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On the strongly regular graphs obtained from quasi-symmetric 2-$(31, 7, 7)$ designs

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Abstract

It is known that the five non-isomorphic quasi-symmetric 2-$(31, 7, 7)$ designs lead to non-isomorphic strongly regular graphs with parameters $(155, 42, 17, 9)$. We will show that there exist no isomorphisms among these graphs and the block graphs of the Steiner triple systems $STS(31)$ except the isomorphism between the block graphs of the point-plane design and the point-line design of $PG(4, 2)$.

1 Introduction

A t-(v, k, λ) design is a pair (X, B), where X is a set of "points" of cardinality v, and B is a collection of k-element subsets of X called "blocks", with the property that any t points are contained in precisely λ blocks. An intersection number is the number of points contained in two blocks. A 2-design is called quasi-symmetric if the intersection number of the design takes just two values. A graph consists of a finite set V of "vertices" together with a set E of "edges", where an edge is a subset of a vertex set of cardinality 2. Two graphs (V, E), (V', E') are isomorphic if there is a bijection $\phi : V \to V'$ such that $(v, w) \in E$ if and only if $(\phi(v), \phi(w)) \in E'$. An automorphism of a graph is an isomorphism from the graph to itself. The set of all automorphisms of a graph forms a group and it is called the automorphism group. A strongly regular graph with parameters (n, k, λ, μ) is a graph with n vertices in which

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the number of common neighbors of x and y is k, λ or μ according as x and y are equal, adjacent or non-adjacent respectively.

Let $D = (X, B)$ be a quasi-symmetric design with the intersection numbers λ_1, λ_2 ($\lambda_1 < \lambda_2$). The block graph of D is a graph whose vertices are the blocks of D and whose edges are the pairs (Y, Z) of blocks with $|Y \cap Z| = \lambda_2$. Then it is known that the block graph is strongly regular (cf. [2, Theorem 5.3], [5, Theorem 37.7]).

Tonchev [8] showed that there are exactly five non-isomorphic quasi-symmetric 2-$(31, 7, 7)$ designs. Five strongly regular graphs with parameters $(155, 42, 17, 9)$ are obtained from the five designs. Then Stoichev [7] proved that the five graphs are non-isomorphic by comparing the automorphism groups. On the other hand, strongly regular graphs with the same parameters are also obtained from the Steiner triple systems $STS(31)$. It is known that if $v \geq 15$, non-isomorphic Steiner triple systems $STS(v)$ lead to non-isomorphic strongly regular graphs [1]. In this paper, we consider the cliques of maximum size in the five graphs obtained from the quasi-symmetric designs by using Magma. We will give an alternative proof of Theorem in Stoichev [7], and further we will prove the following theorem:

Theorem 1. *There exist no isomorphisms among the block graphs from the quasi-symmetric* 2-$(31, 7, 7)$ *designs and the Steiner triple systems* $STS(31)$ *except the isomorphism between the block graphs of the point-plane design and the point-line design of* $PG(4, 2)$.

2 Quasi-symmetric designs and strongly regular graphs

We begin this section with Tonchev's result [8]. Conway and Pless [3] showed that there exist exactly five inequivalent binary extremal doubly-even self-dual $[32, 16, 8]$ codes. We denote these codes by C_1, C_2, C_3, C_4 and C_5, whose components are q_{32}, r_{32}, $2g_{16}$, $8f_4$ and $16f_2$, respectively [3, Table III]. The codewords of weight 8 in C_i form a 3-$(32, 8, 7)$ design D_i by the Assmus–Mattson theorem [2, Theorem 14.11]. As a derived design of D_i, a quasi-symmetric 2-$(31, 7, 7)$ design D'_i is obtained. Its intersection numbers are 1 and 3. In addition, all derived designs of D_i are isomorphic since the automorphism group of D_i acts transitively on the points. It is shown in [8] that the five quasi-symmetric 2-$(31, 7, 7)$ design D'_i $(i = 1, 2, \ldots, 5)$ obtained from D_i are non-isomorphic.

Let G_i be the block graph of D'_i $(i = 1, 2, \ldots, 5)$. The following theorem is proved by Stoichev [7].

Theorem 2. *The five graphs G_1, G_2, G_3, G_4, G_5 are non-isomorphic.*

We will give an alternative proof of the theorem. The generator matrix of the code C_i is obtained from Sloane [6]. We set the generator matrix in Magma as follows:

```
m := KMatrixSpace(GF(2),16,32) ! [1,1,0,0,...];
```

The code "`c`" is defined from the generator matrix "`m`" by using the command "`c:= LinearCode(m);`". All codewords of weight 8 in "`c`" are obtained by using the command "`Words(c,8)`". The design D_i is obtained by the command "`Design`" and furthermore, the derived design D'_i is obtained by the command "`Contraction`". By the following program, the graph G_i is constructed from the generator matrix of the code C_i as in the previous section. In the program, we denote G_1, G_2, G_3, G_4, G_5 by `G1`, `G2`, `G3`, `G4`, `G5`, and the generator matrices of C_1, C_2, C_3, C_4, C_5 by `M1`, `M2`, `M3`, `M4`, `M5`, respectively.

```
SRGFromCode := function(m);

c := LinearCode(m);
d := Design<3,32 | {Support(w) : w in Words(c,8) }>;
d2 := Contraction(d,Point(d,32));

b := BlockSet(d2);
v := #b;
e := {};
for i in [1..v-1] do
 for j in [i+1..v] do
  if #(b.i meet b.j) eq 3 then
   e := e join {{i,j}};
  end if;
 end for;
end for;
g := Graph<v | e>;

return g;

end function;

G1 := SRGFromCode(M1);
G2 := SRGFromCode(M2);
G3 := SRGFromCode(M3);
```

```
G4 := SRGFromCode(M4);
G5 := SRGFromCode(M5);
```

Table 1. Results

graph	maximum size of cliques	♯ of cliques
G_1	7	2015
G_2	15	31
G_3	15	1
G_4	10	21
G_5	10	2

We will consider the sizes and the numbers of maximum cliques of G_1, G_2, G_3, G_4, G_5. Here a clique is an induced complete subgraph. We can use the command "`MaximumClique`" to compute the maximum size of the cliques of the graphs. Moreover we can compute the numbers of the cliques of maximum size by using command "`AllCliques(G,n)`", where G is a graph and n is a size of clique. For example,

```
> #MaximumClique(G1);
7
> #AllCliques(G1,7);
2015
```

The sizes and numbers of maximum cliques of graphs are invariants under isomorphisms of graphs. So the results in Table 1 prove Theorem 2.

3 Steiner triple systems and strongly regular graphs

A 2-$(v,3,1)$ design D is called a Steiner triple system and is denoted by $STS(v)$. It is known that there is an $STS(v)$ if and only if $v \equiv 1, 3 \pmod 6$. The total number of blocks is equal to $v(v-1)/6$ and the number of blocks containing a point is equal to $(v-1)/2$. Also, D is a quasi-symmetric design with the intersection numbers 0 and 1. It follows from [5, Theorem 37.7] that the block graph of an $STS(v)$ $(v > 7)$ is a strongly regular graph with parameters $(\frac{v(v-1)}{6}, \frac{3v-9}{2}, \frac{v+3}{2}, 9)$. Hence the block graph of an $STS(31)$ has parameters $(155, 42, 17, 9)$ which are equal to those of the strongly regular graph obtained from a quasi-symmetric 2-$(31,7,7)$ design. In general, the following theorem is known [1].

Theorem 3. *Let D (resp. D') be an $STS(v)$ and Γ (resp. Γ') be a strongly regular graph with parameters $(\frac{v(v-1)}{6}, \frac{3v-9}{2}, \frac{v+3}{2}, 9)$ obtained from D (resp. D') as the block graph.*

(1) *If $v \geq 19$, then Γ has exactly v maximal cliques of maximum size $\frac{v-1}{2}$.*

(2) *If $v \geq 15$ and D is non-isomorphic to D', then Γ is non-isomorphic to Γ'.*

By Theorem 3(1), if $v = 31$, the number of the maximal cliques of maximum size of Γ is equal to that of the graph G_2 in Table 1. The four graphs G_1, G_3, G_4, G_5 can not be obtained from an $STS(31)$. We will give an example of $STS(31)$ whose block graph is isomorphic to G_2. We note that there are at least 6×10^{16} non-isomorphic $STS(31)$ (cf. [4]). Let V be an $(n+1)$-dimensional vector space over the q-element field $\mathbb{F}_q$. We consider the projective geometry $PG(n, q)$ consisting of the set of all vector subspaces of V. In the case $n = 4$ and $q = 2$, the points and planes of $PG(4, 2)$ form a quasi-symmetric 2-$(31, 7, 7)$ design and the points and lines of $PG(4, 2)$ form a Steiner triple system $STS(31)$. Here, the following proposition is known [2, Exercise 5 of Chapter 5].

Proposition 4. *For $n > 2$, the points and $(n-2)$-flats of $PG(n, q)$ form a quasi-symmetric design, whose block graph is isomorphic to the block graph of the point-line design of $PG(n, q)$.*

By this proposition, the block graph G_2 of the quasi-symmetric 2-$(31, 7, 7)$ design D_2' is isomorphic to the block graph of the point-line design $STS(31)$ of $PG(4, 2)$. By Theorem 3(2), this is the only one isomorphism among the block graphs of the quasi-symmetric 2-$(31, 7, 7)$ designs and the block graphs of Steiner triple systems $STS(31)$.

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