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Some maximal balls in quasi-Fuchsian once punctured torus space*

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Abstract

In this article we exhibit some balls lying in the quasi-Fuchsian space of once punctured tori, which are maximal in the class of balls with the same centers. The centers of our maximal balls lie on the slice determined by the trace equation $y = \bar{x}$.

1 Introduction and statement of the results

Let A and B be loxodromic elements of $PSL(2,\mathbb{C})$ and let x, y and z be the traces of A, B and AB, respectively. Let $G = \langle A, B\rangle$ be the group generated by A and B. We have an interest in the case where G is discrete and the following Markoff equation holds:

$$x^2 + y^2 + z^2 = xyz. \tag{1.1}$$

Let $A_0 = \begin{pmatrix} \sqrt{2}+1 & 0 \\ 0 & \sqrt{2}-1 \end{pmatrix}$ and $B_0 = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}$. Then $\mathrm{trace}(A_0) = \sqrt{8}$, $\mathrm{trace}(B_0) = \sqrt{8}$ and $\mathrm{trace}(A_0B_0) = 4$ and they satisfy (1.1). Identifying matrices with Möbius transformations, it is well known that $G_0 = \langle A_0, B_0\rangle$ is a Fuchsian group of the first kind and $\Omega(G_0)/G_0$ is a pair of once punctured tori, where $\Omega(G_0)$ denotes the region of discontinuity of G_0. For each quasi-Fuchsian group $G = \langle A, B\rangle$ such that $\Omega(G)/G$ is a pair of once punctured tori, there is a quasiconformal mapping f of the extended plane $\mathbb{C} \cup \{\infty\}$ such that $A = fA_0f^{-1}$ and $B = fB_0f^{-1}$. Hence G is a quasiconformal deformation of G_0. The set of all such quasi-Fuchsian groups is called a

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quasi-Fuchsian space of once punctured tori and denoted by $\mathcal{QF}$. By the Bers stability of quasi-Fuchsian groups ([2],[4]), $\mathcal{QF}$ is an open subset of $\mathbb{C}^2$ ([3],[7]). One would like to know what figure does $\mathcal{QF}$ present and how large ball can sit in $\mathcal{QF}$. In this article we investigate the last problem on the special slice determined by trace equation $y = \bar{x}$. First of all we shall prove the following.

Theorem 1.1. *Let $G = \langle A, B\rangle$ be a non-elementary subgroup of $PSL(2,\mathbb{C})$ generated by A and B. Let $x = trace(A), y = trace(B)$ and $z = trace(AB)$ and suppose that they satisfy* (1.1). *Let γ be a real number greater than* 10. *If the inequality*

$$|x-\gamma|^2 + |y-\gamma|^2 < 2 \tag{1.2}$$

holds, then $G \in \mathcal{QF}$.

The trace triple (x, y, z) is a parameter for $G = \langle A, B\rangle$. It is well known and easy to show that if $(\gamma+i, \gamma-i, z)$ is a parameter for $G = \langle A, B\rangle$, then G is not quasi-Fuchsian. Hence we have the following.

Theorem 1.2. *Let $\gamma > 10$ and let $B(\gamma, \gamma; r)$ be an open ball in $\mathbb{C}^2$ with center (γ, γ) and radius r. If $r \leq \sqrt{2}$ and $(x, y) \in B(\gamma, \gamma; r)$ then $G = \langle A, B\rangle$ with a parameter (x, y, z) lies in $\mathcal{QF}$. In other words, $B(\gamma, \gamma; \sqrt{2})$ is maximal in the class of balls with the center (γ, γ) such that each group corresponding to any point in $B(\gamma, \gamma; r)$ lies in $\mathcal{QF}$.*

Remark. The third entry z in the parameter (γ, γ, z) is any of the solutions for the equation $z^2 - \gamma^2 z + 2\gamma^2 = 0$ and the pair (x, y) of the first and the second entries is a local parameter in $\mathbb{C}^2$. See Remark in Section 3.

The following is shown in [6].

Theorem 1.3 ([6]). *Let $G = \langle A, B\rangle$ be a non-elementary subgroup of $PSL(2,\mathbb{C})$ generated by A and B. Let $x = trace(A)$, $y = trace(B)$ and $z = trace(AB)$ and suppose that they satisfy* (1.1). *If $y = \bar{x}$ and $G \in \mathcal{QF}$, then $-1 < \mathrm{Im}\ x < 1$.*

Though it is an easy matter, we extend Theorem 1.2 to the slice determined by $y = \bar{x}$.

Theorem 1.4. *Let $G = \langle A, B\rangle$ be in $\mathcal{QF}$ and assume $y = \bar{x}$, $\mathrm{Re}\ x > 10$. Then $B(x, \bar{x}; \sqrt{2}(1 - |\mathrm{Im}\ x|))$ is maximal in the class of balls with the center $(x, \bar{x})$ such that each group corresponding to any point in $B(x, \bar{x}; r)$ lies in $\mathcal{QF}$.*

The outline of this article is as follows. In Section 2, assuming Theorem

1.2 holds, we prove Theorem 1.4. Our proof of Theorem 1.1 consists of a lot of computations and needs six sections. In Section 3 we make a normalization and derive some equalities. In Section 4 we derive some inequalities which come from the inequality (1.2). Then we check (3.1) of Theorem 3.1 ([5]) in Section 3, which describes a sufficient condition for a group to be quasi-Fuchsian, for $n = 0$, $n = \pm 1$, $n = \pm 2$ and $|n| \geq 3$ in Sections 5, 6, 7 and 8, respectively. These complete the proof of Theorem 1.1 and that of Theorem 1.2. We refer to [1] for discrete groups.

2 Proof of Theorem 1.4.

Assuming Theorem 1.2 holds, we can prove Theorem 1.4. Theorem 1.2 says that it suffices to show

$$B(x, \bar{x}; \sqrt{2}(1 - |\mathrm{Im}\ x|) \subset B(\mathrm{Re}\ x, \mathrm{Re}\ x; \sqrt{2}). \tag{2.1}$$

Assume $(u, v) \in B(x, \bar{x}; \sqrt{2}(1 - |\mathrm{Im}\ x|))$. Then we have by the triangle inequality

$$\begin{aligned}
&\sqrt{|u - \mathrm{Re}\ x|^2 + |v - \mathrm{Re}\ x|^2} \\
&\quad \leq \sqrt{|u - x|^2 + |v - \bar{x}|^2} + \sqrt{|x - \mathrm{Re}\ x|^2 + |\bar{x} - \mathrm{Re}\ x|^2} \\
&\quad < \sqrt{2}(1 - |\mathrm{Im}\ x|) + \sqrt{2}|\mathrm{Im}\ x| \\
&\quad = \sqrt{2}.
\end{aligned}$$

Hence we have (2.1). Clearly, either

$$(x + i(1 - |\mathrm{Im}\ x|), \bar{x} - i(1 - |\mathrm{Im}\ x|)) \quad \text{or} \quad (x - i(1 - |\mathrm{Im}\ x|), \bar{x} + i(1 - |\mathrm{Im}\ x|))$$

is a boundary point of $B(x, \bar{x}; \sqrt{2}(1 - |\mathrm{Im}\ x|)$ and equals $(\mathrm{Re}\ x + i, \mathrm{Re}\ x - i)$. Therefore, the group corresponding to the parameter $(\mathrm{Re}\ x + i, \mathrm{Re}\ x - i, z)$ is not quasi-Fuchsian. Thus we have shown Theorem 1.4.

3 Normalization

In this and consequent sections we shall prove Theorem 1.1. To do this we shall make use the following.

Theorem 3.1 ([5]). *Let* $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ *and* $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $bc \neq 0$, *be loxodromic elements of* $PSL(2, \mathbb{C})$ *such that* $ABA^{-1}B^{-1}$ *is parabolic and let* $G = \langle A, B \rangle$. *If, for each integer* n, *the inequality*

$$\frac{|\alpha^n a| + |\beta^n d|}{|\alpha^n a + \beta^n d|} < \frac{|\alpha| + |\beta|}{|\alpha - \beta|} \tag{3.1}$$

holds, then $G \in \mathcal{QF}$.

Let A, B and γ be as in Theorem 1.1. By the symmetry of (1.1) we may assume

$$\text{Re } x \leq \text{Re } y.$$

This is used in the argument in Section 6 essentially. Conditions $\gamma > 10$ and (1.2) imply that A and B are loxodromic. In order to make use of Theorem 3.1, we normalize A and B as follows:

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \ \alpha\beta = 1, \ |\alpha| = 1 \quad \text{and} \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ bc \neq 0.$$

Then

$$\alpha + \beta = x, \quad a + d = y, \quad \alpha a + \beta d = z. \tag{3.2}$$

Markoff equation (1.1) implies that

$$ad = \left(\frac{\alpha+\beta}{\alpha-\beta}\right)^2.$$

Let

$$x = x_1 + ix_2 \quad \text{and} \quad y = y_1 + iy_2,$$

where x_1, x_2, y_1 and y_2 are real numbers. By (1.2) we see that $x_1, y_1 > 8$ and $|x_2|, |y_2| < 2$. Solving the equation $\alpha + 1/\alpha = x$ with the condition $|\alpha| > 1$, we obtain

$$\alpha = \frac{1}{2}(x_1 + X_1 + i(x_2 + X_2)) \quad \text{and} \quad \beta = \frac{1}{2}(x_1 - X_1 + i(x_2 - X_2)), \tag{3.3}$$

where

$$X_1 = \sqrt{\frac{x_1^2 - x_2^2 - 4 + \sqrt{(x_1^2 - x_2^2 - 4)^2 + 4x_1^2x_2^2}}{2}},$$

(3.4)

$$X_2 = \text{sgn}(x_2)\sqrt{\frac{-(x_1^2 - x_2^2 - 4) + \sqrt{(x_1^2 - x_2^2 - 4)^2 + 4x_1^2x_2^2}}{2}}$$

and $\text{sgn}(x_2) = 1$ if $x_2 \geq 0$ otherwise $\text{sgn}(x_2) = -1$. We put

$$X = \alpha - \beta. \tag{3.5}$$

It is easy to see that

$$X = X_1 + iX_2, \quad X^2 = x^2 - 4 \quad \text{and} \quad ad = \frac{x^2}{X^2} \tag{3.6}$$

and that

$$\alpha = \frac{x+X}{2} \quad \text{and} \quad \beta = \frac{x-X}{2}. \tag{3.7}$$

Next, we shall determine a and d. Solving the equations

$$a + d = y \quad \text{and} \quad ad = \frac{x^2}{X^2}$$

we have

$$a, d = \frac{1}{2}\left(y \pm \frac{\sqrt{x^2y^2 - 4(x^2+y^2)}}{X}\right).$$

We write

$$x^2y^2 - 4(x^2+y^2) = p + iq,$$

where

$$\begin{aligned} p &= (x_1^2 - x_2^2 - 4)(y_1^2 - y_2^2 - 4) - 4x_1x_2y_1y_2 - 16 \quad \text{and} \\ q &= 2((x_1^2 - x_2^2 - 4)y_1y_2 + (y_1^2 - y_2^2 - 4)x_1x_2). \end{aligned} \tag{3.8}$$

Then we put

$$l = \sqrt{\frac{p + \sqrt{p^2+q^2}}{2}}, \quad m = \operatorname{sgn}(q)\sqrt{\frac{-p + \sqrt{p^2+q^2}}{2}} \quad \text{and} \tag{3.9}$$

$$k = l + im.$$

Note that

$$k^2 = x^2y^2 - 4(x^2+y^2). \tag{3.10}$$

If the third parameter of (γ, γ, z) is $z = (\gamma^2 + \sqrt{\gamma^4 - 8\gamma^2})/2$, then

$$a = \frac{1}{2}\left(y + \frac{k}{X}\right) \quad \text{and} \quad d = \frac{1}{2}\left(y - \frac{k}{X}\right), \tag{3.11}$$

and if $z = (\gamma^2 - \sqrt{\gamma^4 - 8\gamma^2})/2$, then

$$a = \frac{1}{2}\left(y - \frac{k}{X}\right) \quad \text{and} \quad d = \frac{1}{2}\left(y + \frac{k}{X}\right).$$

These choices depend on the representation of G in which it is either $\langle A, B\rangle$ or $\langle A, B^{-1}\rangle$. Since $\langle A, B\rangle = \langle A, B^{-1}\rangle$ as groups, we may assume (3.11) holds.

Remark. By (1.1) we see that $|z|^2 - |x||y||z| + |x|^2 + |y|^2 \geq 0$ so that either

$$|z| \geq \frac{|x||y| + \sqrt{|x|^2|y|^2 - 4(|x|^2 + |y|^2)}}{2} \quad \text{or}$$
$$|z| \leq \frac{|x||y| - \sqrt{|x|^2|y|^2 - 4(|x|^2 + |y|^2)}}{2}.$$

In the case $|x - \gamma|^2 + |y - \gamma|^2 < 2$ and $\gamma > 10$, it is easy to see that either $|z| > 32$ or $|z| < 8$. Hence the first two parameters in (x, y, z) determine G uniquely so that the pair (x, y) is a local parameter for G in the ball $B(\gamma, \gamma, z; \sqrt{2})$.

For later use we derive some equalities. By (3.10) we have

$$|k|^4 = |x|^4|y|^4 - 4(|x|^4(y^2 + \bar{y}^2) + |y|^4(x^2 + \bar{x}^2)) + 16|x^2 + y^2|^2. \tag{3.12}$$

By (3.7) and (3.11) we have

$$|\alpha|^2 + |\beta|^2 = \frac{|x|^2 + |X|^2}{2} \quad \text{and} \quad |a|^2 + |d|^2 = \frac{|y|^2|X|^2 + |k|^2}{2|X|^2}. \tag{3.13}$$

Lemma 3.2.

$$\begin{aligned}&|\alpha|^2|a|^2 + |\beta|^2|d|^2\\ &\quad = \frac{(|x|^2 + |X|^2)|xy + k|^2 - 4(x^2 + \bar{x}^2 - 4)|y|^2 - 4(x\bar{y}k + \bar{x}y\bar{k})}{8|X|^2}\end{aligned}$$

and

$$\begin{aligned}&|\beta|^2|a|^2 + |\alpha|^2|d|^2\\ &\quad = \frac{(|x|^2 + |X|^2)|xy - k|^2 - 4(x^2 + \bar{x}^2 - 4)|y|^2 + 4(x\bar{y}k + \bar{x}y\bar{k})}{8|X|^2}.\end{aligned}$$

Proof. By (3.7) and (3.11) we have

$$\begin{aligned}&|\alpha|^2|a|^2 + |\beta|^2|d|^2\\ &\quad = \frac{(|x|^2 + |X|^2)(|y|^2|X|^2 + |k|^2) + (x\bar{X} + \bar{x}X)(yX\bar{k} + \bar{y}\bar{X}k)}{8|X|^2}.\end{aligned}$$

Since $|X|^4 = |x^2 - 4|^2 = |x|^4 - 4(x^2 + \bar{x}^2 - 4)$, we have

$$(|x|^2 + |X|^2)|y|^2|X|^2 = (|x|^2 + |X|^2)|xy|^2 - 4(x^2 + \bar{x}^2 - 4)|y|^2.$$

We also caluculate and obtain

$$(x\bar{X} + \bar{x}X)(yX\bar{k} + \bar{y}\bar{X}k) = (|x|^2 + |X|^2)(xy\bar{k} + \bar{x}\bar{y}k) - 4(x\bar{y}k + \bar{x}y\bar{k}).$$

Hence the first equality holds. The proof of the second is similar. □

4 Inequalities

In this section we collect some inequalities which are derived from (1.2). Writing (1.2) as

$$(x_1 - \gamma)^2 + x_2^2 + (y_1 - \gamma)^2 + y_2^2 < 2 \quad (\gamma > 10), \tag{4.1}$$

we have

$$x_1 > 8, \; y_1 > 8, \; x_2^2 < 2 \quad \text{and} \quad y_2^2 < 2. \tag{4.2}$$

By (4.2) we have

$$x_1^2 > |x|^2 - 2 \quad \text{and} \quad y_1^2 > |y|^2 - 2. \tag{4.3}$$

By the inequalities $2|x_2y_2| \leq x_2^2 + y_2^2$ and

$$x_2^2 + y_2^2 < 2 \tag{4.4}$$

we have

$$|x_2| + |y_2| < 2. \tag{4.5}$$

Lemma 4.1. (1) $||x| - |y|| < 2$, $||x| - y_1| < 2$, $|x_1 - |y|| < 2$ *and* $|x_1 - y_1| < 2$.

(2) $|x|^2 < 2y_1^2 \leq 2|y|^2$ *and* $|y|^2 < 2x_1^2 \leq 2|x|^2$.

Proof. Since

$$2 > |x - \gamma|^2 + |y - \gamma|^2 = 2\left(\gamma - \frac{x_1 + y_1}{2}\right)^2 - \frac{(x_1 + y_1)^2}{2} + |x|^2 + |y|^2$$

$$\geq 2\left(\gamma - \frac{x_1 + y_1}{2}\right)^2 + \frac{(|x| - |y|)^2}{2},$$

we obtain the first inequality of (1). As the special cases of the first we have the rest of (1). By (1) and (4.2) we have

$$|x|^2 < (y_1 + 2)^2 = 2y_1^2 - y_1(y_1 - 4) + 4 < 2y_1^2 \leq 2|y|^2.$$

Similarly we have $|y|^2 < 2x_1^2 \leq 2|x|^2$. Hence we have (2). □

Lemma 4.2. $|x_1y_2 - x_2y_1| < x_1 + y_1$.

Proof. Since $|x_1y_2 - x_2y_1| \leq |y_2|x_1 + |x_2|y_1$, it suffices to show

$$0 < (1 - |y_2|)x_1 + (1 - |x_2|)y_1. \tag{4.6}$$

There are three cases. If $|x_2| \leq 1$ and $|y_2| \leq 1$, then we see by (4.5) that (4.6) holds. Assume that $|x_2| > 1$, so that $|y_2| < 1$ by (4.5). From the equality

$$(x_1 - \gamma)^2 + (y_1 - \gamma)^2 = 2\left(\gamma - \frac{x_1 + y_1}{2}\right)^2 + \frac{(x_1 - y_1)^2}{2},$$

we have

$$\frac{(x_1 - y_1)^2}{2} \leq (x_1 - \gamma)^2 + (y_1 - \gamma)^2.$$

Hence by (4.1) we have

$$x_2^2 < 2 - y_2^2 - \frac{(x_1 - y_1)^2}{2}.$$

Therefore, to show (4.6), it suffices to show $|x_2|y_1 < y_1 + (1 - |y_2|)x_1$ or

$$\left(2 - y_2^2 - \frac{(x_1 - y_1)^2}{2}\right) y_1^2 < ((1 - |y_2|)x_1 + y_1)^2.$$

This inequality is equivalent to

$$0 < (y_1^2 + 2(1 - |y_2|)^2)x_1^2 - 2(y_1^2 - 2(1 - |y_2|))x_1y_1 + (y_1^2 - 2(1 - |y_2|^2))y_1^2.$$

Since

$$(y_1^2 - 2(1 - |y_2|))^2 - (y_1^2 + 2(1 - |y_2|)^2)(y_1^2 - 2(1 - |y_2|^2))$$
$$= -4(1 - |y_2|)^2(|y|^2 - 2) < 0,$$

we have the desired inequality. Proof for the last case $|y_2| > 1$ is the same. □

Lemma 4.3. $|k|^2 < |x|^2|y|^2$.

Proof. In view of (3.12), it suffices to show

$$2|x^2+y^2|^2 < |x|^4(y_1^2-y_2^2)+|y|^4(x_1^2-x_2^2).$$

Since $|x^2+y^2|^2 \le 2(|x|^4+|y|^4)$, it suffices to show

$$0 < |x|^4(y_1^2-y_2^2-4)+|y|^4(x_1^2-x_2^2-4).$$

By (4.2) we see that this inequality holds. □

Lemma 4.4. $|X|^2 < |x|^2$.

Proof. By (3.6) and (4.2) we have

$$|X|^4 = |x|^4 - 8(x_1^2-x_2^2-2) < |x|^4. \qquad \square$$

Lemma 4.5. $|m| < l$.

Proof. In view of (3.9), it sufficies to show $p > 0$. We note that (4.4) implies $|x_2y_2| < 1$. By (3.8), (4.2) and $2x_1y_1 \le x_1^2+y_1^2$ we have

$$p = (x_1^2-x_2^2-4)(y_1^2-y_2^2-4)-4x_1x_2y_1y_2-16 > (x_1^2-6)(y_1^2-6)-4x_1y_1-16$$
$$\ge x_1^2y_1^2-8(x_1^2+y_1^2)+20 = (x_1^2-8)(y_1^2-8)-44 > 0.$$

Thus we have our lemma. □

5 Case $n = 0$

We shall begin on the proof of Theorem 1.1. To do this it suffices to show that (3.1) holds for each integer n. We begin with the case $n = 0$. We make use the notations below in Sections $5 \sim 8$. We put

$$L(n) = \frac{|\alpha^n a|+|\beta^n d|}{|\alpha^n a+\beta^n d|} \quad \text{and} \quad R = \frac{|\alpha|+|\beta|}{|\alpha-\beta|}. \tag{5.1}$$

Then (3.1) is written as

$$L(n) < R \quad \text{or} \quad L(n)^2 < R^2.$$

By (3.5) and (3.13) we have

$$R^2 = \frac{|x|^2+|X|^2+4}{2|X|^2}. \tag{5.2}$$

By (3.2), (3.6) and (3.13) we have

$$L(0)^2 = \left(\frac{|a|+|d|}{|a+d|}\right)^2 = \frac{|a|^2+|d|^2+2|ad|}{|y|^2} = \frac{|y|^2|X|^2+|k|^2+4|x|^2}{2|y|^2|X|^2}. \tag{5.3}$$

Hence the inequality $L(0)^2 < R^2$ is equivalent to

$$|k|^2 < |x|^2|y|^2 - 4|x|^2 + 4|y|^2. \tag{5.4}$$

Squaring both sides of (5.4) and making use of the equality (3.12), a calculation shows that, in order to show (5.4), it suffices to show

$$x_1^2|y|^4 - y_2^2|x|^4 - |x\bar{y} + \bar{x}y|^2 > 0.$$

By (4.2) and Lemma 4.1 we have

$$\begin{aligned} x_1^2|y|^4 - y_2^2|x|^4 - |x\bar{y} + \bar{x}y|^2 &> 64|y|^4 - 2|x|^4 - 4|x|^2|y|^2 \\ &= 2(16|y|^4 - |x|^4) + 4|y|^2(8|y|^2 - |x|^2) > 0. \end{aligned}$$

Thus we have shown that (3.1) holds for $n = 0$.

6 Cases $n = \pm 1$.

We shall first show the following.

Proposition 6.1. *Inequalities $L(1) < R$ and $L(-1) < R$ are equivalent to*

$$|x|^2|y|^2 + 4(|x|^2 + |y|^2) < |2x_1 y + k|^2 \tag{6.1}$$

and

$$|x|^2|y|^2 + 4(|x|^2 + |y|^2) < |2x_1 y - k|^2, \tag{6.2}$$

respectively.

Proof. Making use of the equalities

$$\alpha a + \beta d = \frac{1}{2}(xy + k), \quad ad = \frac{x^2}{X^2}$$

and Lemma 3.2, straightforword calculations give us the following equalities:

$$R^2 = \frac{(|x|^2 + |X|^2 + 4)|xy + k|^2}{8|X|^2|\alpha a + \beta d|^2},$$

$$L(1)^2 = \frac{(|x|^2+|X|^2)|xy+k|^2 - 4((x^2+\bar{x}^2)|y|^2 + \bar{x}y\bar{k} + x\bar{y}k) + 16(|x|^2+|y|^2)}{8|X|^2|\alpha a+\beta d|^2}$$

and

$$R^2 - L(1)^2 = \frac{|x|^2|y|^2 - 4(|x|^2+|y|^2) + |y|^2(x^2+\bar{x}^2) + 2x_1(y\bar{k}+\bar{y}k) + |k|^2}{2|X|^2|\alpha a+\beta d|^2}$$

$$= \frac{|2x_1 y + k|^2 - |x|^2|y|^2 - 4(|x|^2+|y|^2)}{2|X|^2|\alpha a+\beta d|^2}.$$

Hence we see that $L(1) < R$ is equivalent to (6.1). Noting $\beta a + \alpha d = \frac{1}{2}(xy-k)$, similar calculations give us the proof for the case $L(-1) < R$. □

By (3.9), Lemma 4.5 and (4.2) we have

$$y\bar{k} + \bar{y}k = 2(y_1 l + y_2 m) > 2(y_1 - \sqrt{2})l > 0.$$

Hence we have

$$y\bar{k} + \bar{y}k > 0. \tag{6.3}$$

This inequality implies that $|2x_1 y - k|^2 < |2x_1 y + k|^2$. Hence we have

Proposition 6.2. *If the inequality $L(-1) < R$ holds, then also holds $L(1) < R$.*

We shall show $L(-1) < R$ by a sequence of lemmas. For compactness of description we shall use the following notation.

$$I = |x|^2|y|^2 + 4(|x|^2+|y|^2).$$

Lemma 6.3. *Inequality* (6.2) *is equivalent to*

$$4x_1|x\bar{y} - \bar{x}y| < I - |k|^2. \tag{6.4}$$

Proof. Inequality (6.2) is rewritten as follows:

$$2x_1(y\bar{k} + \bar{y}k) < 4x_1^2|y|^2 - (I - |k|^2). \tag{6.5}$$

By (4.2), (4.3) and (2) of Lemma 4.1, we have

$$4x_1^2|y|^2 - (I - |k|^2) > 4x_1^2|y|^2 - |x|^2|y|^2 - 4(|x|^2+|y|^2)$$

$$> 4(|x|^2 - 2)|y|^2 - |x|^2|y|^2 - 12|y|^2 = (3|x|^2 - 20)|y|^2 > 0.$$

Hence by (6.3) we may square both sides of (6.5) and obtain

$$(6.6)\qquad 4x_1^2\{(y\bar{k}+\bar{y}k)^2-4x_1^2|y|^4+2|y|^2(I-|k|^2)\}<(I-|k|^2)^2.$$

By (3.10) we have

$$(y\bar{k}+\bar{y}k)^2=2(x_1^2-x_2^2)|y|^4-4(x^2\bar{y}^2+\bar{x}^2y^2)-8|y|^4+2|y|^2|k|^2.$$

Then a calculation shows that

$$(y\bar{k}+\bar{y}k)^2-4x_1^2|y|^4+2|y|^2(I-|k|^2)=4|x\bar{y}-\bar{x}y|^2.$$

Inserting this equalty into (6.6), we have

$$(4x_1|x\bar{y}-\bar{x}y|)^2<(I-|k|^2)^2,$$

which is equivalent to (6.4). □

Lemma 6.4. *Inequality* (6.4) *is equivalent to*

$$(6.6).\qquad x_1|x_1y_2-x_2y_1|I-4(x_1^2+1)|x_1y_2-x_2y_1|^2<y_1^2|x|^4+x_1^2|y|^4.$$

PROOF. Inequality (6.4) is written as

$$|k|^2<I-4x_1|x\bar{y}-\bar{x}y|.$$

It is easy to see that $I-4x_1|x\bar{y}-\bar{x}y|>0$. Squaring both sides of the above inequality, we have by (3.12)

$$\begin{aligned}&|x|^4|y|^4-4(|x|^4(y^2+\bar{y}^2)+|y|^4(x^2+\bar{x}^2))+16|x^2+y^2|^2\\&<I^2-16x_1|x_1y_2-x_2y_1|I+64x_1^2|x_1y_2-x_2y_1|^2.\end{aligned}$$

It is not difficult to see that this inequality is equivalent to (6.6). □

Lemma 6.5.

$$x_1|x_1y_2-x_2y_1|I-4(x_1^2+1)|x_1y_2-x_2y_1|^2<x_1(x_1+y_1)I-4(x_1^2+1)(x_1+y_1)^2.$$

PROOF. Inequality of the lemma is written as

$$(6.7)\ \ 0<(x_1+y_1-|x_1y_2-x_2y_1|)(x_1I-4(x_1^2+1)(x_1+y_1+|x_1y_2-x_2y_1|)).$$

By Lemma 4.2 we have

$$0<x_1+y_1-|x_1y_2-x_2y_1|.$$

By (4.2) and Lemma 4.1 we have

$$4(x_1^2+1)(x_1+y_1+|x_1y_2-x_2y_1|) < 32x_1^3 < x_1|x|^2|y|^2 < x_1 I.$$

Hence inequality (6.7) holds. □

By Lemmas 6.4 and 6.5 we have the following.

Lemma 6.6. *If the inequality*

$$x_1(x_1+y_1)I - 4(x_1^2+1)(x_1+y_1)^2 < y_1^2|x|^4 + x_1^2|y|^4 \tag{6.8}$$

holds, then (6.6) *holds.*

By our normalization in Section 3 there are two cases.

Lemma 6.7. *If* $x_1 = y_1$, *then* (6.8) *holds.*

Proof. In this case (6.8) reduces to

$$0 < (|x|^2-|y|^2)^2 + 8(2x_1^2+2-(|x|^2+|y|^2)).$$

Inequality (4.4) implies

$$|x|^2+|y|^2 = 2x_1^2+x_2^2+y_2^2 < 2x_1^2+2.$$ □

Lemma 6.8. *If* $x_1 < y_1$, *then*

$$\begin{aligned} 0 &< -2x_1^2(|x|^2|y|^2+4(|x|^2+x_1^2+y_2^2)) + 16(x_1^2+1)x_1^2 + x_1^2|x|^4 + x_1^2|y|^4 \\ &< -x_1(x_1+y_1)I + 4(x_1^2+1)(x_1+y_1)^2 + y_1^2|x|^4 + x_1^2|y|^4, \end{aligned} \tag{6.9}$$

so that (6.8) *holds.*

Proof. Since

$$x_1(x_1+y_1)I - 2x_1^2(|x|^2|y|^2+4(|x|^2+x_1^2+y_2^2)) = x_1(y_1-x_1)(I+8x_1(y_1+x_1)),$$

the second inequality of (6.9) is written as

$$x_1(y_1-x_1)(I+8x_1(y_1+x_1)) < (y_1^2-x_1^2)|x|^4 + 4(x_1^2+1)(3x_1+y_1)(y_1-x_1).$$

Hence, to show the second inequality of (6.9), it suffices to show

$$\begin{aligned} &x_1(|x|^2|y|^2+4|x|^2+4|y|^2+8x_1(y_1+x_1)) \\ &\qquad < (x_1+y_1)|x|^4 + 4(x_1^2+1)(3x_1+y_1). \end{aligned} \tag{6.10}$$

By (1) of Lemma 4.1, (4.2) and (4.4) we have $|x|^2 < x_1^2+2,\ \ |y| < x_1+2,\ \ y_1 < x_1 + 2$ so that

$$x_1(|x|^2|y|^2 + 4|x|^2 + 4|y|^2 + 8x_1(y_1 + x_1)) < x_1^5 + 4x_1^4 + 30x_1^3 + 40x_1^2 + 36x_1.$$

We also have

$$(x_1 + y_1)|x|^4 + 4(x_1^2 + 1)(3x_1 + y_1) > 2x_1^5 + 16x_1^3 + 16x_1.$$

Since $x_1 > 8$,

$$\begin{aligned} &2x_1^5 + 16x_1^3 + 16x_1 - (x_1^5 + 4x_1^4 + 30x_1^3 + 40x_1^2 + 36x_1) \\ &\quad = x_1(x_1^4 - 4x_1^3 - 14x_1^2 - 40x_1 - 20) > 0, \end{aligned}$$

so that (6.10) holds. Thus we have shown the second inequality. To show the first inequality of (6.9), it suffices to show

$$2(|x|^2|y|^2 + 4(|x|^2 + x_1^2 + y_2^2)) < |x|^4 + |y|^4 + 16(x_1^2 + 1).$$

This inequality is equivalent to

$$0 < (|x|^2 - |y|^2)^2 + 8(2 - x_2^2 - y_2^2).$$

Inequality (4.4) implies that this holds. Hence we have the first inequality. □

Thus we have shown that (6.8) holds for each cases so that (3.1) holds for $n = \pm 1$.

7 Case $n = \pm 2$.

We shall first show the following.

Proposition 7.1. *Inequalities $L(2) < R$ and $L(-2) < R$ are equivalent to*

$$\begin{aligned} (7.1)\qquad &-2x_1(x_1(xy\bar{k} + \bar{x}\bar{y}k) - 2(y\bar{k} + \bar{y}k)) \\ &\quad < 2x_1^2(|x|^2|y|^2 - 4|y|^2 + |k|^2) + |x|^2|y|^2 + 4|y|^2 - |k|^2 - 4|x|^2 \end{aligned}$$

and

$$\begin{aligned} (7.2)\qquad &2x_1(x_1(xy\bar{k} + \bar{x}\bar{y}k) - 2(y\bar{k} + \bar{y}k)) \\ &\quad < 2x_1^2(|x|^2|y|^2 - 4|y|^2 + |k|^2) + |x|^2|y|^2 + 4|y|^2 - |k|^2 - 4|x|^2, \end{aligned}$$

respectively.

Proof. Recall that $L(2)^2 = \dfrac{(|\alpha^2 a| + |\beta^2 d|)^2}{|\alpha^2 a + \beta^2 d|^2}$ and $R^2 = \dfrac{|x|^2 + |X|^2 + 4}{2|X|^2}$. Making use of the equality

$$\alpha^2 a + \beta^2 d = (\alpha+\beta)(\alpha a + \beta d) - (a+d) = \frac{x(xy+k) - 2y}{2},$$

we have

$$|\alpha^2 a + \beta^2 d|^2 = \frac{|x|^2|xy+k|^2 - 2(x^2+\bar{x}^2)|y|^2 - 2(x\bar{y}k + \bar{x}y\bar{k}) + 4|y|^2}{4}. \tag{7.3}$$

Making use of (3.13), (7.3), Lemma 3.2 and

$$(|x|^2 + |X|^2)^2 = 2(|x|^2 + |X|^2)|x|^2 - 4(x^2 + \bar{x}^2) + 16,$$

one computes and obtains

$$\begin{aligned}
&(|\alpha|^2 + |\beta|^2)(|\alpha|^2|a|^2 + |\beta|^2|d|^2)\\
&\quad = \frac{1}{8|X|^2}\{(|x|^2 + |X|^2)(|x|^2|xy+k|^2 - 2(x^2+\bar{x}^2)|y|^2 - 2(x\bar{y}k + \bar{x}y\bar{k})\\
&\qquad + 4|y|^2) + 4|y|^2(|x|^2 + |X|^2) - 2(x^2 + \bar{x}^2 - 4)|xy+k|^2\}\\
&\quad = \frac{(|x|^2 + |X|^2)|\alpha^2 a + \beta^2 d|^2}{2|X|^2}\\
&\qquad + \frac{2|y|^2(|x|^2 + |X|^2) - (x^2 + \bar{x}^2 - 4)|xy+k|^2}{4|X|^2}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&(|\alpha^2 a| + |\beta^2 d|)^2\\
&\quad = (|\alpha|^2 + |\beta|^2)(|\alpha|^2|a|^2 + |\beta|^2|d|^2) - (|a|^2 + |d|^2) + 2|ad|\\
&\quad = \frac{(|x|^2 + |X|^2)|\alpha^2 a + \beta^2 d|^2}{2|X|^2}\\
&\qquad + \frac{2|x|^2|y|^2 + 8|x|^2 - 2|k|^2 - (x^2 + \bar{x}^2 - 4)|xy+k|^2}{4|X|^2}\\
&\quad = \left(R^2 - \frac{2}{|X|^2}\right)|\alpha^2 a + \beta^2 d|^2\\
&\qquad + \frac{|x|^2|y|^2 + 4|x|^2 - |k|^2 - (x_1^2 - x_2^2 - 2)|xy+k|^2}{2|X|^2}.
\end{aligned}$$

Therefore we see that inequality $L(2)^2 < R^2$ is equivalent to

$$|x|^2|y|^2 + 4|x|^2 - |k|^2 - (x_1^2 - x_2^2 - 2)|xy+k|^2 < 4|\alpha^2 a + \beta^2 d|^2.$$

Inserting the right hand side of (7.3) into this, it is not hard to see that the above inequality is equivalent to (7.1). We obtain (7.2) similarly. □

By Lemma 4.5 and (4.2) we have

$$y\bar{k} + \bar{y}k = 2(y_1 l + y_2 m) > 2(y_1 l - 2|m|) > 2l(y_1 - 2) > 2l(x_1 - 4)$$

and

$$|y\bar{k} - \bar{y}k| = 2|y_1 m + y_2 l| < 2(y_1|m| + 2l) < 2l(y_1 + 2) < 2l(x_1 + 4).$$

Hence

$$\begin{aligned} x_1(xy\bar{k} + \bar{x}\bar{y}k) - 2(y\bar{k} + \bar{y}k) &= (x_1^2 - 2)(y\bar{k} + \bar{y}k) + ix_1x_2(y\bar{k} - \bar{y}k) \\ &\geq (x_1^2 - 2)(y\bar{k} + \bar{y}k) - x_1|x_2||y\bar{k} - \bar{y}k| \\ &> (x_1^2 - 2)2l(x_1 - 4) - 2x_1 2l(x_1 + 4) \\ &= 2l(x_1^3 - 6x_1^2 - 10x_1 + 8) > 0. \end{aligned}$$

Because of $x_1 > 8$, the last inequality holds. Hence we have

$$x_1(xy\bar{k} + \bar{x}\bar{y}k) - 2(y\bar{k} + \bar{y}k) > 0. \tag{7.4}$$

By Proposition 7.1 and (7.4) we have the following.

Proposition 7.2. *If inequality $L(-2) < R$ holds, then also holds $L(2) < R$.*

Next we shall show $L(-2) < R$. Our tactics is to change (7.2) to another. The square of the right hand side of (7.2) is greater than

$$\begin{aligned} &4x_1^4(|x|^2|y|^2 - 4|y|^2 + |k|^2)^2 \\ &\quad + 4x_1^2(|x|^2|y|^2 - 4|y|^2 + |k|^2)(|x|^2|y|^2 + 4|y|^2 - |k|^2 - 4|x|^2) \end{aligned}$$

and that of the left hand side of (7.2) is equal to

$$4x_1^2(x_1^2(xy\bar{k} + \bar{x}\bar{y}k)^2 - 4x_1(xy\bar{k} + \bar{x}\bar{y}k)(y\bar{k} + \bar{y}k) + 4(y\bar{k} + \bar{y}k)^2).$$

Hence, in order to show (7.2), it suffices to show

$$\begin{aligned} &x_1^2(xy\bar{k} + \bar{x}\bar{y}k)^2 - 4ix_1x_2(y\bar{k} - \bar{y}k)(y\bar{k} + \bar{y}k) - 4(x_1^2 - 1)(y\bar{k} + \bar{y}k)^2 \\ &< x_1^2(|x|^2|y|^2 - 4|y|^2 + |k|^2)^2 + (|x|^2|y|^2 - 4|y|^2 + |k|^2)(|x|^2|y|^2 + 4|y|^2 - |k|^2 - 4|x|^2). \end{aligned} \tag{7.5}$$

Calculations show that

$$(xy\bar{k}+\bar{x}\bar{y}k)^2=(|x|^2|y|^2+|k|^2)^2-16|x^2+y^2|^2,$$
$$(y\bar{k}+\bar{y}k)^2=|y|^4(x^2+\bar{x}^2-8)+4(|x|^4+|y|^4-|x^2+y^2|^2)+2|y|^2|k|^2$$

and

$$(y\bar{k}-\bar{y}k)(y\bar{k}+\bar{y}k)=-4i(|y|^4x_1x_2+4(x_1^2-x_2^2)y_1y_2-4(y_1^2-y_2^2)x_1x_2).$$

Then (7.5) is written as

$$\begin{aligned}0<{}&x_1^2(16|x^2+y^2|^2-8|y|^2(|x|^2|y|^2+|k|^2)+16|y|^4)\\&+16x_1x_2(|y|^4x_1x_2+4(x_1^2-x_2^2)y_1y_2-4(y_1^2-y_2^2)x_1x_2)\\&+4(x_1^2-1)(|y|^4(x^2+\bar{x}^2-8)+4(|x|^4+|y|^4-|x^2+y^2|^2)+2|y|^2|k|^2)\\&+(|x|^4|y|^4-16|y|^4+8|y|^2|k|^2-|k|^4)-4|x|^2(|x|^2|y|^2-4|y|^2+|k|^2)\end{aligned}$$

or

$$\begin{aligned}0<{}&16x_1^2|x|^4+64x_1x_2((x_1^2-x_2^2)y_1y_2-(y_1^2-y_2^2)x_1x_2)-4|y|^4(x^2+\bar{x}^2)\\&+|x|^4|y|^4+16|x^2+y^2|^2-|k|^4-4|x|^2(|x|^2|y|^2+|k|^2+4|x|^2-4|y|^2).\end{aligned}$$

By (3.12) we reduce this to

$$\begin{aligned}&-16x_1x_2((x_1^2-x_2^2)y_1y_2-(y_1^2-y_2^2)x_1x_2)\\&<4x_1^2|x|^4-|x|^2(|x|^2|y|^2+|k|^2)+|x|^4(y^2+\bar{y}^2)-4|x|^4+4|x|^2|y|^2.\end{aligned}\tag{7.6}$$

Therefore, in order to show (7.2), it suffices to show (7.6). Since, by (4.4), $|x_2y_2|\le(x_2^2+y_2^2)/2<1$, we have

$$16x_1y_1|x_2y_2|(x_1^2-x_2^2)<16|x|^3y_1<|x|^4(y_1^2-y_2^2-4)=\frac{|x|^4(y^2+\bar{y}^2)}{2}-4|x|^4.$$

We also have by (4.2)

$$16x_1^2x_2^2(y_1^2-y_2^2)<32|x|^2(y_1^2-y_2^2)<\frac{|x|^4(y^2+\bar{y}^2)}{2}.$$

Hence, in order to show (7.6), it suffices to show

$$0<4x_1^2|x|^4-|x|^2(|x|^2|y|^2+|k|^2)+4|x|^2|y|^2.$$

By Lemmas 4.3 and 4.1 we have

$$\begin{aligned}&4x_1^2|x|^4-|x|^2(|x|^2|y|^2+|k|^2)+4|x|^2|y|^2>4x_1^2|x|^4-2|x|^4|y|^2\\&=2|x|^4(2x_1^2-|y|^2)>0.\end{aligned}$$

Hence (7.6) holds. Therefore, we have shown that (7.2) holds so that we have $L(-2)<R$.

Thus we have shown that (3.1) holds for $n=\pm2$.

8 Case $|n| \geq 3$

We assume $|n| \geq 3$ and put

$$A(n) = |\alpha|^{2n}|a|^2 + |\beta|^{2n}|d|^2 \quad \text{and} \quad f(t) = \frac{2t(t+1)}{t-1} \quad (t = |ad| > 1).$$

Proposition 8.1. *If $f(|ad|) < A(n)$, then $L(n) < R$.*

PROOF. Since

$$L(n)^2 = \frac{|\alpha|^{2n}|a|^2 + 2|ad| + |\beta|^{2n}|d|^2}{|\alpha^n a + \beta^n d|^2} \leq \frac{A(n) + 2|ad|}{A(n) - 2|ad|}.$$

and $R^2 \geq |x|^2/|X|^2 = |ad|$, in order to show $L(n) < R$, it suffices to show

$$\frac{A(n) + 2|ad|}{A(n) - 2|ad|} < |ad|.$$

It is easy to see that this is equivalent to $f(|ad|) < A(n)$. □

Recall that $\gamma > 10$. We shall show the following two inequalities.

$$f(t) \leq \frac{(\gamma + \sqrt{2})^2((\gamma + \sqrt{2})^2 - 2)}{(\gamma + \sqrt{2})^2 - 4}. \tag{8.1}$$

$$A(n) > \frac{(\gamma + \sqrt{2})^8 \left(\gamma + \sqrt{2} - \sqrt{(\gamma + \sqrt{2})^2 - 4}\right)^2}{32((\gamma + \sqrt{2})^2 - 4)}. \tag{8.2}$$

Lemma 8.2. $\dfrac{(\gamma + \sqrt{2})^2}{(\gamma + \sqrt{2})^2 - 4} < |ad| < \dfrac{(\gamma - \sqrt{2})^2}{(\gamma - \sqrt{2})^2 - 4}.$

PROOF. We put $g(w) = 1 - 4/w^2$. Then $g(w)$ is a holomorphic function in the closed disc $D = \{w \in \mathbb{C} \mid |w - \gamma| \leq \sqrt{2}\}$ and does not take the value 0 in D. Hence by the maximal principle of holomorphic functions we have

$$\max_{|w-\gamma|<\sqrt{2}} |g(w)| < \max_{|w-\gamma|=\sqrt{2}} |g(w)| = 1 - \frac{4}{(\gamma + \sqrt{2})^2}$$

and

$$\min_{|w-\gamma|<\sqrt{2}} |g(w)| > \min_{|w-\gamma|=\sqrt{2}} |g(w)| = 1 - \frac{4}{(\gamma - \sqrt{2})^2}.$$

Since $|ad| = |x^2/(x^2-4)| = 1/|g(x)|$, we have the desired inequalities. □

We put

$$\frac{(\gamma+\sqrt{2})^2}{(\gamma+\sqrt{2})^2-4} = t_1 \quad \text{and} \quad \frac{(\gamma-\sqrt{2})^2}{(\gamma-\sqrt{2})^2-4} = t_2.$$

It is easy to see that $1 < t_1 < t_2 < 2$ and that $f(t)$ is a decreasing function in the interval (t_1, t_2). Hence we have

$$f(t) \leq f(t_1) = \frac{(\gamma+\sqrt{2})^2((\gamma+\sqrt{2})^2-2)}{(\gamma+\sqrt{2})^2-4}.$$

This is (8.1). Next, we shall show (8.2). We need two more lemmas.

Lemma 8.3. $|\alpha| > \dfrac{\gamma+\sqrt{2}}{\sqrt{2}}$.

PROOF. Since $|\alpha| + |\beta| \geq |x| > \gamma - \sqrt{2}$ by (4.1), we have

$$|\alpha|^2 - (\gamma-\sqrt{2})|\alpha| + 1 > 0.$$

Since $|\alpha| > 1$, this inequality implies

$$|\alpha| > \frac{\gamma-\sqrt{2}+\sqrt{(\gamma-\sqrt{2})^2-4}}{2}.$$

Now, it is easy to see that

$$\frac{\gamma-\sqrt{2}+\sqrt{(\gamma-\sqrt{2})^2-4}}{2} > \frac{\gamma+\sqrt{2}}{\sqrt{2}}.$$

Hence the desired inequality holds. □

Lemma 8.4. $|a|^2, |d|^2 > \dfrac{(\gamma+\sqrt{2})^2\left(\gamma+\sqrt{2}-\sqrt{(\gamma+\sqrt{2})^2-4}\right)^2}{4((\gamma+\sqrt{2})^2-4)}$.

PROOF. From the inequality $|y-\gamma| < \sqrt{2}$ we have

$$|d| < \gamma + \sqrt{2} + |a|.$$

Hence by Lemma 8.2 we have

$$\frac{(\gamma+\sqrt{2})^2}{(\gamma+\sqrt{2})^2-4} < |ad| < (\gamma+\sqrt{2})|a| + |a|^2,$$

from which we have the inequality for $|a|^2$ easily. The proof for $|d|^2$ is similar. □

If $n \geq 3$ then

$$A(n) > |\alpha|^{2n}|a|^2 \geq |\alpha|^6|a|^2$$

and if $n \leq -3$ then

$$A(n) > |\beta|^{2n}|d|^2 \geq |\alpha|^6|d|^2.$$

Hence, by Lemmas 8.3 and 8.4, we obtain (8.2).

By (8.1) and (8.2) we see that, to show $f(|ad|) < A(n)$, it suffices to show

$$((\gamma+\sqrt{2})^2-2) < \frac{(\gamma+\sqrt{2})^6\left(\gamma+\sqrt{2}-\sqrt{(\gamma+\sqrt{2})^2-4}\right)^2}{32}$$

or

$$2((\gamma+\sqrt{2})^2-2)\left(\gamma+\sqrt{2}+\sqrt{(\gamma+\sqrt{2})^2-4}\right)^2 < (\gamma+\sqrt{2})^6. \tag{8.3}$$

Since

$$2((\gamma+\sqrt{2})^2-2)\left(\gamma+\sqrt{2}+\sqrt{(\gamma+\sqrt{2})^2-4}\right)^2 < 8(\gamma+\sqrt{2})^4$$

and $8 < (\gamma+\sqrt{2})^2$, we see that (8.3) hold. Hence we have $f(|ad|) < A(n)$.

Thus by Proposition 8.1 we see that (3.1) holds for the case $|n| \geq 3$.

Therefore, we have checked (3.1) for all integers n, so that by Theorem 3.1 we have completed the proof of Theorem 1.1. Hence we have also completed the proof of Theorem 1.2.

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