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Putting The Pieces Together: Understanding Robinson's Nonperiodic Tilings

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Suppose that you wish to tile a huge floor using square tiles of equal size with variously colored edges. When you place two tiles next to each other, their edge colors must match, and of course you must leave no gaps anywhere. There are only a finite number of tile types available, but you may use as many tiles of each type as you want. For simplicity, assume that you may not rotate your tiles as you place them. (We could allow rotations simply by including the rotated tiles as new tile types, so this is not a fundamental restriction.) *Can you look at the tile types available and determine whether the task is possible or not?*

If it is possible to cover the whole plane with a given set of tile types, we say that this set will *tile the plane.* Certainly you can envision a situation where the available tile types will not tile the plane. For instance, with only the two tile types illustrated in Figure 1, you would never be able to place one tile above another.

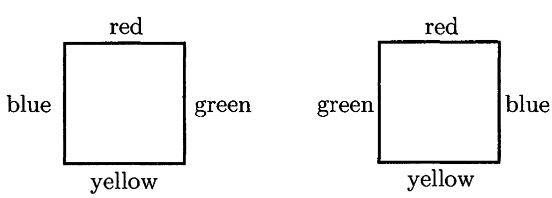

Figure 1

But it is not always so easy to decide. For example, does the set of four tile types shown in Figure 2 tile the plane? In attempting to answer this question, you might begin trying to tile 2×2 squares. If that works, you might try tiling 3×3 squares, and so on. Should you find some size square that you cannot complete, you could conclude that these tile types will not tile the plane. (What is the smallest square that you cannot complete with the tiles in Figure 2?) But even if you managed to tile all squares with a million tiles on a side, maybe the square whose sides take a million and one tiles could not be tiled.

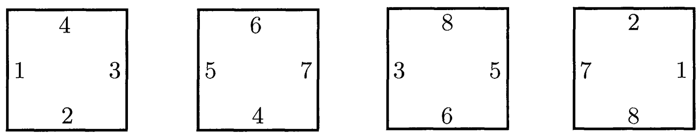

Figure 2. Four tile types; numbers represent edge colors.

In 1961, Hao Wang speculated that this process will always eventually end—either you will find a square that you cannot complete or you will find a "periodic square" [7]. An $n \times n$ square of tiles is *periodic* if its top and bottom rows of tiles are the same and if its left and right columns of tiles are the same. If you can construct a periodic square with your tile types, you can tile the plane, because periodic squares can be stacked end to end vertically and horizontally with the matching edges of each square overlapping. (Equivalently, removing the bottom row and the right column of tiles from each periodic square, we could tile the plane with the resulting $(n-1)\times(n-1)$ squares.)

Wang's conjecture. *Any set of tiles that tiles the plane can be used to tile the plane with periodic squares.*

A tiling of the plane with periodic squares is called a *periodic tiling*. You can tell that a tiling is periodic if there are at least two places you can stand from which the resulting floor pattern looks exactly the same. In fact there are then infinitely many places from which you can see the same pattern, since any translation that moves you from one such spot to another can be repeated to move you to a third spot from which the pattern looks the same. Figure 3 shows part of a periodic tiling. (In this simple example any edge of a tile can meet either possible edge of tiles of the other type.) Can you see the periodic squares? What translations move you between spots from which the pattern looks the same?

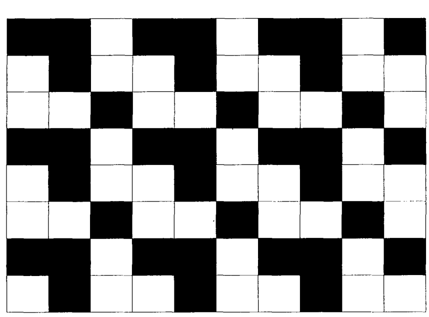

Figure 3

Wang's conjecture seems reasonable because it is true in one dimension. That is, suppose you are tiling not a two-dimensional floor but a one-dimensional strip, laying the tiles end to end. Having only finitely many tile types, you must use at least one tile type twice. But then you have a block of tiles, beginning and ending with the same tile type, which you can use over and over in a periodic tiling of the strip. For instance, given the set of tile types in Figure 4a, the periodic block in Figure 4b tiles the infinite strip periodically, with a three-tile repeating pattern.

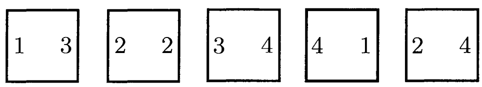

Figure 4a

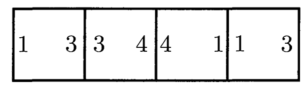

Figure 4b

If Wang's conjecture were true in two dimensions as well, we would have a general method for determining whether a given set of tile types tiles the plane. Simply construct *all* tilings of 2×2 squares, then of 3×3 squares, and so on. If the tiles do not tile the plane, eventually we would find a square that cannot be tiled. If the tiles do tile the plane, the conjecture says that eventually a periodic square would be found. Either way, the search process would eventually terminate, thus allowing us to decide whether the given tile types tile the plane.

Wang's conjecture remained an open question for several years until shown to be false by one of his students, Robert Berger [**1**], who found a set of tiles that tiles the plane but for which no tiling using them is periodic. Berger's original example involved a set of over 20,000 tile types! In 1971 Raphael Robinson [**5**] found a simpler example with just 28 tile types.

So the answer to the original question is "no": We do not know a general method of determining whether a given finite set of tile types will tile the plane. Our main goal here is to understand why Robinson's tiles will tile the plane and yet no tiling with them can be periodic. In a final section we will briefly consider other familiar nonperiodic tilings with nonsquare tiles: the Penrose tilings and pinwheel tilings.

Robinson's Example

Before describing Robinson's tiles, we introduce an improved labeling system that makes it easy to see which tiles can be juxtaposed. When we use square tiles with colored or numbered edges, these labeled edges give us "matching rules." Other markings on the tiles could be used to express the desired matching rules in other ways. We will mark the tiles with arrows that must match head to tail when a juxtaposition is allowed. The color-edged tiles in Figure 1 might thus be relabeled as in Figure 5, to indicate that the two tiles must alternate horizontally and that no vertical juxtapositions are allowed.

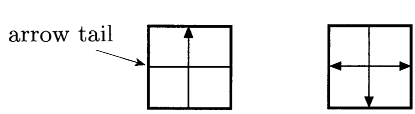

Figure 5

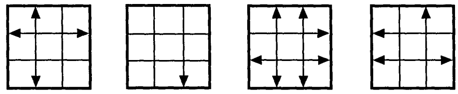

Figure 6

Similarly, the matching rules for the four tiles in Figure 2 could be expressed with arrow markings as in Figure 6.

The 28 tile types in Robinson's example are illustrated in Figure 7. Let's refer to tiles 1 through 4 as *crosses*; these are the tiles with two doubleheaded arrows crossing at the center and a doubleheaded "elbow." The remaining tiles we call *arms*; these tiles contain no doubleheaded arrows.

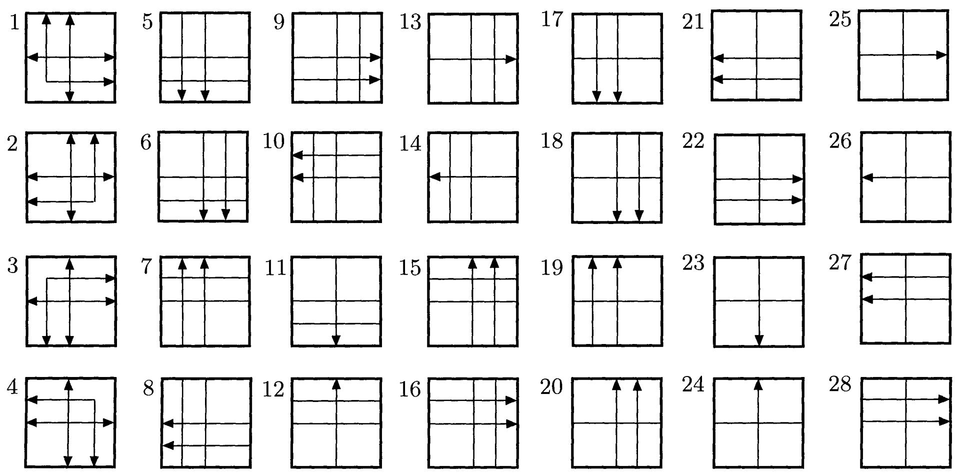

Figure 7. Robinson's tiles.

In addition to requiring that arrowheads and arrow tails always meet, we will require that our tilings satisfy the *alternating cross rule*, illustrated in Figure 8. It is important to note that crosses may appear in other locations besides those specified by this rule. The alternating cross rule is not technically a matching rule, but we could add four more tile types with additional markings to obtain an essentially equivalent set of 32 tiles that obey matching rules. (This is in fact what Robinson did [**5**]. He also pointed out that if we wanted to use only tiles with colored edges, we could do so with a set of 56 tile types.)

cross		cross		cross
cross		cross		cross

Figure 8. *The alternating cross rule*: Crosses must appear in every other position in every other row.

3×3 squares. Let's think about the consequences of the matching rule requiring arrowheads to meet arrow tails. First of all, a cross cannot sit next to another cross (vertically or horizontally), since that would force two arrowheads to touch. Next, consider any two crosses separated by a sequence of arms: either their elbows bend toward one another (the crosses *face each other*, as in Figure 9) or they bend away (the crosses are *back to back*, as in Figure 10). When they are back to back, two configurations are possible: they may be mirror images or inverted.

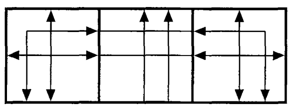

Figure 9. Crosses facing each other.

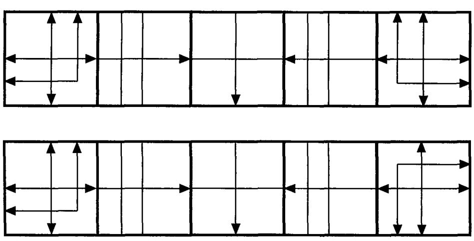

Figure 10. Crosses back to back. Above: mirror images; below: inverted.

These three arrangements of the crosses are the only possibilities. Anything else yields a configuration with different numbers of arrowheads or different locations of arrowheads on the sides of the crosses nearest each other. But then these cannot be connected by a sequence of arms because *on each arm the number and location of arrows on any edge are always the same as on the opposite edge.* For example, consider the two crosses in Figure 11 and try to fill in the middle tile with an arm. It can't be done! A similar argument holds for a pair of crosses appearing vertically with a sequence of arms in between them.

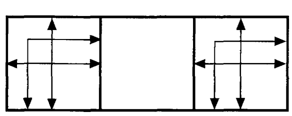

Figure 11

Consequently, adjacent crosses appearing in one of the alternating rows specified by the alternating cross rule must either face each other or be back to back. So any one of these crosses faces another one two units to its left or right. Consider a 3×3 square with two such facing crosses in the corners of its bottom row. We know that its other corners must be crosses (by the alternating cross rule) and that they must face those in the bottom row.

Thus the 3×3 square looks like Figure 12.

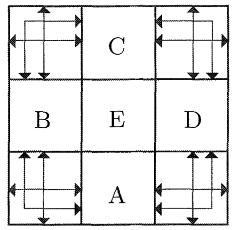

Figure 12

Because crosses cannot sit side by side, we also know that an arm must be in position A. There are three choices for the arm in position A: tiles 5, 6, and 11. Note that all of these arms have arrows pointing away from the center of the 3×3 square. Similarly there are three choices for positions B, C, and D, and the arrowheads in all of these arms point away from the center of the square. So no matter how we choose the four arms for positions A–D, the tile in position E must have arrowheads on all four sides—in other words, it has to be a cross. Moreover, for each of the four crosses in Figure 7 there is a unique 3×3 square with this cross in the central position.

Three of the four possible 3×3 squares are illustrated in Figure 13. Can you construct the fourth?

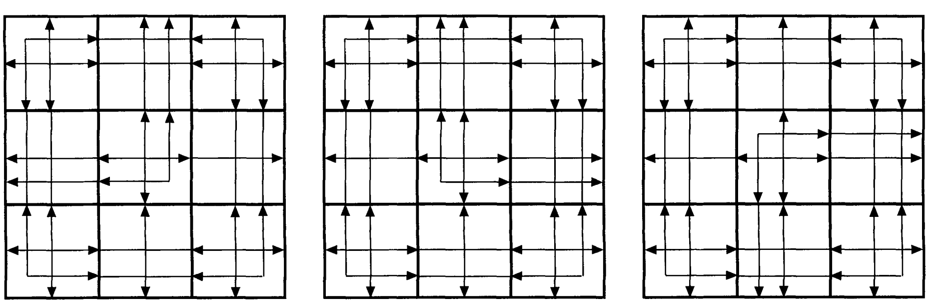

Figure 13

7 × 7 squares. Let's say that a 3×3 square faces in the direction indicated by the elbow in its central cross; for example, the 3×3 square on the left in Figure 13 faces up and to the left, or northwest. *We can extend any* 3×3 *square in the direction it faces*. The northwest-facing square in Figure 13, for instance, can be extended to a 7×7 square with itself in the lower right corner.

How do we perform this extension? First, notice that all the exterior edges of the 3×3 square have arrowheads, which crosses cannot meet; so we know that the top and left sides of our 3×3 square must meet arms. Secondly, by the alternating cross rule, the crosses in the 7×7 square must be located as shown in Figure 14a. Then, because crosses cannot sit next to crosses, we will have arms in between, as indicated.

In fact the crosses will be oriented as in Figure 14b. This is not obvious, but playing with our tiles a bit should make it clear. We have five choices for the arm in position A of Figure 14a: tiles 5, 6, 8, 11, or 21. All have tails on their top edge. Since heads must meet tails, this forces the arm in position B to have its arrowhead(s) point away from the center of the 7×7 square. Similarly, by considering the choices for arm C, we see that arm D also points away from the center of the 7×7 square. So the central tile of the 7×7 square must have arrowheads on at least two sides; that is, it must be a cross.

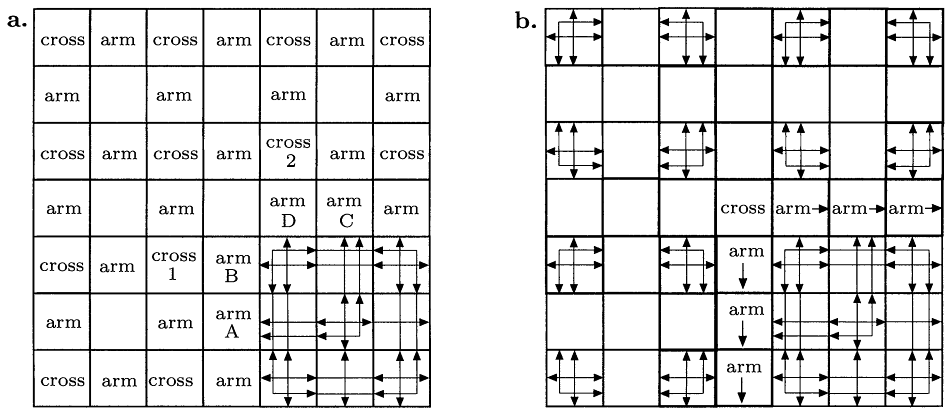

Figure 14

The cross labeled 1 in Figure 14a will be back to back with the cross in the upper left corner of the original 3×3 square, either inverted or a mirror image. If it were inverted then the configuration in Figure 15 would be in the center of our 7×7 square. How would you fill in the arms and the central cross in Figure 15? It can't be done. Thus, cross 1 must be the mirror image of the cross in the upper left corner of the original 3×3 square. A similar argument applies to cross 2. Thus crosses 1 and 2 are forced to appear as in Figure 14b, and they in turn force the remaining crosses in Figure 14a to be oriented as in Figure 14b.

From Figure 14b, we see that each corner must have a 3×3 square whose central cross is determined by the central cross of our initial 3×3 square. The central cross

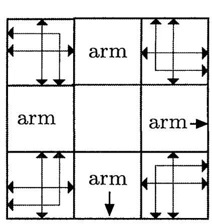

Figure 15. Center of 7×7 square.

of the lower left 3×3 square must face the central cross of the initial 3×3 square; then the central cross of the upper left 3×3 square must face toward the central cross of the lower left 3×3 square; and so on. Finally, because all the 3×3 configurations have arrowheads pointing away from their central cross, so must the horizontal row and vertical column of tiles radiating away from the central cross of the 7×7 square.

Once again, note that each choice for the central cross produces a unique 7×7 square. One possibility is illustrated in Figure 16.

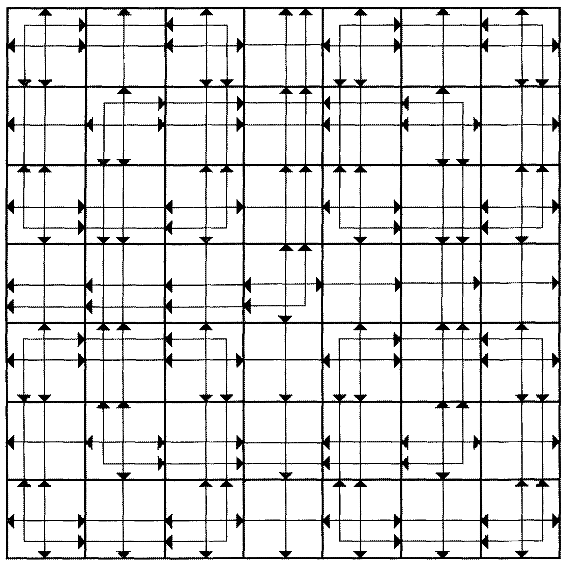

Figure 16

Aperiodic tiling. In a similar way each 7×7 square can be extended in the direction of its central cross to a 15×15 square. This 15×15 square will have two familiar features: a central cross with a row and column of arms pointing away from it, and four corners consisting of 7×7 squares facing each other. The 15×15 square can then be extended to a 31×31 square with a central cross and with its 15×15 corner squares facing each other, and so on.

Because $(2^n - 1) \times (2^n - 1)$ squares can be tiled for all values of n, it follows that the plane can be tiled with this set of tiles! None of the tilings can be periodic, however. No matter how we translate a tiling of the plane formed using these tiles, some tiles will fail to match up. (In other words, we can't stand at two different spots on our infinite floor and have the pattern look exactly the same.) Let's see why this is so.

Imagine that, having tiled the floor using these tiles, you now stand on one of the crosses found in alternate positions in alternate rows. As described before, this cross is part of a 3×3 square with a pattern as in Figure 13, and this 3×3 square lies in some 7×7 square as in Figure 16, and so on. Look at the 3×3 square you are standing on and memorize its pattern. Now, move four units away horizontally or vertically, staying within your 7×7 square. You will be in a new 3×3 square, so the pattern will look similar. But the central cross of your new 3×3 square faces the central cross of your original 3×3 square, so these 3×3 squares cannot look exactly the same! Thus the tiling is not invariant under a translation by 2^2. Nor is it invariant under a translation by 3 or 2; moving horizontally or vertically by these amounts you see a pattern quite different from your original 3×3 square.

Now return to the cross you started on and memorize the pattern of the 7×7 square you are in. Move horizontally or vertically by 8, staying in the 15×15 square that contains your 7×7 square. Again you are in a 7×7 square, so the pattern is familiar. But, again, the center cross of the new 7×7 faces the central cross of the original 7×7 square, so the two squares do not look exactly the same. So the tiling is not invariant under a translation by 2^3. It is also not invariant under a translation by 5, 6, or 7, because under such translations the pattern does not even resemble your original 7×7 square.

By similar arguments, the tiling is not invariant under horizontal or vertical translation by any value 2^n or any value $m \neq 2^n$. Therefore no tiling that satisfies the alternating cross rule is periodic.

Other Facts and Examples

We have shown that Robinson's tiles provide an example of a set that tiles the plane but whose tiling is not invariant under a translation. In fact, no tiling constructed using this set of tiles is invariant under a rotation [**5**].

Interestingly, although no tiling constructed with this set of tiles is periodic, they are *almost periodic*; that is, if you choose a large enough period, an arbitrarily large percentage of the tiles will repeat. For example, the corner crosses of the 3×3 squares repeat horizontally and vertically with period 4. These crosses comprise one quarter of the total tiles. To see this, divide any $2^n \times 2^n$ block into 2×2 blocks; each will contain exactly one such cross. So, one quarter of the tiles repeat with period 4. Similarly, we can divide any $2^n \times 2^n$ block into 4×4 blocks, each containing exactly one 3×3 square. The 3×3 squares are determined by their central crosses and they repeat with period 8. So 9/16th of the tiles repeat with period 8. In general, $[(2^n - 1)/2^n]^2$ of the tiles repeat with period 2^{n+1}.

Penrose tilings. Another famous tiling of the plane was discovered by Roger Penrose in 1973 [**2**], [**4**]. A later modification of the Penrose tiling uses finitely many different rotations of only two tile shapes, *kites* and *darts*. The vertices of each tile alternate in color, as indicated by the open and filled circles in Figure 17, and the tilings are required to satisfy the matching rule that colored vertices match.

The Penrose tilings are not periodic under any translation; however, they can be periodic under rotations by 72°. The Penrose tilings are almost periodic in the sense that the pattern seen in any arbitrarily large block repeats within a bounded translation. The Penrose tiles are useful in understanding the geometric properties of quasicrystals [**6**]; they have actually been patented.

Pinwheel tilings. Figure 18 shows pinwheel tilings, which feature a single tile shape, a 1-2-$\sqrt{5}$ right triangle. These tilings, unlike our earlier ones, involve infinitely many different tile types because in each tiling the triangle occurs in infinitely many different orientations [**3**]. The pinwheel tilings are nonperiodic under translations and rotations; and they too are almost periodic in the sense described above.

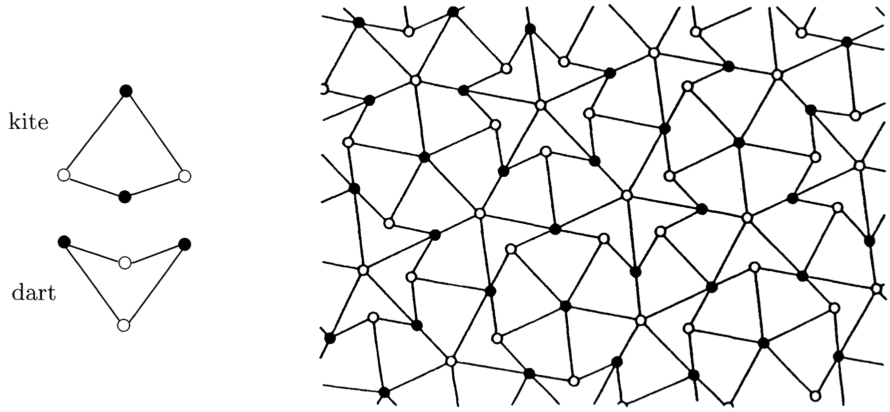

Figure 17. A Penrose tiling with kites and darts.

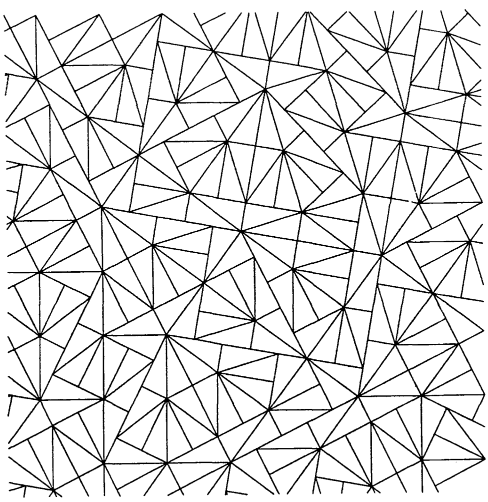

Figure 18. A pinwheel tiling.

We recommend [**2**] for further exploration of many other interesting tiling examples.

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