Hardy-type inequalities for the generalized Mehler transform

勘　甚　裕　一
（金沢大学　理工研究域　機械工学系）
佐　藤　邦　夫
（山形大学　理工学研究科数物学分野）

山形大学紀要（自然科学）第17巻第4号別刷
平成25年（2013）2月

Bull. of Yamagata Univ., Nat Sci., Vol.17, No.4, Feb. 2013

Hardy-type inequalities for the generalized Mehler transform

Yuichi KANJIN and Kunio SATO

Abstract

We establish Hardy-type inequalities for the generalized Mehler transform on the real Hardy space H^p, $0 < p < 1$.

1. Introduction and Results

Let $0 < p \le 1$ and $H^p(\mathbb{R})$ be the real Hardy space, that is, the space of the boundary distributions $f(x) = \Re F(x)$ of the real parts $\Re F(z)$ of functions $F(z)$ in the Hardy space $H^p(\mathbb{R}^2_+) = \{F(z);$ analytic in $\mathbb{R}^2_+$ and $\|F\|_{H^p(\mathbb{R}^2_+)} = \sup_{t>0}(\int_{-\infty}^{\infty} |F(x+it)|^p dx)^{1/p} < \infty\}$ on the upper half plane $\mathbb{R}^2_+ = \{z = x + it\,;\, t > 0\}$, with the norm $\|f\|_{H^p} = \|F\|_{H^p(\mathbb{R}^2_+)}$. Then, the Fourier transform $\hat{f}$ of $f \in H^p(\mathbb{R})$ is a continuous function and satisfies the inequality

$$\int_{-\infty}^{\infty} |\hat{f}(\xi)|^p |\xi|^{p-2}\, d\xi \le C\|f\|_{H^p}^p,$$

which is well-known as Hardy's inequality for $H^p(\mathbb{R})$ (cf. [7, Corollary 7.23], [21, p.128]).

The aim of this paper is to establish an analogue of this inequality for the generalized Mehler transform.

The generalized Mehler transform is defined as follows. Let m be a real number such that $m \le 1/2$, and define

$$K^m(x, y) = k_m(x)(\sinh y)^{1/2} P^m_{-1/2+ix}(\cosh y),$$

where

$$k_m(x) = \left|\frac{\Gamma(1/2 - m - ix)}{\Gamma(-ix)}\right|, \tag{1}$$

and $P^m_{-1/2+ix}(z)$ is the Legendre function of order m and degree $-1/2 + ix$, which is given by using the hypergeometric function as follows:

$$P^m_{-1/2+ix}(z) = \frac{1}{\Gamma(1-m)}\left(\frac{z+1}{z-1}\right)^{m/2} F(1/2 - ix, 1/2 + ix; 1 - m; 1/2 - z/2).$$

2010 *Mathematics Subject Classification*. Primary 44A20, 42A38; Secondary 42B30, 43A90.

Key words and phrases. Hardy' sinequality, Generalized Mehler transform.

The first author was supported by Grant-in-Aid for Science Research (C)(No.24540167), Japan Society for the Promotion of Science.

The following transforms

$$\mathcal{G}^m(f;y) = \int_0^\infty f(x)K^m(x,y)\,dx,$$
$$\mathcal{H}^m(g;x) = \int_0^\infty g(y)K^m(x,y)\,dy.$$

are called the generalized Mehler transform. We remark that if $f, g \in L^1[0,\infty)$, then the values $\mathcal{G}^m(f;y), \mathcal{H}^m(g;x)$ exist for every $x, y > 0$ since $|K^m(x,y)| \leq C, x > 0, y > 0, m \leq 1/2$ (cf. [20]). Let us call $\mathcal{G}^m$ and $\mathcal{H}^m$ the *G-type transform* of order m and the *H-type transform* of order m, respectively. It is known that $K^{1/2}(x,y) = \sqrt{2/\pi}\cos xy$ and $K^{-1/2}(x,y) = \sqrt{2/\pi}\sin xy$. Thus the H-type and G-type transforms of order $1/2$ are the cosine transform and, those transforms of order $-1/2$ are the sine transform. The above classical Hardy inequality leads to the following inequalities

$$\int_0^\infty |\mathcal{G}^{\pm 1/2}(f,y)|^p y^{p-2}\,dy \leq C\|f\|^p_{H^p(\mathbb{R})},$$
$$\int_0^\infty |\mathcal{H}^{\pm 1/2}(f,y)|^p y^{p-2}\,dy \leq C\|f\|^p_{H^p},$$

where $f \in H^p(\mathbb{R})$ with supp $f \subset [0,\infty)$ and $0 < p \leq 1$.

In this paper, we shall investigate Hardy-type inequalities for the G-type and H-type transforms of arbitrary order $m < 1/2$ on the space

$$H^p[0,\infty) = \{\ f \in H^p(\mathbb{R})\ :\text{supp}\ f \subset [0,\infty)\ \},\quad 0 < p \leq 1,$$

and obtain the following:

Theorem 1. (i) *Let $-m + 1/2 > 0$ and $0 < p \leq 1$. Then, there exists a constant C such that*

$$\int_1^\infty |\mathcal{G}^m(f;y)|^p y^{p-2}\,dy \leq C\|f\|_{H^p[0,\infty)},\quad f \in H^p[0,\infty).$$

(ii) *Let $-m + 1/2 > 0$ and $0 < p \leq 1$. Suppose that $[1/p] \leq [-m + 1/2]$. Then, there there exists a constant C such that*

$$\int_0^1 |\mathcal{G}^m(f;y)|^p y^{p-2}\,dy \leq C\|f\|_{H^p[0,\infty)},\quad f \in H^p[0,\infty).$$

Theorem 2. (i) *Let $-m + 1/2 > 0$ and $0 < p \leq 1$. Suppose that $1/p - 1 < -m + 1/2$. Then, there exists a constant C such that*

$$\int_1^\infty |\mathcal{H}^m(g;x)|^p x^{p-2}\,dx \leq C\|g\|_{H^p[0,\infty)},\quad g \in H^p[0,\infty).$$

If $-m + 1/2 = 1, 2, 3, \ldots$, then the above inequality holds for every p with $0 < p \leq 1$.

(ii) *Let $-m + 1/2 > 0$ and $1/2 < p \leq 1$. Suppose that $1/p - 1 < -m + 1/2$. Then, there there exists a constant C such that*

$$\int_0^1 |\mathcal{H}^m(g;x)|^p x^{p-2}\,dx \leq C\|g\|_{H^p[0,\infty)},\quad g \in H^p[0,\infty).$$

Collorary 1. *Let* $1/2 < p \leq 1$ *and* $-m + 1/2 = 1, 2, 3, \ldots$. *Then, there exist constants* C *such that*

$$\int_0^\infty |\mathcal{G}^m(f;y)|^p y^{p-2}\, dy \leq C\|f\|_{H^p[0,\infty)}, \quad f \in H^p[0,\infty),$$

and

$$\int_0^\infty |\mathcal{H}^m(g;x)|^p x^{p-2}\, dy \leq C\|g\|_{H^p[0,\infty)}, \quad g \in H^p[0,\infty).$$

There are several results related to Hardy's inequality. A Hardy-type inequality for the Hankel transform is in [11], and the inequalities for Hermite and Laguerre expansions are in [10] and [12]. Hardy's inequality associated with the $n-1$ dimensional unit sphere in $\mathbb{R}^n, n \geq 3$ is in [4], and the ones for higher-dimensional Hermite and special Hermite expansions are in [18]. Some other inequalities of Hardy-type will be found in Colzani and Travaglini [5], Thangavelu [22], Betancor and Rodríguez-Mesa [2], Guadalupe and Kolyada [8], Kanjin and Sato [13], Sato [19], Balasubramanian and Radha [1].

We give some facts about the generalized Mehler transform. The usual generalized Meheler transform pair is the following:

$$\begin{aligned} g(u) &= \int_0^\infty f(x) P^m_{-1/2+ix}(u)\, dx, \\ f(x) &= \pi^{-1} x \sinh \pi x \; \Gamma(1/2 - m + ix)\Gamma(1/2 - m - ix) \\ &\qquad \cdot \int_1^\infty g(u) P^m_{-1/2+ix}(u)\, dx. \end{aligned}$$

Conditions for the inversion of this pair will be found, for example, in [15]. According to [20], we reformulate this pair. We note that

$$k_m^2(x) = \pi^{-1} x \sinh \pi x \; \Gamma(1/2 - m + ix)\Gamma(1/2 - m - ix),$$

and then we have

$$\begin{aligned} g(\cosh y)(\sinh y)^{1/2} &= \int_0^\infty \frac{f(x)}{k_m(x)} K^m(x,y)\, dx, \\ \frac{f(x)}{k_m(x)} &= \int_0^\infty g(\cosh y)(\sinh y)^{1/2} K^m(x,y)\, dy. \end{aligned}$$

Rewriting $g(\cosh y)(\sinh y)^{1/2}$ and $f(x)/k_m(x)$ with $g(y)$ and $f(x)$, again, we have H-type and G-type transforms.

The generalized Mehler transform is a special case of the Jacobi transform. We follow the notations of Koornwinder [14]. Let $\phi_\lambda^{(\alpha,\beta)}(t)$ be the Jacobi functions:

$$\phi_\lambda^{(\alpha,\beta)}(t) = F\big((\alpha+\beta+1-i\lambda)/2, (\alpha+\beta+1+i\lambda)/2; \alpha+1; \sinh^2 t\big).$$

Put

$$\Delta_{\alpha,\beta}(t) = (2\sinh t)^{2\alpha+1}(2\cosh t)^{2\beta+1}.$$

The Jacobi transform of a function f is defined by

$$\hat{f}(\lambda) = \int_0^\infty f(t)\phi_\lambda^{(\alpha,\beta)}(t)\Delta_{\alpha,\beta}(t)\, dt.$$

Let G be a connected noncompact semisimple Lie group with finite center, and fix a maximal compact subgroup K. Associated to G there are constants $p, q =$

$0, 1, 2, \ldots$ determined by the geometry of the symmetric space G/K such that $n = \dim(G/K) = p + q + 1$. Let

$$\alpha = \frac{p+q-1}{2} = \frac{n-2}{2}, \qquad \beta = \frac{q-1}{2},$$

that is,

$$p = 2(\alpha - \beta), \quad q = 2\beta + 1, \quad n = 2\alpha + 2.$$

Then the Jacobi functions $\phi_\lambda^{(\alpha,\beta)}(t)$ and the Jacobi transform appear as the spherical functions and the spherical transform on G/K. The Plancherel theorem for the Jacobi transform is as follows:

$$\int_0^\infty |f(t)|^2 \Delta_{\alpha,\beta}(t)\, dt = \frac{1}{2\pi} \int_0^\infty |\hat{f}(\lambda)|^2 |c(\lambda)|^{-2}\, d\lambda$$

if $\alpha > -1$ and $\alpha \pm \beta + 1 \geq 0$. Here,

$$c(\lambda) = \frac{2^{\rho - i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma\big((i\lambda+\rho)/2\big)\Gamma\big((i\lambda+\alpha-\beta+1)/2\big)}, \quad \rho = \alpha + \beta + 1.$$

There are relations between the generalized Mehler transform and the Jacobi transform. Let

$$\alpha = \beta = -m, \qquad x = \lambda/2, \quad y = 2t.$$

Then we have the following.

$$\begin{aligned}
&\Delta_{\alpha,\beta}(t) = (2\sinh y)^{-2m+1}, \\
&\phi_\lambda^{(\alpha,\beta)}(t) = 2^{-m}\Gamma(-m+1)(\sinh y)^m P^m_{-1/2+ix}(\cosh y), \\
&\hat{f}(\lambda) = \frac{2^{-2m}\Gamma(-m+1)}{k_m(x)} \mathcal{H}^m(g;x), \quad g(y) = 2^{-m}(\sinh y)^{-m+1/2} f(y/2), \\
&|c(\lambda)|^{-2} = \frac{2^{4m}\pi}{\Gamma(-m+1)^2} k_m^2(x).
\end{aligned}$$

In this case, the Plancherel theorem is as follows: If $m \leq 1/2$, then

$$\int_0^\infty |g(y)|^2\, dy = \int_0^\infty |\mathcal{H}^m(g;x)|^2\, dx, \quad g \in L^2((0,\infty), dy),$$

and

$$\int_0^\infty |f(x)|^2\, dy = \int_0^\infty |\mathcal{G}^m(f;y)|^2\, dy, \quad f \in L^2((0,\infty), dx).$$

A main tool for the proof of the theorems is the atomic decomposition characterization of the real Hardy spaces. Let $0 < p \leq 1$ and

$$N = [1/p] - 1$$

where the notation $[x]$ means that the greatest integer not exceeding x. An H^p atom is a real valued function $a(x)$ on $\mathbb{R}$ so that (i) $a(x)$ is supported in an interval $[c, c+h]$, (ii) $|a(x)| \leq h^{-1/p}$ a.e. x, and (iii) $\int_{\mathbb{R}} a(x)x^k\, dx = 0$ for all $k = 0, 1, 2, \cdots, N$. The elements $f \in H^p[0,\infty)$ are characterized as follows: $f \in H^p(\mathbb{R})$ and $\mathrm{supp} f \subset [0,\infty)$ if and only if $f = \sum_{j=0}^\infty \lambda_j a_j$, where every a_j is an H^p atom with $\mathrm{supp}\, a_j \subset [0,\infty)$ and $\sum_{j=0}^\infty |\lambda_j|^p < \infty$. Moreover, the norm $\|f\|_{H^p[0,\infty)}$ is equivalent to $\inf(\sum_{j=0}^\infty |\lambda_j|^p)^{1/p}$, the infimum being taken over all such decompositions, and the series $\sum_{j=0}^\infty \lambda_j a_j$ converges in H^p norm, consequently, also in the sense of tempered distributions. For this characterization, we refer to [17].

The case $p = 1$ is in [7, III.7]. Related results are in [21, III.5.22], [3], [6], [9] and [16].

Because of the above characterization, we will be able to deduce the theorems from estimation of higher derivatives of the kernel $K^m(x, y)$. The estimation will be stated in the following section, and the proof of the theorems will be give in the section 4.

2. Main Estimates

For the proof of the theorems, we need to know about asymptotic behavior of the higher order derivatives $\partial^j K^m(x,y)/\partial x^j$ and $\partial^j K^m(x,y)/\partial y^j$, $j = 0, 1, 2, \dots$ in variables x and y. Schindler [20] has obtained precise asymptotic formulas of $K^m(x, y)$ and the first order derivatives $\partial K^m(x,y)/\partial x$ and $\partial K^m(x,y)/\partial y$. These formulas are enough to obtain our theorems in the case $p = 1$. We would like to consider Hardy-type inequalities for all p with $0 < p \leq 1$. This forces us to estimate the higher order derivatives. Our main estimates are the following Lemma 1 and Lemma 2 in which the letter C means positive constants independent of x and y not necessarily the same at each occurrence.

Lemma 1. *Let $-m + 1/2 > 0$, and put $M = [-m + 1/2]$. Then the following inequalities hold:*
For $0 < x < 1,\ 0 < y < 1$:

$$\left|\frac{\partial^j}{\partial x^j} K^m(x,y)\right| \leq C y^{-m+1/2}, \qquad j = 0, 1, 2, \dots. \tag{2}$$

For $0 < x < 1,\ 1 \leq y$:

$$\left|\frac{\partial^j}{\partial x^j} K^m(x,y)\right| \leq C y^j, \qquad j = 0, 1, 2, \dots. \tag{3}$$

For $1 \leq x,\ 1 \leq y$:

$$\left|\frac{\partial^j}{\partial x^j} K^m(x,y)\right| \leq C y^j, \qquad j = 0, 1, 2, \dots. \tag{4}$$

For $1 \leq x,\ 0 < y < 1$:

$$\left|\frac{\partial^j}{\partial x^j} K^m(x,y)\right| \leq C \cdot \begin{cases} y^j, & j = 0, 1, 2, \dots, M, \\ y^{-m+1/2}, & j = M + 1, \dots. \end{cases} \tag{5}$$

Lemma 2. *Let $-m + 1/2 > 0$, and put $M = [-m + 1/2]$, $\delta = -m + 1/2 - M$. Then the followig inequalities hold:*
For $0 < x < 1,\ 0 < y < 1$:

$$\left|\frac{\partial^j}{\partial y^j} K^m(x,y)\right| \leq C x, \qquad j = 0, 1, 2, \dots, M, \tag{6}$$

$$\left|\frac{\partial^M}{\partial y^M} K^m(x,y) - \frac{\partial^M}{\partial y^M} K^m(x,\xi)\right| \leq C x |y - \xi|^\delta, \quad 0 < \xi < 1. \tag{7}$$

For $0 < x < 1,\ 1 \leq y$:

$$\left|\frac{\partial^j}{\partial y^j} K^m(x,y)\right| \leq C x, \qquad j = 1, 2, 3, \dots. \tag{8}$$

For $1 \le x,\ 1 \le y$:

$$\left| \frac{\partial^j}{\partial y^j} K^m(x,y) \right| \le C x^j, \qquad j = 0, 1, 2, \ldots . \tag{9}$$

For $1 \le x, 0 < y < 1$:

$$K^m(x,y) = \tilde{k}_m(x)(xy)^{1/2} J_{-m}(xy) + E_m(x,y), \tag{10}$$

$$|\tilde{k}_m(x)| \le C, \qquad \left| \frac{\partial^j}{\partial y^j} E_m(x,y) \right| \le C x^j, \quad 0 \le j < -m + 3/2,$$

and if $-m + 1/2 = 1, 2, 3, \ldots,$ *then*

$$\left| \frac{\partial^j}{\partial y^j} K^m(x,y) \right| \le C x^j, \qquad j = 0, 1, 2, \ldots . \tag{11}$$

The above estimates are obtained by reexamining and refining the arguments that Schindler [20] used to get the asymptotic formulas for $K^m(x,y)$, $\partial K^m(x,y)/\partial x$ and $\partial K^m(x,y)/\partial y$. The work is routine, but a little hard. The details are omitted in this paper.

3. The generalized Mehler transform for H^p with $0 < p \le 1$

Let $0 < p \le 1$ and $-m + 1/2 > 0$. We shall discuss defining the transforms $\mathcal{G}^m(f;y)$ and $\mathcal{H}^m(f;x)$ of $f \in H^p[0,\infty)$. We use the fact that an element of the Lipschitz space $\Lambda_{1/p-1}(\mathbb{R})$ defines a continuous linear functional of $H^p(\mathbb{R})$ (cf. [7, III.5]).

Fix $y > 0$. We take a function κ_y^m in x such that

$$\kappa_y^m \in \Lambda_{1/p-1}(\mathbb{R}), \qquad \kappa_y^m(x) = K^m(x,y), \qquad x > 0,$$

and the transform $\mathcal{G}^m(f;y)$ of $f \in H^p[0,\infty)$ ($\subset H^p(\mathbb{R})$) is defined by

$$\mathcal{G}^m(f;y) = < \kappa_y^m, f >, \qquad y > 0,$$

where the existence of such a function κ_y^m will be discussed below. Then for an atom $a \in H^p[0,\infty)$, we have

$$\mathcal{G}^m(a;y) = < \kappa_y^m, a > = \int_0^\infty a(x) K^m(x,y)\, dx,$$

and for the atomic decomposition $f = \sum_{j=0}^\infty \lambda_j a_j(x)$ of $f \in H^p[0,\infty)$,

$$\mathcal{G}^m(f;y) = \sum_{j=0}^\infty \lambda_j < \kappa_y^m, a_j > = \sum_{j=0}^\infty \lambda_j \mathcal{G}^m(a_j;y).$$

We see that the transform $\mathcal{G}^m(f;y)$ is independent of the choice of an extension $\kappa_y^m \in \Lambda_{1/p-1}(\mathbb{R})$. In the same way, for fix $x > 0$, we take a function κ_x^m in y such that

$$\kappa_x^m \in \Lambda_{1/p-1}(\mathbb{R}), \qquad \kappa_x^m(y) = K^m(x,y), \qquad y > 0,$$

and the transform $\mathcal{H}^m(f;x)$ of $f \in H^p[0,\infty)$ is defined by

$$\mathcal{H}^m(f;x) = < \kappa_x^m, f >, \qquad x > 0,$$

where we shall show that it is possible to take a function κ_x^m. Then for an atom $a \in H^p[0,\infty)$, we have

$$\mathcal{H}^m(a;x) = < \kappa_x^m, a > = \int_0^\infty a(y) K^m(x,y)\, dy,$$

and for the atomic decomposition $f = \sum_{j=0}^{\infty} \lambda_j a_j(y)$ of $f \in H^p[0,\infty)$,

$$\mathcal{H}^m(f;x) = \sum_{j=0}^{\infty} \lambda_j < \kappa_x^m, a_j >= \sum_{j=0}^{\infty} \lambda_j \mathcal{H}^m(a_j;x).$$

The transform $\mathcal{H}^m(f;y)$ is independent of the choice of an extension $\kappa_x^m \in \Lambda_{1/p-1}(\mathbb{R})$

Let us discuss the existence of extensions κ_y^m and κ_x^m. Fix a positive y. We examine the kernel

$$\begin{aligned} K^m(x,y) =& k_m(x)(\sinh y)^{1/2} P^m_{-1/2+ix}(\cosh y) \\ =& k_m(x) \frac{1}{\Gamma(1-m)} \frac{(\cosh y + 1)^m}{(\sinh y)^{m-1/2}} \\ & \cdot F(1/2 - ix, 1/2 + ix; 1-m; (1-\cosh y)/2) \end{aligned}$$

as a function in x. We note here that for fixed z in the plane $\mathbb{C}$ cut along $[1,\infty]$, the hyper geometric function $F(\alpha,\beta;\gamma;z)$ is an entire function of α and β, and a meromorphic function of γ, with simple poles at the points $\gamma = 0, -1, -2, \dots$. Thus we see that the function $(\sinh y)^{1/2} P^m_{-1/2+ix}(\cosh y)$ is an entire function in x. The function $k_m(x)$ satisfies

$$\begin{aligned} k_m(x) &= \left| \frac{(1-ix)(-ix)\Gamma(1/2-m-ix)}{\Gamma(2-ix)} \right| \\ &= |(1-ix)(-ix)| \left| \frac{\Gamma(1/2-m-ix)}{\Gamma(2-ix)} \right| \\ &= x\sqrt{x^2+1} \left| \frac{\Gamma(1/2-m-ix)}{\Gamma(2-ix)} \right|, \qquad x > 0. \end{aligned}$$

Since $\Gamma(1/2-m-ix)/\Gamma(2-ix)$ is a holomorphic function with no zeros in $|x| < 3/2$, it follows that $|\Gamma(1/2-m-ix)/\Gamma(2-ix)| \in C^\infty(-3/2,3/2)$. By these considerations, we can take $\kappa_y^m \in C^\infty(\mathbb{R})$ such that

$$\kappa_y^m(x) = \begin{cases} K^m(x,y), & x > 0, \\ 0, & x < -\eta, \end{cases}$$

where η is a positive constant. By Lemma 1, we see that $\kappa_y^m \in \Lambda_\rho(\mathbb{R})$ for every $\rho > 0$.

Fix a positive x. By the properties of the hyper geometric functions, we see that there exists a function $h_x(y) \in C^\infty(\mathbb{R})$ such that

$$(\sinh y)^{1/2} P^m_{-1/2+ix}(\cosh y) = (\sinh y)^{-m+1/2} h_x(y), \qquad y > 0,$$

and then for a positive constant $\eta > 0$ there exists a function p_x^m such that

$$p_x^m(y) = \begin{cases} (\sinh y)^{-m+1/2} h_x(y), & y > -\eta, \\ 0, & y \le -2\eta, \end{cases}$$

and $p_x^m \in C^\infty(\mathbb{R} \setminus \{0\})$ if $-m+1/2 \neq 0,1,2,\dots$, and $p_x^m \in C^\infty(\mathbb{R})$ if $-m+1/2 = 0,1,2,\dots$. By Lemma 2, we see that

$$\left| \frac{\partial^j}{\partial y^j} (\sinh y)^{1/2} P^m_{-1/2+ix}(\cosh y) \right| \le C_{j,m}(x), \qquad y > 0, \quad j = 0,1,2,\dots.$$

Thus we have that for $-m+1/2=1,2,3,\dots$,

$$\left|\frac{\partial^j}{dy^j}p_x^m(y)\right| \le C'_{j,m}(x), \qquad -\infty<y<\infty, \quad j=0,1,2,\dots,$$

and that $\kappa_x^m \in \Lambda_\rho(\mathbb{R})$ for every $\rho>0$, where

$$\kappa_x^m(y)=k_m(x)p_x^m(y), \qquad -\infty<y<\infty.$$

Here, $C_{j,m}(x), C'_{j,m}(x)$ are constants independent of y and depending on m, j and x. In the case $-m+1/2\neq 1,2,3,\dots$, we see that

$$\text{(12)} \qquad \left|\frac{\partial^j}{dy^j}p_x^m(y)\right| \le \begin{cases} C'_{j,m}(x), & -\eta<y<\eta, \quad j=0,1,2,\dots,M, \\ C''_{j,m}(x), & \eta\le|y|, \quad j=0,1,2,\dots, \end{cases}$$

where $M=[-m+1/2]$. Put $\delta=-m+1/2-M>0$. Then it is easy to see that

$$\text{(13)} \qquad \left|\frac{\partial^M}{dy^M}p_x^m(y)-\frac{\partial^M}{dy^M}p_x^m(y')\right| \le C|y-y'|^\delta, \qquad y,y'\in(-\eta,\eta).$$

The inequalities (12) and (13) lead to $\kappa_x^m \in \Lambda_\rho(\mathbb{R})$ for every ρ with $0<\rho\le -m+1/2$.

Summarizing the above discussion, we have the following.

Lemma 3. (i) *Let $0<p\le 1$ and $-m+1/2>0$. Then, the G-transform $\mathcal{G}^m$ is well-defined on $H^p[0,\infty)$.*

(ii − 1) *Let $0<p\le 1$ and suppose $1/p-1\le -m+1/2$. Then, the H-transform $\mathcal{H}^m$ is well-defined on $H^p[0,\infty)$.*

(ii − 2) *If $-m+1/2=0,1,2,\dots$, then the H-transform $\mathcal{H}^m$ is well-defined on $H^p[0,\infty)$ for every p with $0<p\le 1$.*

4. Proofs of Theorems

We shall turn to proofs of the theorems. Let $f\in H^p[0,\infty), 0<p\le 1$. Then we have $f=\sum_{j=0}^\infty \lambda_j a_j$, where every a_j is an H^p atom with $\operatorname{supp} a_j \subset [0,\infty)$ and $\sum_{j=0}^\infty |\lambda_j|^p<\infty$. Moreover, the norm $\|f\|_{H^p[0,\infty)}$ is equivalent to $\inf(\sum_{j=0}^\infty|\lambda_j|^p)^{1/p}$, the infimum being taken over all such decompositions. Because of the decomposition, to prove the theorems it is enough to show that for H^p-atoms a with $\operatorname{supp} a\subset[0,\infty)$,

$$\text{(14)} \qquad \int_A^B |\mathcal{G}^m(a;y)|^p y^{p-2}\,dy \le C_1, \qquad \int_A^B |\mathcal{H}^m(a;x)|^p x^{p-2}\,dx \le C_2$$

with constants C_1 and C_2 independent of atoms a under the conditions we need for p and m, where $(A,B)=(0,1)$ or $(A,B)=(1,\infty)$. For the continuity of the transforms leads to

$$\text{(15)} \qquad \mathcal{G}^m(f;y)=\sum_{j=0}^\infty \lambda_j\mathcal{G}^m(a_j;y), \quad \mathcal{H}^m(f;x)=\sum_{j=0}^\infty \lambda_j\mathcal{H}^m(a_j;x),$$

and if (14) holds, then we have that

$$\int_A^B |\mathcal{G}^m(f;y)|^p y^{p-2}\,dy \le \sum_{j=0}^{\infty} |\lambda_j|^p \int_A^B |\mathcal{G}^m(a_j;y)|^p y^{p-2}\,dy$$
$$\le C_1 \sum_{j=0}^{\infty} |\lambda_j|^p \le C_1' \|f\|_{H^p}^p,$$

and $\int_A^B |\mathcal{H}^m(f;y)|^p y^{p-2}\,dy \le C_2' \|f\|_{H^p}^p$, where C_1' and C_2' are constants independent of $f \in H^p[0,\infty)$.

Proof of Theorem 1 (i). Let $0 < p \le 1$ and $-m + 1/2 > 0$. Let a be an H^p-atom with the support interval $[c'-h, c'] \subset [0,\infty)$. We put $N = [1/p] - 1$. The vanishing mean property of atoms leads to

$$(16)\qquad |\mathcal{G}^m(a;y)| \le \int_{c'-h}^{c'} |a(x)| \left| \frac{\partial^{N+1}}{\partial x^{N+1}} K^m(c_1, y) \right| |x - c'|^{N+1}\,dx,$$

where $c' - h < x < c_1 < c'$. We are supposing $y \ge 1$ and so by Lemma 2, (8) and (9) we have

$$|\mathcal{G}^m(a;y)| \le C \int_{c'-h}^{c'} |a(x)| y^{N+1} |x - c'|^{N+1}\,dx$$
$$(17)\qquad \le C' y^{\lambda} \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)}, \qquad \lambda = N+1,$$

where C and C' are constants independent of a and y. The last inequality follows from the following small lemma which will be given for later convenience, and three more simple lemmas will be also stated here.

Lemma 4. *Let a be an H^p-atom with the support interval $[c, c+h] \subset [0,\infty)$. Let $\lambda > 0$. Then the following inequality holds:*

$$\int_0^{\infty} |a(x)| (y|x - c'|)^{\lambda}\,dx \le y^{\lambda} \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)},$$

where c' is an arbitrary point with $c \le c' \le c + h$.

Proof. It follows from $\|a\|_2 \le h^{-1/p+1/2}$, that is, $h \le \|a\|_2^{-2p/(2-p)}$ that

$$\int_0^{\infty} |a(x)| (y|x - c'|)^{\lambda}\,dx \le y^{\lambda} \|a\|_2 \left(\int_c^{c+h} |x - c'|^{2\lambda}\,dx \right)^{1/2}$$
$$\le y^{\lambda} \|a\|_2 h^{\lambda+1/2} \le y^{\lambda} \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)}.$$

□

Lemma 5. *Let $0 < p \le 1$. Then for an arbitrary λ with $1/p - 1 < \lambda$ and any $a \in L^2[0,\infty)$,*

$$\int_0^R \left(y^{\lambda} \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)} \right)^p y^{p-2}\,dy = \frac{1}{p(\lambda+1) - 1},$$

where R satisfies

$$(18)\qquad \|a\|_2^p R^{-(2-p)/2} = 1.$$

Proof. It follows that

$$\int_0^R \left(y^\lambda \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)}\right)^p y^{p-2}\,dy = \|a\|_2^{p(1+\frac{-2p}{2-p}(\lambda+1/2))} \int_0^R y^{p(\lambda+1)-2}\,dy$$

$$= \frac{1}{p(\lambda+1)-1}\|a\|_2^{p(1+\frac{-2p}{2-p}(\lambda+1/2))} R^{p(\lambda+1)-1}$$

$$= \frac{1}{p(\lambda+1)-1}\left\{\|a\|_2^p R^{\{p(\lambda+1)-1\}/(1+\frac{-2p}{2-p}(\lambda+1/2))}\right\}^{1+\frac{-2p}{2-p}(\lambda+1/2)}$$

$$= \frac{1}{p(\lambda+1)-1}.$$

Here, we used the fact that the power to the last R is equal to $-(2-p)/2$. □

Lemma 6. *Let* $0 < p \le 1$ *and* $-m + 1/2 > 0$. *Then for any* $a \in L^2[0,\infty)$ *and a constant* R *satisfying* (18),

$$\int_R^\infty |\mathcal{G}^m(a;y)|^p y^{p-2}\,dy \le 1, \qquad \int_R^\infty |\mathcal{H}^m(x)|^p x^{p-2}\,dy \le 1.$$

Proof. By Plancherel's theorem, we have that

$$\int_R^\infty |\mathcal{G}^m(a;y)|^p y^{p-2}\,dy \le \left(\int_R^\infty |\mathcal{G}^m(a;y)|^2\,dy\right)^{p/2}\left(\int_R^\infty y^{-2}\,dy\right)^{(2-p)/2}$$

$$\le \|a\|_2^p R^{-(2-p)/2} = 1.$$

In the same way, we have the H-transform case. □

Lemma 7. *Let* $I(x), J(x)$ *be nonnegative functions on* $(0,\infty)$.

(i) *If* $I(x) \le J(x)$ *for* $0 < x < 1$, *then the inequality*

$$\int_0^1 I(x)\,dx \le \int_0^R J(x)\,dx + \int_R^\infty I(x)\,dx$$

holds for every $R > 0$.

(ii) *If* $I(x) \le J(x)$ *for* $1 \le x$, *then the inequality*

$$\int_1^\infty I(x)\,dx \le \int_0^R J(x)\,dx + \int_R^\infty I(x)\,dx$$

holds for every $R > 0$.

We go back to the proof. By (17) and Lemma 7, we have that for every $R > 0$,

$$\int_1^\infty |\mathcal{G}^m(a;y)|^p y^{p-2}\,dy \le \int_0^R \left(C' y^\lambda \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)}\right)^p y^{p-2}\,dy + \int_R^\infty |\mathcal{G}^m(a;y)|^p y^{p-2}\,dy.$$

Taking R with (18), we have by Lemma 5 and Lemma 6 that

$$\int_1^\infty |\mathcal{G}^m(a;y)|^p y^{p-2}\,dy \le C,$$

where C is a constant independent of a. Here, we need the condition

$$1/p - 1 < \lambda = N + 1 = [1/p],$$

and it is trivially satisfied. This completes the proof of Theorem 1 (i).

Proof of Theorem 1 (ii). Let $0 < p \le 1$ and $-m + 1/2 > 0$. In the same way as the above, we have (16). Now we are dealing with the case $0 < y < 1$, and our assumption is that $N + 1 \le M = [-m + 1/2]$. Thus by the estimates (2) and (5), we have (17) for $0 < y < 1$. It follows from Lemma 7 that

$$\int_0^1 |\mathcal{G}^m(a;y)|^p y^{p-2}\,dy \le \int_0^R \left(C' y^\lambda \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)}\right)^p y^{p-2}\,dy + \int_R^\infty |\mathcal{G}^m(a;y)|^p y^{p-2}\,dy,$$

and taking R with (18), by Lemma 5 and 6 we have

$$\int_0^1 |\mathcal{G}^m(a;y)|^p y^{p-2}\,dy \le C,$$

where C is a constant independent of a. The condition $1/p - 1 < N + 1 = [1/p]$ is automatically satisfied.

Proof of Theorem 2 (i). Let $0 < p \le 1$ and $-m + 1/2 > 0$, and put $N = [1/p] - 1$, $M = [-m + 1/2]$. We divide a matter into two cases $N + 1 \le M$ and $M < N + 1$.

Let us deal with the case $N + 1 \le M$. Let a be an H^p-atom with the support interval $[c - h, c] (\subset [0, \infty))$. We first suppose that $c - h < 1 < c$. By the vanishing mean property of atoms, we have that

$$\begin{aligned} |\mathcal{H}^m(a;x)| &\le \int_{c-h}^c |a(y)| \left|\frac{\partial^{N+1}}{\partial y^{N+1}} K^m(x, c_2)\right| |y-1|^{N+1}\,dy \\ &= \left\{\int_{c-h}^1 + \int_1^c\right\} |a(y)| \left|\frac{\partial^{N+1}}{\partial y^{N+1}} K^m(x, c_2)\right| |y-1|^{N+1}\,dy \\ &= J_1(x) + J_2(x), \quad \text{say,} \end{aligned}$$

where $c - h < y < c_2 < 1$ or $1 < c_2 < y < c$. We are now treating the case $1 \le x$. It follows from Lemma 2 (9) and Lemma 4 that

$$J_2(x) \le C \int_1^c |a(y)| (x|y-1|)^{N+1}\,dy \le C x^\lambda \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)}, \quad \lambda = N + 1, \tag{19}$$

where C is independent of x and a. For $J_1(x)$, since $N + 1 \le M$, Lemma 2 (10) with $j = N + 1$ leads to

$$\begin{aligned} J_1(x) \le{}& C_1 \int_{c-h}^1 |a(y)| \left| \frac{\partial^{N+1}}{\partial y^{N+1}} \{(xy)^{1/2} J_{-m}(xy)\}\Big|_{y=c_2} \right| |y-1|^{N+1}\,dy \\ &+ C_2 \int_{c-h}^1 |a(y)| (x|y-1|)^{N+1}\,dy = C_1 J_{10}(x) + C_2 J_{11}(x), \quad \text{say,} \end{aligned}$$

and $J_{11}(x) \le x^\lambda \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)}$, $\lambda = N + 1$, where C_1 and C_2 are independent of x and a. For the term $J_{10}(x)$, by using the estimate

$$\sup_{t>0} \left| \frac{\partial^j}{\partial t^j} t^{1/2} J_\alpha(t) \right| < \infty, \quad j = 0, 1, 2, \ldots, [\alpha + 1/2], \quad \alpha \ge -1/2$$

([11, Lemma 1, (8)]), we have that

$$J_{10}(x) \le C \int_{c-h}^1 |a(y)| (x|y-1|)^{N+1}\,dy \le C x^\lambda \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)}, \quad \lambda = N + 1,$$

where C is independent of x and a. Therefore we have

(20) $$|\mathcal{H}^m(a;x)| \le Cx^\lambda \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)}, \quad \lambda = N+1, \quad 1 \le x$$

with a constant C independent of x and a for an H^p-atom a with the support interval $[c-h,c]$ satisfying $c-h<1<c$. For the case $1 \le c-h$, we also have the above estimate (20) in the same way as the argument for $J_2(x)$, and for the case $c \le 1$, we have (20) in the same way as the argument for $J_1(x)$. Lemma 7 leads to

$$\int_1^\infty |\mathcal{H}^m(a;x)|^p x^{p-2}\,dx \le \int_0^R \left(Cx^\lambda \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)}\right)^p x^{p-2}\,dx + \int_R^\infty |\mathcal{H}^m(a;x)|^p x^{p-2}\,dx, \qquad \lambda = N+1$$

for any $R>0$ and every H^p-atom a with the support interval contained in $[0,\infty)$. Noting $1/p-1<\lambda$ and taking R with (18), we have by Lemma 5 and Lemma 6 that

(21) $$\int_1^\infty |\mathcal{H}^m(a;x)|^p x^{p-2}\,dx \le C, \quad N+1 \le M$$

with a constant C independent of a.

Next we treat the case $M<N+1$. We first examine the case $-m+1/2 = 1,2,3,\dots$. Because of (9) and (11), we have by the vanishing mean properties and Lemma 4 that

$$\begin{aligned}|\mathcal{H}^m(a;x)| &\le \int_{c-h}^c |a(y)| \left|\frac{\partial^{N+1}}{\partial y^{N+1}} K^m(x,c_2)\right| |y-c|^{N+1}\,dy \\ &\le \int_{c-h}^c |a(y)|(x|y-c|)^{N+1}\,dy \le x^\lambda \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)}, \quad \lambda = N+1,\end{aligned}$$

where $c-h<y<c_2<c$ and a is an H^p-atom with the support interval $[c-h,c](\subset [0,\infty))$. In the same way as the above argument, we have

(22) $$\int_1^\infty |\mathcal{H}^m(a;x)|^p x^{p-2}\,dx \le C, \quad M<N+1, \quad -m+1/2 = 1,2,3,\dots,$$

where C is independent of a.

Let us consider the case $-m+1/2 \ne 1,2,3,\dots$. In this case we suppose that $1/p-1 < -m+1/2$. Since $M<N+1$, it follows that $-m+1/2<N+1$. By the assumption $1/p-1<-m+1/2$, we have $N<-m+1/2$. Thus, in this case, $N<-m+1/2<N+1$ and $M=N$ hold. Let a be an H^p-atom with the support interval $[c-h,c](\subset [0,\infty))$. We first deal with the case $c-h<1<c$. We have that

$$\mathcal{H}^m(a;x) = \int_{c-h}^c a(y)\left(\frac{\partial^M K^m}{\partial y^M}(x,\xi) - \frac{\partial^M K^m}{\partial y^M}(x,1)\right)(y-1)^M\,dy,$$

and that

$$\begin{aligned}|\mathcal{H}^m(a;x)| &\le \left\{\int_{c-h}^1 + \int_1^c\right\} |a(y)| \left|\frac{\partial^M K^m}{\partial y^M}(x,\xi) - \frac{\partial^M K^m}{\partial y^M}(x,1)\right| |y-1|^M\,dy \\ &= J_3(x)+J_4(x), \quad \text{say,}\end{aligned}$$

where $c-h<y<\xi<1$ or $1<\xi<y<c$. Since $M=N$, it follows that

$$J_4(x) = \int_1^c |a(y)| \left|\frac{\partial^{N+1}K^m}{\partial y^{N+1}}(x,\xi')\right| |y-1|^{N+1}\,dy, \quad 1<\xi'<y<c.$$

We are now dealing with the case $1 \le x$. By Lemma 2 (9), we have that

$$J_4(x) \le C \int_1^c |a(y)|(x|y-1|)^{N+1}\, dy$$

with a constant C independent of x and a, and by Lemma 4 that

$$J_4(x) \le Cx^\lambda \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)}, \quad \lambda = N+1.$$

For $J_3(x)$, it follows from Lemma 2 (10) that

$$\begin{aligned} J_3(x) \le C_1 \int_{c-h}^1 |a(y)| \Bigg| \frac{\partial^M}{\partial y^M}\{(xy)^{1/2}J_{-m}(xy)\}\Big|_{y=\xi} \\ - \frac{\partial^M}{\partial y^M}\{(xy)^{1/2}J_{-m}(xy)\}\Big|_{y=1}\Bigg| |y-1|^M\, dy \\ + C_2 \int_{c-h}^1 |a(y)| \left|\frac{\partial^{M+1}E_m}{\partial y^{M+1}}(x,\xi')\right| |y-1|^{M+1}\, dy \\ = C_1 J_{30}(x) + C_2 J_{31}(x), \text{ say,} \end{aligned}$$

where C_1 and C_2 are independent of x and a. Since $M = N$, it follows from Lemma 4 that

$$J_{31}(x) \le C \int_{c-h}^1 |a(y)|(x|y-1|)^{N+1}\, dy \le Cx^\lambda \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)}, \quad \lambda = N+1$$

with a constant C independent x and a. By using the estimate [11, Lemma 1, (9)], we have

$$\left| \frac{\partial^M}{\partial y^M}\{(xy)^{1/2}J_{-m}(xy)\}\Big|_{y=\xi} - \frac{\partial^M}{\partial y^M}\{(xy)^{1/2}J_{-m}(xy)\}\Big|_{y=1}\right| \le Cx^M |x\xi - x|^{-m+1/2-M},$$

where $c-h < y < \xi < 1$ and C is independent of x. Thus it follows Lemma 4 that

$$J_{30}(x) \le C \int_{c-h}^1 |a(y)|(x|y-1|)^{-m+1/2}\, dy \le Cx^{\lambda'} \|a\|_2^{1+\frac{-2p}{2-p}(\lambda'+1/2)}, \quad \lambda' = -m+1/2$$

with a constant C independent of x and a. Thus for an H^p-atom a with the support interval $[c-h, c]$ satisfying $c-h < 1 < c$ we have

$$|\mathcal{H}^m(a;x)| \le C_1 x^{\lambda'} \|a\|_2^{1+\frac{-2p}{2-p}(\lambda'+1/2)} + C_2 x^\lambda \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)}, \quad \lambda = N+1,\ \lambda' = -m+1/2, \qquad 1 \le x \tag{23}$$

with constants C_1 and C_2 independent of x and a. For the case $1 \le c-h$, we make the same argument for $J_4(x)$, and have

$$|\mathcal{H}^m(a;x)| \le Cx^\lambda \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)}, \quad \lambda = N+1, \qquad 1 \le x.$$

For the case $c \le 1$, the same argument for $J_3(x)$ leads to

$$|\mathcal{H}^m(a;x)| \le Cx^{\lambda'} \|a\|_2^{1+\frac{-2p}{2-p}(\lambda'+1/2)}, \quad \lambda = -m+1/2, \qquad 1 \le x.$$

Therefore for any atoms we have (23). It follows that for every $R > 0$,

$$\int_1^\infty |\mathcal{H}^m(a;x)|^p x^{p-2}\,dx \le \int_0^R \left(C_1 x^{\lambda'} \|a\|_2^{1+\frac{-2p}{2-p}(\lambda'+1/2)}\right)^p x^{p-2}\,dx + \int_0^R \left(C_2 x^{\lambda} \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)}\right)^p x^{p-2}\,dx + \int_R^\infty |\mathcal{H}^m(a;x)|^p x^{p-2}\,dx, \quad \lambda = N+1,\ \lambda' = -m+1/2.$$

Taking R with (18) and noting $1/p-1 < \lambda(= N+1 = [1/p])$ and $1/p-1 < -m+1/2$, we have by Lemma 5 and Lemma 6 that

$$\int_1^\infty |\mathcal{H}^m(a;x)|^p x^{p-2}\,dx \le C, \qquad M < N+1, \quad 1/p-1 < -m+1/2 \ne 1,2,3,\dots \tag{24}$$

with a constant C independent of a. The inequalities (21), (22) and (24) complete the proof of Theorem 2 (i).

Proof of Theorem 2 (ii). Assume that $-m+1/2 > 0$ and $1/2 < p \le 1$. It is clear that $N = [1/p]-1 = 0$. Let a be an H^p-atom with the support interval $[c-h, c](\subset [0,\infty))$.

We treat the case $c-h < 1 < c$, first. Noting that

$$\mathcal{H}^m(a;x) = \int_{c-h}^c a(y)(K^m(x,y) - K^m(x,1))\,dy,$$

we have

$$\begin{aligned}|\mathcal{H}^m(a;x)| &\le \int_{c-h}^c |a(y)||K^m(x,y)-K^m(x,1)|\,dy\\ &= \left\{\int_{c-h}^1 + \int_1^c\right\} |a(y)||K^m(x,y)-K^m(x,1)|\,dy\\ &= J_5(x) + J_6(x), \quad \text{say.}\end{aligned}$$

We are now supposing that $0 < x < 1$. For $J_6(x)$, it follows from Lemma 2 (8) and Lemma 4 that

$$\begin{aligned}J_6(x) &= \int_1^c |a(y)||K^m(x,y)-K^m(x,1)|\,dy = \int_1^c |a(y)|\left|\frac{\partial K^m}{\partial y}(x,\xi)\right||y-1|\,dy\\ &\le C\int_1^c |a(y)|(x|y-1|)\,dy \le Cx^\lambda \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)}, \quad \lambda = 1,\end{aligned}$$

where $1 < \xi < y < c$ and C is independent of x and a. For $J_5(x)$, we divide a matter into two cases $M = [-m+1/2] = 0$ and $M \ge 1$. Let $M \ge 1$. Because of Lemma 2 (6), the same argument for $J_6(x)$ leads to

$$\begin{aligned}J_5(x) &= \int_{c-h}^1 |a(y)||K^m(x,y)-K^m(x,1)|\,dy\\ &= \int_{c-h}^1 |a(y)|\left|\frac{\partial K^m}{\partial y}(x,\xi)\right||y-1|\,dy\\ &\le C\int_{c-h}^1 |a(y)|(x|y-1|)\,dy \le Cx^\lambda \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)}, \quad \lambda = 1.\end{aligned}$$

We next deal with the case $M = 0$. We remark $0 < -m + 1/2 < 1$. It follows from Lemma 2 (7) and Lemma 4 that

$$J_5(x) = \int_{c-h}^{1} |a(y)||K^m(x,y) - K^m(x,1)|\,dy = \int_{c-h}^{1} |a(y)|\; x|y-1|^{\delta}\,dy$$
$$\leq C \int_{c-h}^{1} |a(y)|(x|y-1|)^{\delta}\,dy \leq C x^{\lambda} \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)}, \quad \lambda = -m + 1/2.$$

We used that $x < x^{\delta}$ $(0 < x < 1)$ since $1 > \delta = -m + 1/2 - M = -m + 1/2 > 0$. Thus for an H^p-atom a with the support interval $[c-h, c]$ satisfying $c - h < 1 < c$ we have

$$|\mathcal{H}^m(a;x)| \leq C_1 x^{\lambda'} \|a\|_2^{1+\frac{-2p}{2-p}(\lambda'+1/2)} + C_2 x^{\lambda} \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)}, \qquad (25)$$
$$\lambda = 1,\ \lambda' = -m + 1/2, \qquad 0 < x < 1$$

with constants C_1 and C_2 independent of x and a.

For the case $1 \leq c - h$, by the same argument for $J_6(x)$ we have

$$|\mathcal{H}^m(a;x)| \leq C x^{\lambda} \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)}, \quad \lambda = 1, \qquad 0 < x < 1$$

with a constant C independent of x and a. For the case $c \leq 1$, in a similar way of the argument for $J_5(x)$ we have (25). Therefore we have (25) for any atom. It follows from Lemma 7 that for every $R > 0$,

$$\int_0^1 |\mathcal{H}^m(a;x)|^p x^{p-2}\,dx$$
$$\leq \int_0^R \left(C_1 x^{\lambda'} \|a\|_2^{1+\frac{-2p}{2-p}(\lambda'+1/2)} \right)^p x^{p-2}\,dx + \int_0^R \left(C_2 x^{\lambda} \|a\|_2^{1+\frac{-2p}{2-p}(\lambda+1/2)} \right)^p x^{p-2}\,dx$$
$$+ \int_R^{\infty} |\mathcal{H}^m(a;x)|^p x^{p-2}\,dx, \qquad \lambda = 1,\ \lambda' = -m + 1/2.$$

We take R as it satisfies (18). Noting that $1/p - 1 < -m + 1/2$ and $1/p - 1 < 1$, we have by Lemma 5 and Lemma 6 that

$$\int_0^1 |\mathcal{H}^m(a;x)|^p x^{p-2}\,dx \leq C$$

with a constant C independent a, which completes the proof of Theorem 2 (ii), and the proofs of the theorems complete.

References

[1] R. Balasubramanian and R. Radha, Hardy-type inequalities for Hermite expansions, J. Inequal. Pure Appl. Math. **6** (2005), No. 1, Article 12, 4pp. (electronic).

[2] J. J. Betancor and L. Rodríguez-Mesa, On Hankel transformation, convolution operators and multipliers on Hardy type spaces, J. Math. Soc. Japan **53** (2001), 687–709.

[3] D.-C. Chang, S. G. Krantz and E. M. Stein, Hardy spaces and elliptic boundary value problems, Proceedings of the Madison Symposium on Complex Analysis, Contemporary Math. No. 137, 119-131, American Mathematical Society 1992.

[4] R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, Bull. Amer. Math. Soc. **83** (1977), 569-645.

[5] L. Colzani and G. Travaglini, Hardy-Lorentz spaces and expansions in eigenfunctions of the Laplace-Beltrami operator on compact manifolds, Colloq. Math. **58** (1990), 305–315.

[6] K. Forsman, Atomic decompositions in Hardy spaces on bounded Lipschitz domains, Function spaces and applications (Lund, 1986), 206-222, Lecture Notes in Math., 1302, Springer, Berlin-New York, 1988.

[7] J. Garcia-Cuerva and J. L. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North-Holland, Amsterdam, 1985.

[8] J. J. Guadalupe and V. I. Kolyada, A transplantation theorem for ultraspherical polynomials at critical index, Studia Math. **147** (2001), 51–72.

[9] A. Jonsson, P. Sjögren and H. Wallin, Hardy and Lipschitz spaces on subsets of $\mathbb{R}^n$, Studia Math. **80** (1984), 141-166.

[10] Y. Kanjin, Hardy's inequalities for Hermite and Laguerre expansions, Bull. London Math. Soc. **29** (1997), 331–337.

[11] Y. Kanjin, On Hardy-type inequalities and Hankel transforms, Monatsh. Math. **127**(1999), 311-319.

[12] Y. Kanjin, Hardy's inequalities for Hermite and Laguerre expansions revisited, J. Math. Soc. Japan **63** (2011), 753-767.

[13] Y. Kanjin and K. Sato, Hardy's inequality for Jacobi expansions, Math. Inequal. Appl. **7** (2004), 551–555.

[14] T. H. Koornwinder, Jacobi functions and analysis on non-compact semisimple Lie groups, Special Functions: Group Theoretical Aspects and Applications, R. A. askey et al., Eds., 1–85, 1984.

[15] B. N. Mandal and Nanigopal Mandal, Integral expansions related to Mehler-Fock type transforms, Addioson Wesley Longman limited, 1997.

[16] A. Miyachi, H^p spaces over open subsets of $\mathbb{R}^n$, Studia Math. **95** (1990), 205-228.

[17] A. Miyachi, Private communication (1997).

[18] R. Radha and S. Thangavelu, Hardy's inequalities for Hermite and Laguerre expansions, Proc. Amer. Math. Soc. **132** (2004), 3525–3536.

[19] K. Sato, Paley's inequality and Hardy's inequality for the Fourier-Bessel expansions, J. Nonlinear convex Anal. **6** (2005), 441–451.

[20] S. Schindler, Some transplantation theorems for the generalized Mehler transform and related asymptotic expansions, Trans. Amer. Math. Soc. **155** (1971), 257–291.

[21] E. M. Stein, Harmonic analysis, real-variable methods, orthogonality, and oscillatory integrals, Princeton University Press, Princeton, New Jersey, 1993.

[22] S. Thangavelu, On regularity of twisted spherical means and special Hermite expansions, Proc. Indian Acad. Sci. Math. Sci. **103** (1993), 303–320.

Mathematics Section, Division of Innovative Technology and Science, Graduate School of Natural Science and Technology, Kanazawa University Kanazawa 920-1192, Japan
E-mail address: `kanjin@staff.kanazawa-u.ac.jp`

Department of Basic Technology, Faculty of Engineering, Yamagata University, Yonezawa 992-8510, Japan
E-mail address: `sato@yz.yamagata-u.ac.jp`