric ansatz

$$ds_5^2 = ds_4^2 + e^{2\tilde{\sigma}}\left(d\xi + B\right)^2 \tag{2.4}$$

the lower-dimensional action can be written as

$$S_4 = C_0 L_\xi \int d^4x\, e^{\tilde{\sigma}} \sqrt{-G_D}\Big[R_D - \frac{4}{3}\left(\nabla\Phi\right)^2 + V\left(\Phi\right) - \frac{1}{4}e^{-8\Phi/3}F_D^2 - \frac{m^2}{2}A_D^2 +$$

$$4\nabla\Phi\cdot\nabla\tilde{\sigma} - \frac{e^{2\tilde{\sigma}}}{4}\left(dB\right)^2 - \frac{1}{2}e^{-8\Phi/3}\left(\nabla A_\xi\right)^2 - \frac{m^2}{2}A_\xi^2 + \mathcal{L}_{int}\left(A_\xi, B, A\right)\Big]\,. \tag{2.5}$$

Note that the line element in (2.2) can be written as

$$ds_5^2 = \frac{1}{r^2\kappa^{2/3}}\left(\frac{-dt^2}{r^2} + \Omega^2 r^4\left(d\xi + \frac{dt}{r^4\Omega^2}\right)^2 - \frac{dt^2}{\Omega^2 r^4}\right) + \kappa^{1/3}\left(\frac{d\vec{x}^2 + dr^2}{r^2}\right)\,. \tag{2.6}$$

Scaling t by $\Omega Q^{1/2}$ and scaling ξ by $Q^{-1/2}/\Omega$ in the above expression we get

$$ds_5^2 = \frac{1}{r^2\kappa^{2/3}}\left(\kappa Q\frac{-dt^2}{r^4} + r^4/Q\left(d\xi + Q\frac{dt}{r^4}\right)^2\right) + \kappa^{1/3}\left(\frac{d\vec{x}^2 + dr^2}{r^2}\right)\,. \tag{2.7}$$

Under this rescaling $A_t \to \Omega Q^{1/2}A_t$. Hence, the higher-dimensional line element can be written as

$$ds_5^2 = ds_D^2 + e^{2\tilde{\sigma}}\left(d\xi + B\right)^2 \tag{2.8}$$

$$e^{2\tilde{\sigma}} = \frac{r^2}{Q\kappa^{2/3}}, \qquad B = \frac{Q}{r^4}dt\,. \tag{2.9}$$

The 4−dimensional line element in the above expression is

$$ds_D^2 \equiv \left(G_D\right)_{\mu\nu}dx^\mu dx^\nu = \kappa^{1/3}\left(-\frac{Qdt^2}{r^6} + \frac{d\vec{x}^2 + dr^2}{r^2}\right) \tag{2.10}$$

If we now define $e^{2\sigma} = \Omega^2 e^{2\tilde{\sigma}}$ and take the scaling limit, $\Omega \to 0$, holding σ fixed, we are left with an extremum of the much-simpler action

$$S_4 = C_0 L_\xi \int d^4x\, e^{\sigma} \sqrt{-G_D}\Big[R_D - 2\Lambda - \frac{e^{2\sigma}}{4}\left(dB\right)^2\Big]\,, \tag{2.11}$$

where we have chosen units so that the cosmological constant is $\Lambda = -6$. In the above limit, $A = 0$, $\Psi = 0$, $\Phi = 0$ and $\kappa = 1$; the 4-dimensional solution is (2.10) with $\kappa = 1$. Note

that Q is related to the chemical potential. When $Q \to \infty$, the Schrödinger symmetries become an exact symmetry of the above system; however, the metric becomes degenerate in the $Q \to \infty$ limit. Rewriting the 'string frame' action (2.11) in 4d Einstein frame (and throwing away the fields A, Ψ, Φ which vanish) we see that

$$ds_E^2 = e^\sigma \left(-Q\frac{dt^2}{r^6} + \frac{d\vec{x}^2 + dr^2}{r^2} \right), \quad B = Q\frac{dt}{r^4}, \quad e^\sigma = \frac{r}{\sqrt{Q}} \tag{2.12}$$

is a solution of the simple action

$$S_4^E = \mathfrak{K} \int d^4x \sqrt{-g_4} \left[R_4 - 2\Lambda e^{-\sigma} - \frac{e^{3\sigma}}{4} (dB)^2 - \frac{3}{2} (\partial\sigma)^2 \right] , \tag{2.13}$$

where we have named $\mathfrak{K} \equiv C_0 L_\xi$ the effective 4d coupling. Note that the apparent strong coupling behavior of the action for the gauge field B at the boundary ($g_{\text{eff}}^{-2} \sim e^{3\sigma} \sim r^3 \to 0$) is an artifact of dimensional reduction.

2.3   Symmetry Generators

Let us try to understand how the Schrödinger symmetry group is realized by the above action and asymptotics. It is clear that the symmetries of the Schrödinger group are not realised as isometries in the lower-dimensional theory: the putative symmetry generators in the lower-dimensional theory do not solve the Killing equation. The metric is of the Lifshitz form [32] and seems to have scaling symmetry with dynamical exponent $z = 3$. What equation determines the symmetry generators of the lower-dimensional action? It is not hard to guess that the appropriate symmetry generators should solve the equation obtained by dimensional reduction of the higher-dimensional Killing equation.

So the symmetry generators of the lower-dimensional theory are:

- Particle Number: The $U(1)$ gauge charge associated with the massless gauge field B is the particle number. This acts by $B \to B + d\lambda$, and by phase rotations on charged fields in the bulk, of which we should include one or more. Let us introduce such a field $\Phi \equiv |\Phi|e^{i\varphi}$ of charge ℓ; we take Φ to vanish in the solution shown above. ℓ is the mass of the associated particle.

- Translations and rotations are realized as usual by isometries.

- Galilean boosts act as follows:

$$t \to t, \quad \vec{x} \to \vec{x} - \vec{v}t, \quad \varphi \to \varphi + \ell \left(\frac{1}{2}v^2 t + \vec{v} \cdot \vec{x} \right), \tag{2.14}$$

where φ is the phase of a field of charge ℓ under the particle-number gauge symmetry. The role previously played by ξ in the Schrödinger geometry is now played by the phase φ of charged bulk fields. In summary, the boost generator is:

$$K^i = -t\partial_i + \text{gauge shift}$$

where the gauge transformation parameter is $\lambda = \frac{1}{2}v^2 t + \vec{v} \cdot \vec{x}$. Note the similarity to the action in quantum mechanics given in Eqn. (1.4).

- Scale symmetry acts by

$$D = -2t\partial_t - x^i\partial_i - r\partial_r;$$

The generators of these symmetries satisfy the Schrödinger algebra.

The asymptotic profiles of the fields are *not* preserved by these transformations, but one can show, as follows, that they are nevertheless (asymptotic) symmetries of the system. The higher dimensional (5D) Killing equation can be written as

$$\delta_\eta \left(G_D \right)_{AB} = \mathcal{L}_\eta \left(G_D \right)_{AB} = 0 . \tag{2.15}$$

If the above equation is only true as $r \to 0$ (which we denote by ≈ 0), as is the case in the solutions described above, then the symmetry is realized only asymptotically. The lower dimensional (4D) metric (g) does not satisfy the 4d Killing equation, *i.e.* $\delta_\eta g_{\mu\nu} \neq 0$. The above equation (2.15), however, implies[3]

$$\delta_\eta \left(e^{-\sigma} g_{\mu\nu} + e^{2\sigma} B_\mu B_\nu \right) \approx 0 \tag{2.19}$$

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[3]The transformation rules for the lower-dimensional fields can be obtained from the transformation rule for the higher-dimensional metric:

$$\delta_\eta g_{\mu\nu} = [g_{\mu\rho}\partial_\nu\eta^\rho + g_{\rho\nu}\partial_\mu\eta^\rho + \eta^\rho\partial_\rho g_{\mu\nu}] \tag{2.16}$$

$$\delta_\eta B_\mu = [B_\rho\partial_\mu\eta^\rho + \eta^\rho\partial_\rho B_\mu] + \partial_\mu\eta^{D+1} \tag{2.17}$$

$$\delta_\eta e^{2\sigma} = [\eta^\rho\partial_\rho e^{2\sigma}] . \tag{2.18}$$

Note that the quantities within [ ] are the changes due to the coordinate transformations in the lower dimensions, while the transformation of x^{D+1} generates field transformations.

$$\delta_\eta \left(e^{2\sigma} B_\mu \right) \approx 0 \tag{2.20}$$

$$\delta_\eta \left(e^{2\sigma} \right) \approx 0 \tag{2.21}$$

These quantities have the transformation properties of tensors. We also know (from its higher-dimensional origin) that the action can be written as a functional of these quantities, that is

$$S_D[g, B, \sigma] = S[e^{-\sigma} g_{\mu\nu} + e^{2\sigma} B_\mu B_\nu, e^{2\sigma} B_\mu, e^{2\sigma}] . \tag{2.22}$$

When the symmetries are realized as isometries, $\delta_\eta S$ vanishes as a consequence of $\delta_\eta G_{AB}$ vanishing. In the present case, $\delta_\eta S$ will vanish as a consequence of $\delta_\eta \left(e^{-\sigma} g_{\mu\nu} + e^{2\sigma} B_\mu B_\nu \right)$, $\delta_\eta \left(e^{2\sigma} B_\mu \right)$ and $\delta_\eta \left(e^{2\sigma} \right)$ vanishing.

In the solutions described above, these quantities only vanish asymptotically near the boundary. Note that we do not know a solution of the lower-dimensional system (2.13) which exactly preserves the Schrödinger symmetry. This is perhaps unsurprising given that such a solution would correspond to the vacuum of a Galilean-invariant system, a very boring state indeed. Rather, the surprising fact is that the previous holographic realizations [1, 2] did provide such a solution.

From the form of B in (2.12), we see that the solution (2.10) has non-zero chemical potential ($\mu \neq 0$), but charge density zero. This can happen for example if the chemical potential is smaller than the particle mass.

2.4   Wave equation

The wave equation for a probe scalar field with charge ℓ under the Kaluza Klein gauge field (and mass m^2) takes the following form

$$\left(-\omega^2 r^6 + m^2 + r^2 (2\ell\omega + k^2) \right) \Phi - r^{d+3} \partial_r \left(r^{-d-1} \partial_r \Phi \right) = 0 . \tag{2.23}$$

Notice that the first term in this equation is unimportant for the boundary behavior ($r \to 0$), but does spoil the Schrödinger invariance of the equation (and renders us and Mathematica unable to solve it analytically).

The D-dimensional (Einstein-frame) action that produces this equation of motion is perhaps surprising:

$$S_{\text{probe}}[\Phi] = \int \sqrt{g} \left[|(\partial - i\ell B)\Phi|^2 - \left(\ell^2 e^{-3\sigma} + m^2 e^{-\sigma} \right) |\Phi|^2 \right] . \tag{2.24}$$

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This coupling to the background scalar σ is required in order that solutions for Φ represent the Schrödinger symmetry.

3   Black hole solution

The following is a black hole solution of (2.13) that asymptotes to the solution written in (2.12):

$$ds_E^2 = e^\sigma \left(-Qf\frac{dt^2}{r^6} + \frac{d\vec{x}^2}{r^2} + \frac{dr^2}{r^2 f} \right), \quad B = Q\frac{(1+f)dt}{2r^4}, \quad e^{2\sigma} = \frac{r^2}{Q} \tag{3.1}$$

where $f = 1 - r^4/r_H^4$. The above solution can be obtained (by dimensional reduction) from the following five dimensional solution of Einstein's equation (with negative cosmological constant):

$$ds_5^2 = \frac{Q}{4r_H^8}r^2 dt^2 + \frac{d\vec{x}^2}{r^2} + \frac{(1+f)d\xi dt}{r^2} + \frac{dr^2}{fr^2} + \frac{r^2}{Q}d\xi^2 \tag{3.2}$$

The scaling symmetry of Sch algebra relates solutions with different values of Q and r_H. These solutions are also related to each other through the lightcone symmetry: $t \to \lambda t, \xi \to \lambda^{-1}\xi$. Even though compactification of ξ direction breaks this symmetry, this "symmetry" relates a system with chemical potential μ and temperature T to the system with chemical potential μ/λ^2 and temperature T/λ. This transformation along with the conformal Ward identity fixes the form (as a function of chemical potential and temperature) of the free energy [5].

A curious feature of the lower dimensional solution is the fact that the gauge field is non-vanishing at the horizon. This is not an indication that the solution is irregular. In fact, this solution was obtained from a regular solution in more dimensions. The gauge field obtained by dimensional reduction of the solution in [5, 6, 7] also has this feature.

Note also that t is not a time-like direction in (3.2). However, dimensional reduction of the higher-dimensional system results in t becoming a time-like direction. This feature can also be seen in rotating black hole solutions.

3.1   Thermodynamics

The temperature of the black hole in (3.1) is $T = \frac{\sqrt{Q}}{2\pi r_H^3}$ and the entropy density is

$$s = \frac{\mathfrak{K}}{\sqrt{Q}r_H} \ . \tag{3.3}$$

The energy density, pressure and free energy can be computed from the regularized action. It is possible to regularize the on-shell action and boundary stress tensor with the following boundary counterterms:

$$S_{ct} = \mathfrak{K} \int_{bdy} d^3x \sqrt{\gamma} \left(-4e^{-\frac{1}{2}\sigma} + e^{3\sigma} \frac{1}{2} n^r B^\mu F_{r\mu} \right) . \tag{3.4}$$

Note that the second term in (3.4) changes the boundary condition on the gauge field from Dirichlet to Neumann; this means that we are in the canonical ensemble (fixed ρ). In its origin as the dimensional reduction of the previous $d + 3$-dimensional Schrödinger solution, this 'Neumannizing' term results from the dimensional reduction of the Gibbons-Hawking term.

In the higher-dimensional system, the number density is given by the momentum along the ξ direction. In the lower dimensional system this momentum appears as the charge density of the black hole which is given by

$$\rho = \frac{N}{L_x L_y} = \frac{\mathfrak{K}}{L_x L_y} \int_{bdy} d^2x \sqrt{\gamma} n^r e^{3\sigma} F_r^t \sim Q^{-1} \tag{3.5}$$

Using this we find

$$-\mathcal{F} = \mathcal{P} = \mathcal{E} = \frac{1}{2} \mathfrak{K} \rho^{2/3} T^{4/3} \sim \frac{T^4}{\mu^2} \tag{3.6}$$

where $\mathcal{C}$ is a numerical constant. The chemical potential, read off from $\frac{\partial \mathcal{F}}{\partial \rho}$, is

$$\mu \sim \frac{T^{4/3}}{\rho^{1/3}} \quad \text{or} \quad \rho \sim \frac{T^4}{\mu^3} . \tag{3.7}$$

The form of the thermodynamic quantities are the same as that in [5, 6, 7]. In the following section, we will present a solution describing a NRCFT with a finite density at zero temperature, which has a non-zero free energy (unlike the $T \to 0$ limit of (3.6)).

Note added in v2: The solution (3.1) and the solution (2.12) with periodic imaginary time are saddle points of the same action. However, for any $T > 0$, the black hole solution (3.1) has a smaller on-shell action and hence its contribution dominates. Like in planar AdS, the would-be Hawking-Page transition is at $T = 0$ (where the two solutions coincide); unlike in AdS, here this does not follow from scale invariance. However, it has been brought to our attention (by Tom Faulkner) that the 5d uplift of the solution in this section is in fact isometric to the AdS_5 black brane solution; this explains the similarity in the phase diagram.

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4   A Holographic Mott Insulator?

Let us now look at a marginal deformation of the above system; this will lead to an interesting new family of solutions. We will do this by adding a massless scalar field Ψ in the bulk. The corresponding operator will turn out to be marginally relevant in the presence of finite density. Consider the following action with two scalar fields:

$$S_4^E = \mathfrak{K} \int d^4 x (-g_4)^{1/2} \left[R_4 - 2\Lambda e^{-\sigma} - \frac{e^{3\sigma}}{4} (dB)^2 - \frac{3}{2} (\partial \sigma)^2 - \frac{1}{2} (\partial \Psi)^2 \right] \ . \qquad (4.1)$$

The following background is a saddle point of the above action which has asymptotic *Sch* symmetry:

$$ds_E^2 \left(\widehat{Sch} \right) = e^\sigma \left(-Q K_x^2 \frac{dt^2}{r^6} + K_x \frac{d\vec{x}^2}{r^2} + \frac{dr^2}{r^2} \right)$$

$$B = Q \frac{dt}{r^4}, \quad e^{2\sigma} = \frac{r^2}{Q}, \quad e^{2\Psi/\sqrt{5}} = \frac{1 + \varsigma^2 r^4/Q^2}{1 - \varsigma^2 r^4/Q^2} \qquad (4.2)$$

where $K_x^2 = 1 - \varsigma^4 r^8/Q^4$. The geometry is cut off at $r = r_0 = \sqrt{Q/\varsigma}$ where $K_x(r_0) = 0$. There is a curvature singularity at $r = r_0$, which we resolve below. ς is a dimensionless parameter describing the source for the operator dual to Ψ; this is a marginally relevant operator whose running produces the dimensional trasmutation scale r_0 in the solution. This solution has non-zero energy, presure, density and free energy[4], but has zero entropy. Note that the number density, identical to the calculation of (3.5), is Q^{-1} and the chemical potential is $\varsigma^2 Q^{-1}$.

The curvature singularity at $r = r_0$ can be resolved by dimensional oxidation. In general, dimensional reduction of a regular solution along a circle action with degenerate fibers can result in a curvature singularity in the lower dimensional metric [33]. In the next subsection we will show that such a resolution is available here.

The equations of motion have a symmetry which takes $\Psi \to -\Psi$. If we identify Ψ with the dilaton field, as we will in the next subsection, this transformation is an S-duality transformation. In the solution obtained by the action of this transformation (this reverses the sign of ς^2), the coupling dual to Ψ is marginally *irrelevant*. The phase diagram is thus similar to the BCS RG flow, where an attractive/repulsive coupling is relevant/irrelevant. We note, however, that even in our gravity solution for the marginally irrelevant case, the

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[4] $\mathcal{E} = \mathcal{P} = -\mathcal{F} \sim \varsigma^2 Q^{-2}$

flow ends at a finite location in the bulk; perhaps this can be attributed to the strong coupling in the dual frame.

4.1   Lift to eleven dimensions and mass gap

We begin our journey to a smooth uplift of the solution (4.2) by noting that the four dimensional action in (4.1) can be obtained as a consistent truncation of the following five dimensional action:

$$S_5^E = C_0 \int d^5x (-g_5)^{1/2} \Big[R_5 - 2\Lambda - \frac{1}{2} (\partial\Psi)^2 \Big] . \tag{4.3}$$

In particular, the equations of motion of S_5^E with ansatz

$$ds_5^2 = e^{-\sigma} ds_E^2 \left(\widehat{Sch}\right) + e^{2\sigma} (d\xi + B)^2 = ds_E^2 (Sch) + e^{2\sigma} (d\xi + B)^2 \tag{4.4}$$

are the equations of motion of (4.1); we have defined

$$ds_E^2 (Sch) \equiv e^{-\sigma} ds_E^2 \left(\widehat{Sch}\right) = -QK_x^2 \frac{dt^2}{r^6} + K_x \frac{d\vec{x}^2}{r^2} + \frac{dr^2}{r^2} . \tag{4.5}$$

In fact, the action in (2.13) was obtained by dimensional reduction of a five dimensional system related to (4.3) by turning off Ψ.

We pause on our path to eleven dimensions to make some comments about the geometry (4.3). The asymptotics of the 5d metric are precisely AdS with a light-like identification:

$$ds^2 = \frac{2d\xi dt + d\vec{x}^2 + dr^2}{r^2}; \tag{4.6}$$

this is the realization of Schrödinger symmetry described in [30, 31]. Note that with the ansatz in (4.4), a gauge transformation of the B field which does not fall off at the boundary has a dramatic effect on the asymptotics. For example, the transformation $B \to B + \alpha dt$ is equivalent in the higher-dimensional description to a redefinition of the ξ coordinate by $\xi \to \xi + \alpha t$. This violates the periodicity of the ξ coordinate and is not an equivalence relation. Such a transformation is precisely what would be required in order to set the gauge field to zero at the IR boundary $B_t(r_0) = 0$.

Some evidence that this solution is not the result of Melvinization of a relativistic geometry is the fact that the free energy is finite at zero temperature and finite chemical potential; this is hard to get from a $T \to 0$ limit of $F \sim -T^4/\mu^2$.

The five-dimensional action (4.3) can in turn be obtained as a consistent truncation of type IIB supergravity [5]. Specifically, the action in (4.3) can be obtained from the consistent truncation of [5] by turning off the massive vector as well as the breathing and squashing modes u, v. This allows us to lift the solution in (4.2) to the following solution of type IIB supergravity:

$$ds_{10}^2 = ds_E^2 \left(Sch \right) + e^{2\sigma} \left(d\xi + B \right)^2 + ds^2 \left(S^5 \right),$$

$$F_5 = 4 \left(\Omega_5 + \star \Omega_5 \right), \text{ and } \Phi = \Psi \tag{4.7}$$

where Φ is the IIB dilaton. However, there is a curvature singularity at $r = r_0$ even in this ten dimensional metric. The presence of this singularity is related to the non-trivial profile of the dilaton, consistent with our interpretation above in terms of dimensional transmutation. It is convenient to think of the dilaton as the radius of a compact direction in eleven dimensions [35]. This suggests that the singularity in the 10-D metric can be resolved by lifting it to M-theory. The details of the lift are described in Appendix B. After performing the lift, we get the following 11-dimensional solution, which is regular:

$$ds_{11}^2 = e^{-\Psi/6} \left[ds_E^2 \left(Sch \right) + e^{2\sigma} \left(d\xi + B \right)^2 + ds^2 \left(\mathbb{CP}^2 \right) + d\chi_1^2 \right] + e^{4\Psi/3} d\chi_2^2$$

$$F_4 = 2J \wedge J + 2J \wedge d\chi_1 \wedge d\chi_2 \tag{4.8}$$

where J is the Kähler form on $\mathbb{CP}^2$. We can now get two solutions of type IIA from this 11 dimensional solution - (a) by reducing along χ_1 and (b) by reducing along χ_2. The first reduction produces a regular metric (with a smoothly shrinking circle) and a constant dilaton, while the second system has a metric with a curvature singularity and non-trivial dilaton profile. The second system is related to the type IIB solution in (4.7) by T-duality. The two type-II solutions are related by S-duality.

Note that in the presence of fermion fields (as in eleven-dimensional supergravity), the regularity of the solution (4.8) requires antiperiodic boundary conditions around the χ_2 circle for the fermions, since in the neighborhood of r_0, χ_2 is merely an angle in polar coordinates in $\mathbb{R}^2$. This explicitly breaks any supersymmetries.

$\varsigma \equiv Q/r_0^2$ is a dimensionless parameter. It can be considered a perturbation of the non-normalizible falloff of Ψ, which from the IIB frame, is the string coupling. This encodes a marginally relevant deformation of the boundary theory. In vacuum, it is exactly marginal.

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It is driven marginally relevant by the finite density, and runs strong at $r = r_0$, producing this confining groundstate.

A finite temperature solution can be obtained simply by periodically identifying the Euclidean time direction in this solution. It is not clear that this is the thermodynamically favored solution[5]. If it is, then it implies that $e^{-E_{gap}/T}$ effects do not appear in observables in this state; this is consistent with an energy gap of order N^2. At $\varsigma \to 0$, a finite-temperature solution with a horizon is the one given in Section 3.

4.2   What is a translation-invariant insulator?

The fact that the geometry ends smoothly in the IR (at r_0) strongly suggests that the excitations of this groundstate are gapped [37]. More precisely, regularity requires the boundary condition $\partial_r \varphi|_{r=r_0} = 0$ on any smooth 11-dimensional field φ. This *real* boundary condition in the IR implies the vanishing of the spectral density $\mathrm{Im}\langle \mathcal{O}\mathcal{O} \rangle$ of the dual operator $\mathcal{O}$, up to a discrete series of delta functions associated with normal modes. In particular, this applies to the bulk gauge field B which couples to the particle number current j, and implies a gap in the spectral density for j. This spectral density determines the conductivity.

Hence this solution is dual to a system at finite density with a gap for the charged excitations. We emphasize that the distinction between this solution and an ordinary confining groundstate of the dual gauge theory [37] is the presence of a nonzero charge density.

Such a thing can be called a Mott insulator. From the point of view of the dual field theory, it is the strong interactions that prevent the charge from moving. It is certainly not a band insulator or an Anderson insulator – indeed this system is translationally invariant. This raises a thorny point: translation invariance plus finite charge density implies that the center of mass of the system will accelerate in an external field, and hence $\mathrm{Re}\,\sigma(\omega) \propto \delta(\omega)$ – the DC conductivity is infinite. The system is actually a perfect conductor.

What we mean by calling the system an insulator is that we believe it would be an insulator if we pinned it down, for example by a boundary condition. We have not figured out how to show that the thing is actually an insulator in the above sense. The answer

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[5]We note that in contrast with the Hawking-Page transition [36, 37], in our case a double Wick rotation $t \to i\chi_2, \chi_2 \to -it$ does not provide a finite temperature deconfined solution with the same asymptotics, because of the $dtd\xi$ term in the metric. The ability to do this previously was a result of Lorentz invariance of the asymptotics.

for the conductivity $\mathrm{Re}\,\sigma(\omega) \propto \delta(\omega)$ is not enough: the clean free Fermi gas also gives this answer, and obviously that is a metal. In that case, and quite generally [38][6], adding static impurities just turns the delta function into a transport peak. In real systems (*i.e.* with decent UV behavior) there is a sum rule that says that the spectral weight from the delta function has to be redistributed somehow upon adding a momentum sink. A possible concern is that the conclusion (*i.e.* whether the spectral weight gets redistributed away from $\omega = 0$) might depend on *how* the center of mass mode is frozen.

Even zero compressibility is not enough, at least in the presence of long-range forces (which presumably the dual field theory has): the 'jellium model' of a metal (in which the lattice of ions is approximated by a fixed uniform density of background charge) is incompressible if Coulomb interactions are included, but is also clearly a metal. Further analysis is required to test our conjecture. The application of an electric field of finite wavenumber may be the simplest approach.

There are several known examples of translation-invariant insulators[7]. Quantum Hall states are insulators which preserve continuous translation invariance; the translation-invariance delta function in $\sigma(\omega)$ is shifted from zero to the cyclotron frequency because the (charged) center of mass mode is subjected to a magnetic field (this is known as Kohn's theorem). In contrast, the state discussed in this paper is not subjected to an external magnetic field.

In strongly-correlated lattice models, the particles can fractionalize in such a way as to produce an integer number of fractionalized particles per unit cell, which can then realize an ordinary band insulator. The arguments of [39, 40, 41] show that in a gapped system with a conserved particle number (not spontaneously broken) at incommensurate filling, either translation invariance is broken or the system exhibits groundstate degeneracy on a torus. All of the examples mentioned above realize the latter option.

For realizing a finite-density insulating state which preserves continuous (non-magnetic) translations, it is crucial that the high-energy excitations of our system are not particles, but rather CFT excitations. If the system at the scale of the chemical potential were described in terms of charged particles, a state where the charges were localized would have to (spontaneously) break translation invariance, since the particles have to sit somewhere. It would be interesting to find a "slave unparticle" construction of such a state.

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[6]We thank Sean Hartnoll for bringing this argument to our attention.
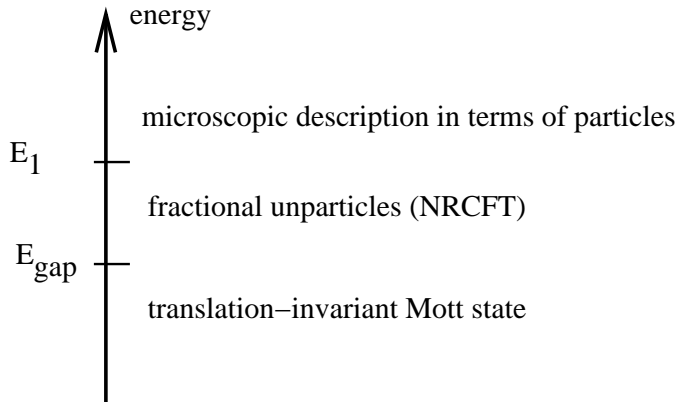[7]We thank T. Grover, S-S. Lee, T. Senthil and B. Swingle for very helpful conversations on these issues.

Figure 1: A scheme for realization a translation-invariant insulator. In the gravity dual, we are only addressing physics below the scale E_1.

A state in which $\mathrm{Im}\langle\rho(\omega)\rho(-\omega)\rangle$ vanishes below some gap must be incompressible[8]. Our system, as currently presented, does not have such a gap (there are zero-energy excitations, at least those associated with translation invariance), and naively the compressibility is finite. Indeed it seems to be a consequence of the scale invariance Ward identity that $\mu \propto \rho$, which is what we find with an appropriate choice of boundary counterterms. What this constraint on the compressibility has to say about our proposal for what would happen if one pinned this system down is not clear to us at present. One point to note is that we are forced to study the system at fixed particle number rather than fixed chemical potential (see §3.1). In general, an incompressible system ($d\rho/d\mu = 0$) does not have a homogeneous groundstate at fixed particle number. For example, consider a Mott insulator of repulsive bosons on a lattice. In general only certain values of the number density will admit homogenous groundstates states (*i.e.* those values with integer filling fraction) and at other values the system will phase separate. The phase separated solutions are compressible. It may be that the solution we have is describing such a mixed state. Another possible resolution is that the nonzero compressibility arises in our system via neutral modes – *i.e.* in our construction, it's not clear that all excitations have nonzero particle number.

A final disclaimer about our use of the name "Mott insulator" is that there is no local moment physics in our problem so far. It would be interesting to include spin degrees of freedom.

[8]We thank D. Son and S. Sachdev for emphasizing this to us. As we learned from Son, this follows by comparing the compressibility sum rule and the f-sum rule.

5   Examples with multiple species

In the Schrödinger metric of [1, 2], the ξ-momentum is dual to the particle number $\hat{N}$ of the dual NRCFT. Compactifying the ξ direction on a circle of radius L_ξ gives a spectrum of $\hat{N}$ which is just a tower of integer multiples of a fixed mass scale L_ξ^{-1}. In a system with multiple species of different mass (for example, in a pile of atoms consisting of several species) the mass operator will have a spectrum which is not just a tower of integer multiples of a fixed mass. Here we would like to find gravity duals with similar spectra. We can do this by adding more dimensions analogous to ξ [9].

We mention in passing that a usefully liberating perspective on the realization of the particle number was provided by [17]: in the analog of 'global coordinates' discussed there the particle-number circle is fibered non-trivially over the $t, \vec{x}$ directions.

We can solve a reasonable set of equations of motion with a nice metric with two ξ directions if we add a second gauge field. The Lagrangian is just

$$L = R + 2\Lambda - \frac{1}{4}F_1^2 - \frac{1}{2}m_1^2 A_1^2 - \frac{1}{4}F_2^2 - \frac{1}{2}m_2^2 A_2^2 \tag{5.1}$$

with (for $d = 2$)

$$m_1^2 = 4z, \quad m_2^2 = -4(z-2) \tag{5.2}$$

$$\Lambda = \frac{1}{2}\left(26 - 7z + z^2\right) \tag{5.3}$$

(for $z = 2$, $\Lambda = 8$). The z-dependence of Λ is a novel development, compared to previous Schrödinger solutions.

The solution is

$$ds^2 = -r^{-2z}dt^2 + r^{-2}(-2d\xi_+ dt + d\vec{x}^2 + dr^2) + d\xi_-^2 r^{2z-4} \tag{5.4}$$

(the symmetries are discussed below) with

$$A_1 = \Omega_1 r^{-z}dt \tag{5.5}$$

$$A_2 = \Omega_2 r^{z-2}d\xi_-.$$

We believe that it is not possible to source the stress tensor for this metric with a single gauge field (which solves its own equations of motion).

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[9]The discussion in this section was motivated by a question asked by Pavel Kovtun, and we thank him for discussions. This question was independently asked by Petr Hořava.

Note that the mass-squared of the second gauge field is negative for many z's of interest $(z > 2)$. According to (5.2), the second gauge field is massless for $z = 2$; we will comment below on some subtleties with this case.

5.1 Symmetries

This metric is invariant under galilean boosts just like the usual Schrodinger metric, with no action on ξ_-,

$$\xi_+ \to \xi_+ + \vec{v} \cdot \vec{x} - \frac{1}{2}v^2 t, \xi_- \to \xi_-. \tag{5.6}$$

It is scale invariant with $\xi_\pm$ both scaling like $length^{2-z}$:

$$\vec{x} \to \lambda\vec{x}, \ t \to \lambda^z t, \ r \to \lambda r, \ \xi_\pm \to \xi_\pm \lambda^{2-z} \ . \tag{5.7}$$

Interestingly, for $z = 2$, the $g_{\xi_-\xi_-}$ coefficient is 1. And, finally, $[K_i, P_j] = i\delta_{ij}\hat{N}$ with

$$\hat{N} = i\partial_{\xi_+} \ . \tag{5.8}$$

So, if we set $\xi_\pm = \xi^1 \pm \xi^2$ and compactify

$$\xi_1 \simeq \xi_1 + L_1, \quad \xi_2 \simeq \xi_2 + L_2 \tag{5.9}$$

then the spectrum of $\hat{N}$ is

$$\left\{ \frac{n_1}{L_1} + \frac{n_2}{L_2} | n_{1,2} \in \mathbb{Z} \right\} ; \tag{5.10}$$

in particular $\frac{L_1}{L_2}$ needn't be rational. We can think of $i\partial_{\xi_1}$ and $i\partial_{\xi_2}$ as the conserved particle numbers of the individual particle species; only their sum appears in the Schrödinger algebra.

The isometries of this spacetime include $P_i = i\partial_{x^i}$, $K^j = ix^j\partial_{\xi_+} + it\partial_{x^j}$ and the Schrödinger algebra says: $[P_i, K^j] = i\delta_i^j\hat{N}$ so we have $\hat{N} = i\partial_{\xi_+}$. For $z = 2$, there is trivially a special conformal symmetry which acts on ξ_+ in the same way as on ξ in the usual Schrödinger spacetime, and does not act on ξ_-.

It would be interesting to realize a system with arbitrarily many ξ-directions (*i.e.* species). Note that the new realization of the Galilean algebra described in the rest of this paper offers a simple possibility: one can just introduce a collection of gauge fields (perhaps coming from some p-form reduced on representatives of some rank $p-1$ cohomology group of a compactification space *e.g.* as described recently in [42]) and associate them with conserved

particle numbers of various species. The Galilean boost will act by some linear combination of the gauge generators; this combination is the total mass appearing in the Galilean symmetry algebra.

The wave equation is qualitatively the same as in the one-species case [1, 2].

5.2 $z \to 2$

Note that the interesting case $z = 2$ is actually quite degenerate here. The Einstein equations determine the coefficients $\Omega_{1,2}$ in the solutions for the gauge fields to be (for $d = 2$!)

$$\Omega_1^2 = 2\frac{z-1}{z} , \qquad \Omega_2^2 = 2\frac{z-1}{z-2} . \tag{5.11}$$

Notice that Ω_2 has a pole at $z = 2$. But the stress tensor it produces is finite, because both the field strength and the mass go to zero as $z \to 2$! That is, we must take a scaling limit where $z \to 2, \Omega_2 \to \infty$ holding fixed $(z-2)\Omega_2^2$ to which the stress tensor of A_2 is proportional.

6 Comments on the superfluid state

The ground state of most assemblies of ultracold atoms, bosonic or fermionic, is a superfluid [43]. It is natural to ask whether the zero-temperature, finite-density solution found in [7] describes such a state. That it does not can be seen as follows. If shifts in the ξ-direction correspond to the particle-number symmetry, then the gravity dual of a superfluid ground state must somehow break translation invariance in the ξ direction in the IR region of the geometry. This is because the ground state wave function of a superfluid is localized in the space conjugate to the particle number.

A precedent for the required gravity description is the spontaneous breaking of the $U(1)_R$ symmetry in the Klebanov-Strassler [44] and Maldacena-Nunez [45] solutions, where it is indeed some isometry of the bulk geometry which is broken (to a discrete subgroup) by the exact solution in the IR region of the geometry. A possibility to keep in mind is that the symmetry may be broken by something other than the metric, *e.g.* some other field.

We note that it is not clear that the twisted DLCQ theories, to which the stringy embeddings of Schrödinger spaces found in [5, 6, 7] are dual, indeed have superfluid ground states. If not, how does the dual field theory avoid breaking the particle number symmetry at zero

temperature? The fact that the only zero-temperature solution we know [7] is singular leaves open the likely possibility that there is a better, more correct solution with the same leading asymptotics which does describe a superfluid.

In the new realizations of the Schrödinger symmetry described in this paper, the question of spontaneous breaking of the particle number symmetry becomes much more similar to the (well-developed) study of holographic superconductors in gravity duals of relativistic CFTs [46, 47]. We note in particular that the system of §4 has a dimensionless parameter ς which controls the strength of the coupling. Upon the addition of a charged scalar to the bulk, we anticipate that varying ς will produce a quantum phase transition from the "Mott" "insulator" phase described here to a superfluid phase [48].

7   Conclusions

In this paper we have introduced a new class of gravity duals of Galilean-invariant CFTs. This requires somewhat novel asymptotics. In particular, the bulk gauge field which represents the particle number symmetry becomes strongly coupled at the UV boundary. In the examples constructed by dimensional reduction, this strong coupling of the gauge field is resolved by the lift; this is the statement that the ξ direction becomes null at the boundary of an asymptotically-Schrödinger geometry. It is an interesting open problem to characterize the resolution independent of the lift.

In the most interesting new solution we found (described in Section 4), there was also a singularity at the IR end of the geometry. This curvature singularity was resolved by a lift to 11-dimensional supergravity; we emphasize that the shrinking circle in the IR is *not* the particle-number direction ξ. It would be interesting to characterize which singularities of this kind can be resolved (see *e.g.* [33, 34]). A necessary criterion for a resolution by oxidation is that the geometry be conformal to a regular metric. It would be most useful for our purposes to be able to describe the resolution without resorting to dimensional oxidation.

Solutions of related Einstein-Maxwell-dilaton systems have been studied recently in [49, 50, 51, 52], mainly with *AdS* asymptotics in mind. It is possible that the near-infrared solutions studied in these papers can be integrated to the asymptotics described here.

Finally, we comment that although our goal in this paper was to rid ourselves of the extra

dimension ξ conjugate to the particle number, the 11-dimensional supergravity solution in Section 4 does indeed include such a dimension. Further, the regular 4d black hole solutions (without a ξ direction) which we found (in Section 3 and Appendix A) all have equations of state similar to that following from DLCQ (1.2). It will be of great interest to find black hole solutions, with the asymptotics described here, which have other equations of state.

Acknowledgements

A Black hole solution in a system with two scalars

In this appendix we present another system in which we have found black holes with asymptotic Schrödinger symmetry. Its action is rather contrived.

Let us consider the following action

$$S_D^E = \int d^D x (-g_D)^{1/2} \left[R_D - 2\Lambda e^{-\sigma} - \frac{e^{3\sigma}}{4} (dB)^2 - \frac{3}{2} (\partial\sigma)^2 - \frac{1}{2} (\partial\Psi)^2 + V_2(\sigma, \Psi) \right] \quad \text{(A.1)}$$

where

$$V_2(\sigma, \Psi) = \left[12 e^{-\sigma} \left(\sinh \left(\Psi/\sqrt{5} \right) \right)^3 + 16 \sqrt{Q}/r_0 \left(\tanh \left(\Psi/\sqrt{5} \right) \right)^{-9/4} \left(\sinh \left(\Psi/\sqrt{5} \right) \right)^5 \right] C_1^2 \tag{A.2}$$

The following background is a saddle point of this action

$$ds_E^2 = e^\sigma \left(-Qf K_x^2 \frac{dt^2}{r^6} + K_x \frac{d\vec{x}^2}{r^2} + \frac{dr^2}{f r^2} \right)$$

$$B = Q\frac{(1 - r^4/r_H^4)dt}{r^4} + B(r_H), \quad e^{2\sigma} = \frac{r^2}{Q}, \quad \Psi = \sqrt{5}\tanh^{-1}\left(\frac{r^4}{r_0^4}\right) \tag{A.3}$$

where

$$K_x^2 = 1 - r^8/r_0^8 \quad \text{and} \quad f = 1 - C_1^2\frac{r^4}{\sqrt{r_0^8 - r^8}}. \tag{A.4}$$

The above system has asymptotic Schrödinger symmetry, realized as in Section 2.3. The free energy of this system has the same form as the black hole in the system with one scalar *i.e.*,

$$\mathcal{F} \sim -\frac{T^4}{\mu^2}. \tag{A.5}$$

B Uplifting to M-theory

In this appendix we exhibit a useful sector of type IIA supergravity as a consistent truncation of eleven-dimensional supergravity[10], slightly generalizing the construction in [53]. Let us consider the following ansatz for the eleven-dimensional line element and the four-form flux:

$$ds^2 = g_{MN}dx^M dx^N = G_{\mu\nu}dx^\mu dx^\nu + e^{2\aleph}dz_{10}^2$$

$$\tilde{F}_4 = F_4 + H_3 \wedge dz_{10} \tag{B.1}$$

where g is the eleven-dimensional metric, G is the ten-dimensional metric, $\tilde{F}_4$ is the eleven-dimensional four-form flux, F_4 and H_3 are the ten-dimensional four-form and three-form flux. With this ansatz, the Bianchi identity becomes

$$d\tilde{F}_4 = 0 \quad \Leftrightarrow \quad dF_4 = 0 \quad \text{and} \quad dH_3 = 0. \tag{B.2}$$

The equation of motion for the eleven-dimensional four-form field strength can be written as

$$d \star \tilde{F}_4 = \frac{1}{4}\tilde{F}_4 \wedge \tilde{F}_4 \Leftrightarrow d\left(e^\aleph \star F_4\right) = \frac{1}{2}H_3 \wedge F_4 \quad \text{and} \quad d\left(e^{-\aleph} \star H_3\right) = \frac{1}{4}F_4 \wedge F_4. \tag{B.3}$$

The components of eleven-dimensional Ricci tensor $(\tilde{R}_{\mu\nu})$ are given by

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - \nabla_\mu\nabla_\nu\aleph - \nabla_\mu\aleph\nabla_\nu\aleph \tag{B.4}$$

$$\tilde{R}_{\mu 10} = 0 \tag{B.5}$$

$$\tilde{R}_{1010} = (\nabla_\mu\nabla^\mu\aleph + \nabla_\mu\aleph\nabla^\mu\aleph) \tag{B.6}$$

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[10]We will ignore fermions and assume that the Ramond-Ramond vector A_1 is turned off.

where $R_{\mu\nu}$ is the ten-dimensional Ricci scalar. After some algebra, the eleven-dimensional Einstein equations can be written as

$$\left(R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R \right) - \nabla_\mu \nabla_\nu \aleph - \nabla_\mu \aleph \nabla_\nu \aleph + \left(\nabla_\mu \nabla^\mu \aleph + \nabla_\mu \aleph \nabla^\mu \aleph \right) G_{\mu\nu} = T^F_{\mu\nu} + T^H_{\mu\nu} \quad \text{(B.7)}$$

$$R - \frac{1}{48}F_4^2 + \frac{1}{12}e^{-2\aleph}H_3^2 = 0 \quad . \quad \text{(B.8)}$$

The above equations can be obtained from the following action[11]

$$S_{10} = \int d^{10}x \sqrt{G} e^{\aleph} \left(R - \frac{1}{48}F_4^2 - \frac{e^{-2\aleph}}{12}H_3^2 \right) + \frac{1}{2} \int B_2 \wedge F_4 \wedge F_4 \quad \text{(B.10)}$$

where $H_3 = dB_2$. Let us redefine $\tilde{G}_{\mu\nu} = e^{-\aleph}G_{\mu\nu}$ and $\aleph = 2\Phi/3$. In terms of the redefined variables, the action in (B.10) can be written as

$$S_{10} = \int d^{10}x \sqrt{\tilde{G}} e^{-2\Phi} \left(\tilde{R} + 4\left(\partial\Phi \right)^2 - \frac{e^{2\Phi}}{48}F_4^2 - \frac{1}{12}H_3^2 \right) + \frac{1}{2} \int B_2 \wedge F_4 \wedge F_4 \quad . \quad \text{(B.11)}$$

This action is the bosonic part of the type IIA supergravity action (with A_1 turned off) in string frame. Any solution of the ten dimensional action in (B.11) can be oxidized to give a solution of eleven-dimensional supergravity. The 10-D Einstein frame metric metric is related to the string frame metric $\tilde{G}$ through the following Weyl transformation $g_E = e^{\Phi/2}\tilde{G}$.

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[11]We have made use of the following formula

$$\frac{1}{\sqrt{g}}\frac{\delta}{\delta g_{ab}} \int d^d x \sqrt{g} X R = X \left(R_{ab} - \frac{1}{2}Rg_{ab} \right) - \nabla_a \nabla_b X + g_{ab}\nabla_c \nabla^c X \quad . \quad \text{(B.9)}$$

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