# A NOTE ON THE PROBABILITY OF GENERATING ALTERNATING OR SYMMETRIC GROUPS 

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#### Abstract

We improve on recent estimates for the probability of generating the alternating and symmetric groups $\mathrm{A}_{n}$ and $\mathrm{S}_{n}$. In particular we find the sharp lower bound, if the probability is given by a quadratic in $n^{-1}$. This leads to improved bounds on the largest number $h\left(\mathrm{~A}_{n}\right)$ such that a direct product of $h\left(\mathrm{~A}_{n}\right)$ copies of $\mathrm{A}_{n}$ can be generated by two elements.


## 1. Introduction

For a group $X=\mathrm{S}_{n}$ or $\mathrm{A}_{n}$, we write $p(X)$ for the probability that two elements of $X$ generate a group that contains $\mathrm{A}_{n}$. In [1], Dixon proved that $p\left(\mathrm{~S}_{n}\right) \rightarrow 1$ as $n \rightarrow \infty$. In [2] he sharpened this statement to

$$
p\left(\mathrm{~S}_{n}\right)=1-\frac{1}{n}-\frac{1}{n^{2}}-\frac{4}{n^{3}}-\frac{23}{n^{4}}-\frac{171}{n^{5}}-\frac{1542}{n^{6}}+O\left(n^{-7}\right)
$$

For many applications, numerical results are needed, rather than asymptotics. In 5 Maróti and Tamburini proved explicit upper and lower bounds

$$
1-\frac{1}{n}-\frac{13}{n^{2}}<p(X) \leqslant 1-\frac{1}{n}+\frac{2}{3 n^{2}}
$$

In this present note, we find the best possible lower bound of this type, and a close-to-optimal upper bound.
Theorem 1.1. Let $X=\mathrm{A}_{n}$ or $X=\mathrm{S}_{n}$ with $n \geqslant 5$. Then

$$
1-\frac{1}{n}-\frac{8.8}{n^{2}} \leqslant p(X)<1-\frac{1}{n}-\frac{0.93}{n^{2}}
$$

Equality holds in the lower bound if and only if $n=6$.
In fact, for $n \geqslant 14$, we prove that $1-\frac{1}{n}-\frac{7.5}{n^{2}}<p(X)<1-\frac{1}{n}-\frac{0.93}{n^{2}}$. The result for smaller $n$ comes from the values for $p(X)$ in Table 1 (taken from [7, Table 4.1]).

Hall [3] considered the largest number $h(S)$ such that a direct product of $h(S)$ copies of a non-abelian finite simple group $S$ can be generated by two elements, and proved that $h(S)=p(S)|S| / \mid$ Out $(S) \mid$. The function $h(S)$ has received considerable attention recently; we refer the reader to [5 for more discussion and references and to [6] for lower bounds on $h(S)$ for all non-abelian finite simple groups $S$. The new bounds above yield:

Corollary 1.2. Let $n$ be an integer with $n \geqslant 14$. Then

$$
\left(1-\frac{1}{n}-\frac{7.5}{n^{2}}\right)\left(\frac{n!}{4}\right)<h\left(\mathrm{~A}_{n}\right)<\left(1-\frac{1}{n}-\frac{0.93}{n^{2}}\right)\left(\frac{n!}{4}\right)
$$

Let $m(S)$ denote the minimal index of a proper subgroup of a group $S$. In [4], it is proved that there exist absolute constants $c_{1}$ and $c_{2}$ such that $1-c_{1} / m(S)<p(S)<1-c_{2} / m(S)$, for all non-abelian finite simple groups $S$. Our main theorem immediately yields the following bounds on $c_{1}$ and $c_{2}$.
Corollary 1.3. For $n \geqslant 5$,

$$
1-\frac{2.468}{n}<p\left(\mathrm{~A}_{n}\right)<1-\frac{1.186}{n}
$$

and hence $c_{1} \geqslant 2.468$ and $c_{2} \leqslant 1.186$.

[^0]
## 2. Proof of Theorem 1.1

Definition 2.1. For $X=\mathrm{A}_{n}$ or $\mathrm{S}_{n}$ we let $p_{\text {intrans }}(X)$ and $p_{\text {trans }}(X)$ be the probability that two elements chosen randomly from $X$ generate a subgroup of an intransitive maximal subgroup of $X$, or a subgroup of a transitive maximal subgroup of $X$ other than $\mathrm{A}_{n}$, respectively.
Lemma 2.2. Let $X=\mathrm{A}_{n}$ or $\mathrm{S}_{n}$ with $n \geqslant 14$. Then

$$
p_{\text {intrans }}(X)<\frac{1}{n}+\frac{2.7}{n^{2}} .
$$

Proof. We prove the result for $\mathrm{S}_{n}$, the arguments for $\mathrm{A}_{n}$ are identical. Let $x, y \in \mathrm{~S}_{n}$ and suppose that $Y:=\langle x, y\rangle$ is contained in an intransitive maximal subgroup. Then $Y$ is contained in a subgroup conjugate to $\mathrm{S}_{k} \times \mathrm{S}_{n-k}$ for some $1 \leqslant k \leqslant\left\lfloor\frac{n-1}{2}\right\rfloor$.

Let $k \in\{1, \ldots, n-1\}$. Then the probability that $Y \leqslant \mathrm{~S}_{k} \times \mathrm{S}_{n-k}$ is bounded by

$$
\binom{n}{k}\left(\frac{k!(n-k)!}{n!}\right)^{2}=\binom{n}{k}^{-1}
$$

So the probability that $Y \leqslant \mathrm{~S}_{1} \times \mathrm{S}_{n-1}$ is at most $\frac{1}{n}$, and the probability that $Y \leqslant \mathrm{~S}_{2} \times \mathrm{S}_{n-2}$ and $Y$ is transitive on the orbit of size 2 is bounded by

$$
\frac{3}{4} \frac{2}{n(n-1)}=\frac{3}{2 n(n-1)}
$$

Similarly, the probability that $Y \leqslant \mathrm{~S}_{3} \times \mathrm{S}_{n-3}$ and $Y$ is transitive on the orbit of length 3 is

$$
\frac{13}{18}\binom{n}{3}^{-1}=\frac{13}{3 n(n-1)(n-2)}
$$

Now the probability that $Y \leqslant \mathrm{~S}_{k} \times \mathrm{S}_{n-k}$ for some $4 \leqslant k \leqslant\left\lfloor\frac{n-1}{2}\right\rfloor$ is

$$
\sum_{k=4}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{1}{\binom{n}{k}} \leqslant \sum_{k=4}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{1}{\binom{n}{4}} \leqslant \frac{12(n-7)}{n(n-1)(n-2)(n-3)}
$$

We now observe that, since $n \geqslant 14$,

$$
\frac{3}{2 n(n-1)}+\frac{13}{3 n(n-1)(n-2)}+\frac{12(n-7)}{n(n-1)(n-2)(n-3)}<\frac{2.7}{n^{2}}
$$

which completes the proof.
Lemma 2.3. Let $X=\mathrm{A}_{n}$ or $\mathrm{S}_{n}$, with $n \geqslant 14$. Then

$$
p_{\text {intrans }}(X)>\frac{1}{n}+\frac{0.93}{n^{2}} .
$$

Proof. We observe that $p_{\text {intrans }}(X)$ is bounded below by the probability that a random pair of elements of $X$ generate a subgroup with a fixed point, or with an orbit of size 2 . For $X=S_{n}$, we bound $p_{\text {intrans }}(X)$ by doing inclusion-exclusion to depth 2 on the union of the sets $\left(\mathrm{S}_{n}\right)_{\alpha}$, with $1 \leqslant \alpha \leqslant n$, and $\left(\mathrm{S}_{n}\right)_{\{\alpha, \beta\}} \backslash\left(\mathrm{S}_{n}\right)_{(\alpha, \beta)}$, with $1 \leqslant \alpha<\beta \leqslant n$. We find that $p_{\text {intrans }}(X)$ is greater than

$$
\frac{1}{n}+\frac{3}{4} \frac{2(n-2)!}{n!}-\frac{(n-2)!}{2 n!}-\frac{3}{4}\binom{n}{1}\binom{n-1}{2}\left(\frac{2(n-3)!}{n!}\right)^{2}-\left(\frac{3}{4}\right)^{2} \frac{\binom{n}{2}\binom{n-2}{2}}{2}\left(\frac{4(n-4)!}{n!}\right)^{2}
$$

Thus

$$
p_{\text {intrans }}(X) \geqslant \frac{1}{n}+\frac{8 n^{2}-52 n+75}{8 n(n-1)(n-2)(n-3)}
$$

which, since $n \geqslant 14$, is greater than $\frac{1}{n}+\frac{0.93}{n^{2}}$.
Proof of Theorem 1.1. For the upper bound we use Lemma 2.3. For the lower bound, note that

$$
1-p(X)=p_{\text {intrans }}(X)+p_{\text {trans }}(X)
$$

It follows from the proofs of [5, Lemmas 3.1 and 4.3] that $p_{\text {trans }}(X) \leqslant \frac{4.8}{n^{2}}$. Combining this with Lemma 2.2 gives the theorem.

In Table 1 we record the value of $p\left(\mathrm{~A}_{n}\right)$ and $p\left(\mathrm{~S}_{n}\right)$ for $n \leqslant 13$, together with our lower and upper bounds as stated in Theorem 1.1. All values are correct to three decimal places.

Table 1. Precise values and bounds for $p(X)$

| $n$ | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $p\left(\mathrm{~A}_{n}\right)=$ | 0.633 | 0.588 | 0.726 | 0.739 | 0.848 | 0.875 | 0.893 | 0.902 | 0.913 |
| $p\left(\mathrm{~S}_{n}\right)=$ | 0.633 | 0.588 | 0.795 | 0.796 | 0.859 | 0.875 | 0.894 | 0.903 | 0.913 |
| $p(X) \geqslant$ | 0.448 | 0.588 | 0.677 | 0.737 | 0.780 | 0.812 | 0.836 | 0.855 | 0.871 |
| $p(X) \leqslant$ | 0.763 | 0.808 | 0.839 | 0.861 | 0.878 | 0.891 | 0.902 | 0.911 | 0.918 |

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