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A note on almost contact Riemannian 3-manifolds

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Abstract

We investigate curvatures of normal almost contact Riemannian 3-manifolds. In particular, we show that Kenmotsu 3-manifolds of constant scalar curvature are of constant curvature -1.

Introduction

In [6], K. Kenmotsu introduced a class of almost contact Riemannian manifolds. The almost contact Riemannian manifolds introduced by Kenmotsu are called *Kenmotsu manifolds*. Kenmotsu showed that locally symmetric Kenmotsu manifolds are of constant curvature -1. This fact means that local symmetry is a strong restriction for Kenmotsu manifolds.

In stead of local symmetry, U. C. De [4] studied Kenmotsu manifolds $M = (M; \varphi, \xi, \eta, g)$ satisfying

$$\varphi^2\{(\nabla_W R)(X,Y)Z\} = 0 \tag{1}$$

for all X, Y, Z, $W \in \mathfrak{X}(M)$ orthogonal to ξ. He showed that if M satisfies (1) for all vector fields on M, then M is Einstein. In dimension 3, De showed that a Kenmotsu 3-manifold M satisfies (1) for all vector fields orthogonal to ξ if and only if M is of constant scalar curvature.

In this paper we point out that Kenmotsu 3-manifolds of constant scalar curvature are of constant curvature -1. Thus De's condition on Kenmotsu 3-manifolds implies local symmetry.

1 Preliminaries

Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Denote by R the Riemannian curvature of M:

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad X, Y \in \mathfrak{X}(M).$$

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Here $\mathfrak{X}(M)$ is the Lie algebra of all vector fields on M. A tensor field F of type $(1,3)$;

$$F : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

is said to be *curvature-like* provided that F has the symmetric properties of R. For example,

$$(X \wedge Y)Z = g(Y,Z)X - g(Z,X)Y, \quad X, Y \in \mathfrak{X}(M) \tag{2}$$

defines a curvature-like tensor field on M. Note that the curvature R of a Riemannian manifold (M, g) of constant curvature c satisfies the formula $R(X,Y) = c(X \wedge Y)$.

A Riemannian manifold (M, g) is said to be *locally symmetric* if $\nabla R = 0$. Clearly every Riemannian manifolds of constant curvature is locally symmetric.

In dimension 3, the Riemannian curvature R is determined by the Ricci tensor. In fact, R is expressed as

$$\begin{aligned} R(X,Y)Z &= \rho(Y,Z)X - \rho(Z,X)Y \\ &\quad + g(Y,Z)SX - g(Z,X)SY - \frac{\mathrm{s}}{2}(X \wedge Y)Z, \end{aligned} \tag{3}$$

where ρ is the Ricci tensor, S is the corresponding Ricci operator and s is the scalar curvature of M, respectively.

2 Almost contact Riemannian manifolds

Let M be an odd-dimensional manifold. An *almost contact structure* on M is a quadruple of tensor fields (φ, ξ, η, g), where φ is an endomorphism field, ξ is a vector field, η is a one form and g is a Riemannian metric, respectively, such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{4}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \quad X, Y \in \mathfrak{X}(M). \tag{5}$$

An $(2n+1)$-dimensional manifold together with an almost contact structure is called an *almost contact Riemannian manifold* (or *almost contact manifold*) . The *fundamental* 2*-form* Φ of M is defined by

$$\Phi(X,Y) = g(X, \varphi Y), \quad X, Y \in \mathfrak{X}(M).$$

If an almost contact Riemannian manifold $(M; \varphi, \xi, \eta, g)$ satisfies the condition:

$$\rho = ag + b\eta \otimes \eta \tag{6}$$

for some functions a and b, then M is said to be η*-Einstein*.

The formulae (3) and (6) imply the following result.

Proposition 2.1 *Let M be an η-Einstein almost contact Riemannian 3-manifold. Then its Riemannian curvature R is given by*

$$R(X,Y)Z = \left(2a - \frac{\mathrm{s}}{2}\right)(X \wedge Y)Z - [(b\xi) \wedge \{(X \wedge Y)\xi\}]Z. \tag{7}$$

An almost contact Riemannian manifold M is said to be *normal* if it satisfies $[\varphi, \varphi] + 2\mathrm{d}\eta \otimes \xi = 0$, where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ.

Proposition 2.2 ([7]) *An almost contact Riemannian 3-manifold is normal if and only if there exist functions α and β such that*

$$(\nabla_X \varphi)Y = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\}. \tag{8}$$

We call the pair (α, β) of functions the *type* of a normal almost contact Riemannian 3-manifold M. More generally, an almost contact manifold of dimension $2n+1 \geq 3$ is said to be *trans-Sasakian* if there exist functions α and β such that (8) (see [9]).

In particular, a normal almost contact Riemannian 3-manifold is said to be a

- *Sasakian manifold* if $(\alpha, \beta) = (1, 0)$,
- *Kenmotsu manifold* if $(\alpha, \beta) = (0, 1)$,
- *coKähler manifold* if $(\alpha, \beta) = (0, 0)$.

Let $(M; \varphi, \xi, \eta, g)$ be a normal almost contact Riemannian 3-manifold. Then from (4) and (8), we have

$$\nabla_X \xi = -\alpha \varphi X + \beta\{X - \eta(X)\xi\}, \quad X, Y \in \mathfrak{X}(M). \tag{9}$$

In particular we have $\nabla_\xi \xi = 0$. Hence on trans-Sasakian manifolds, integral curves (trajectories) of ξ are geodesics.

Next, we consider η-Einstein normal almost contact Riemannian 3-manifolds.

Proposition 2.3 ([3]) *Let M be a normal almost contact Riemannian 3-manifold of type (α, β). Then M is η-Einstein if and only if*

$$g(\mathrm{grad}\,\beta - \varphi \mathrm{grad}\,\alpha, X) = 0$$

for all $X \in \mathfrak{X}(M)$ orthogonal to ξ. In this case,

$$\rho = \left\{\frac{\mathrm{s}}{2} + \mathrm{d}\beta(\xi) - (\alpha^2 - \beta^2)\right\} g + \left\{-\frac{\mathrm{s}}{2} - 3\mathrm{d}\beta(\xi) + 3(\alpha^2 - \beta^2)\right\} \eta \otimes \eta.$$

Corollary 2.1 *The Riemannian curvature of a Sasakian 3-manifold is given by*

$$R(X,Y)Z = \frac{\mathrm{s}-4}{2}(X \wedge Y)Z + \frac{\mathrm{s}-6}{2}[\xi \wedge \{(X \wedge Y)\xi\}]Z.$$

Corollary 2.2 *The Riemannian curvature of a Kenmotsu* 3*-manifold is given by*

$$R(X,Y)Z = \frac{\mathrm{s}+4}{2}(X \wedge Y)Z + \frac{\mathrm{s}+6}{2}[\xi \wedge \{(X \wedge Y)\xi\}]Z.$$

Corollary 2.3 *The Riemannian curvature of a coKähler* 3*-manifold is given by*

$$R(X,Y)Z = \frac{\mathrm{s}}{2}[\xi \wedge \{(X \wedge Y)\xi\}]Z.$$

3 Kenmotsu 3-manifolds

Let (N, h, J) be a Riemannian 2-manifold together with the compatible orthogonal complex structure J. Take a direct product $M = \mathbb{E}^1(t) \times N$ of real line $\mathbb{E}^1(t)$ and N. We denote π and σ the natural projections onto the first and second factors,

$$\pi : M \to \mathbb{E}^1, \quad \sigma : M \to N,$$

respectively. On the direct product M, we equip a Riemannian metric g defined by

$$g = \mathrm{d}t^2 + f(t)^2 \pi^* h.$$

Here f is a positive function on $\mathbb{E}^1(t)$. The resulting Riemannian manifold (M, g) is denoted by $\mathbb{E}^1 \times_f N$ and called the *warped product* with base $\mathbb{E}^1$ and fibre N. The function f is called the *warping function*.

On the warped product $M = \mathbb{E}^1 \times_f N$, we define the vector field ξ by $\xi = \frac{\partial}{\partial t}$. Then the Levi-Civita connection ∇ of M is given by (*cf.* [8]):

$$\begin{aligned}
\nabla_{\overline{X}^{\mathrm{v}}} \overline{Y}^{\mathrm{v}} &= (\overline{\nabla}_{\overline{X}} \overline{Y})^{\mathrm{v}} - \frac{1}{f} g(\overline{X}^{\mathrm{v}}, \overline{Y}^{\mathrm{v}}) f' \, \xi, \\
\nabla_{\xi} \overline{X}^{\mathrm{v}} &= \nabla_{\overline{X}^{\mathrm{v}}} \xi = \frac{f'}{f} \overline{X}^{\mathrm{v}}, \\
\nabla_{\xi} \xi &= 0.
\end{aligned}$$

Here the superscript v means the vertical lift operation of vector fields from N to M. Define φ by $\varphi X = \{J(\sigma_* X)\}^{\mathrm{v}}$. Then we get

$$\nabla_X \xi = \beta(X - \eta(X)\xi),$$

$$(\nabla_X \varphi)Y = \beta\{g(\varphi X, Y) - \eta(Y)\varphi X\}, \quad \beta = f'/f.$$

Hence $M = \mathbb{E}^1 \times_f N$ is a normal almost contact Riemannian 3-manifold of type $(0, \beta)$. In particular $\mathbb{E}^1 \times_f N$ is a Kenmotsu manifold if and only if $f(t) = ce^t$ for some positive constant c. Take a local orthonormal frame field $\{\bar{e}_1, \bar{e}_2\}$ of (N, h) such that $\bar{e}_2 = J\bar{e}_1$. Then we obtain a local orthonormal frame field $\{e_1, e_2, e_3\}$ by

$$e_1 = \frac{1}{f}\bar{e}_1^{\mathrm{v}}, \quad e_2 = \frac{1}{f}\bar{e}_2^{\mathrm{v}} = \varphi\, e_1, \quad e_3 = \xi.$$

Then sectional curvatures of M are given by

$$K(e_1 \wedge e_2) = \frac{1}{f^2}\{\kappa - (f')^2\}, \quad K(e_1 \wedge e_3) = K(e_2 \wedge e_3) = -\frac{f''}{f},$$

where κ is the Gaussian curvature of N. The Ricci tensor components $\rho_{ij} = \rho(e_i, e_j)$ are given by

$$\rho_{11} = \rho_{22} = \frac{\kappa}{f^2} - \frac{f''}{f} - \left(\frac{f'}{f}\right)^2, \quad \rho_{33} = -\frac{2f''}{f}$$

The local structure of Kenmotsu manifolds is described as follows.

Lemma 3.1 ([6]) *A Kenmotsu* 3*-manifold M is locally isomorphic to a warped product $I \times_f N$ whose base $I \subset \mathbb{E}^1(t)$ is an open interval, N is a surface and warping function $f(t) = ce^t$, $c > 0$. The structure vector field is $\xi = \partial/\partial t$.*

Proposition 3.1 *A Kenmotsu* 3*-manifold is of constant scalar curvature if and only if M is of constant curvature* -1.

(*Proof.*) For every point $p \in M$, there exists a neighbourhood U_p of p such that U_p is a warped product $(-\epsilon, \epsilon) \times_f N$ of an open interval $(-\epsilon, \epsilon)$ and a Riemannian 2-manifold of Gaussian curvature κ with warping function $f(t) = ce^t$. The scalar curvature s over U_p is computed as

$$\mathrm{s}|_{U_p} = -6 + 2\kappa c^{-2} e^{-2t}.$$

Thus the differential ds is computed as

$$\frac{1}{2}\mathrm{ds} = c^{-2} e^{-2t} \mathrm{d}\kappa - 2\kappa c^{-2} e^{-2t} \mathrm{d}t.$$

Hence $\mathrm{ds} = 0$ if and only if $\kappa = 0$. This implies that U_p is of constant curvature -1. ■

Corollary 3.1 *A Kenmotsu* 3*-manifold satisfies the condition* (1) *for all X, Y, Z, $W \in \mathfrak{X}(M)$ orthogonal to ξ if and only if M is locally symmetric.*

(*Proof.*) De [4] showed that M satisfies (1) for all $X, Y, Z, W \in \mathfrak{X}(M)$ orthogonal to ξ if and only if M is of constant scalar curvature. As we have seen above, M is of constant scalar curvature if and only if M is of constant curvature -1. ■

Note that all the examples of Kenmotsu 3-manifold exhibited in [4, Example 5.1, 5.2, 5.3] are of constant curvature -1.

References

[1] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math. 509 (1976), Springer-Verlag, Berlin-Heidelberg-New-York.

[2] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics, 203, Birkhäuser Boston, Inc., Boston, 2002.

[3] J. T. Cho, J. Inoguchi, and J. E. Lee, Pseudo-symmetric contact 3-manifolds III, Coll. Math. **114** (2009), 77–98.

[4] U. C. De, On Φ-symmetric Kenmotsu manifolds, Int. Elec. J. Geom. 1 (2008), 33-38.

[5] D. Janssens and L. Vanhecke, Almost constant structures and curvature tensors, Kōdai Math. J. 4 (1981), 1–27.

[6] K. Kenmotsu, A class of almost contact Riemannian manifolds, Tôhoku Math. J. 24 (1972), 93–103.

[7] Z. Olszak, Normal almost contact metric manifolds of dimension three, Ann. Pol. Math. 47 (1986), 41–50.

[8] B. O'Neill, *Semi-Riemannian Geometry with Application to Relativity*, Academic Press, Orland, 1983.

[9] J. A. Oubiña, New classes of almost contact metric structures, Publ. Math. Debrecen 32 (1985), 187–193.