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Locally Conformal Almost Cosymplectic Manifolds Endowed with a Skew-Symmetric Killing Vector Field

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Abstract

We study a locally conformal almost cosymplectic manifold M which carries a horizontal skew-symmetric Killing vector field X. Such X defines a relative conformal cosymplectic transformation of the conformal cosymplectic 2-form Ω of M and the square of its length is both an isoparametric function and an eigenfunction of the Laplacian.

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1 Preliminaries

Let (M, g) be an oriented n-dimensional Riemannian C^∞-manifold and ∇ be the covariant differential operator with respect to the metric tensor g. Let ΓTM be the set of sections of the tangent bundle and $\flat : TM \to T^*M$ and $\natural = \flat^{-1}$ the classical musical isomorphisms defined by g. We denote by $A^q(M, TM)$ the set of all vector valued q-forms, $q < \dim M$.

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A vector field U is said to be *exterior concurrent* if it satisfies

$$\nabla^2 U = \alpha \wedge dp \in A^2(M, TM), \quad \alpha \in \Lambda^1(M, TM), \tag{1.1}$$

where $\alpha = \lambda U^\flat$ for a certain $\lambda \in \Lambda^0$ and it is called a *concurrence form* ([MRV], [PRV], [R2]).

In (1.1), α is called the *concurrence form* and is defined by

$$\alpha = \lambda U^\flat, \quad \lambda \in \Lambda^0 M.$$

A function $f : M \to \mathbf{R}$ is *isoparametric* if $\|\nabla f\|$ and $div(\nabla f)$ are functions of f ([W]).

Let $\mathcal{O} = \{e_A \,|\, A = 1, \ldots, n\}$ be a local field of orthonormal frame over M and let $\mathcal{O}^* = \{\omega^A\}$ be its associated coframe. Then the soldering form dp is expressed by $dp = \omega \otimes e$. Also, the Cartan's structure equations written in indexless manner are

$$\nabla e = \theta \otimes e, \tag{1.2}$$

$$d\omega = -\theta \wedge \omega, \tag{1.3}$$

$$d\theta = -\theta \wedge \theta + \Theta. \tag{1.4}$$

In the above equations, θ (resp. Θ) are the *local connection forms* in the tangent bundle TM (resp. the *curvature forms* on M).

A $(2m+1)$-dimensional *locally conformal almost cosymplectic manifold M with structure* $(\phi, \Omega, \xi, \eta, g)$ is defined by

$$d\Omega = 2\omega \wedge \Omega, \quad \eta = \omega \wedge \eta,$$

for certain 1-form ω, where ϕ is an endomorphism of the tangent bundle TM of square -1, Ω is the structure 2-form, which is called a *locally conformal almost cosymplectic* 2*-form*, Ω a conformal cosymplectic 2-form of rank $2m$, ξ the Reeb vector field and η the Reeb covector field.

It is known that the 1-form ω from the above equation is a closed 1-form which is called the *characteristic form* associated with the locally conformal almost cosymplectic structure ([MMR]).

In addition, if M is endowed with a quasi-Sasakian structure defined by a field ϕ of endomorphism of its tangent space and ω satisfies $\omega = -\eta$, then M is called an *almost cosymplectic* -1*-manifold.* Let $D_p^\top$ (resp. $D_p^\perp$) be a set of all tangent vectors at p which are orthogonal to (resp. proportional to) ξ_p. Then we may split the tangent space T_pM of M at $p \in M$ as $T_pM = D_p^\top \oplus D_p^\perp$.

We can construct the distribution $D : p \to D_p^\top = \{X; \eta_p(X_p) = 0\}$, called the *horizontal distribution* and the distribution $D^\perp : p \to D_p^\perp = \{\xi_p\}$, called the *vertical distribution.*

In almost cosymplectic -1-manifold M, one has the following (see, for instance, [MMR], [OR])

$$d\Omega = -2\eta \wedge \Omega, \quad \Omega(Z, Z') = g(\phi Z, Z'), \tag{1.5}$$

$$(\nabla_{Z'}\phi)Z = \eta(Z)\phi Z' + g(\phi Z, Z')\xi, \tag{1.6}$$

$$\nabla\xi = -dp + \eta \otimes \xi, \tag{1.7}$$

$$d\eta = 0. \tag{1.8}$$

A vector field X is called a *horizontal skew-symmetric Killing vector field with generatives* ξ if it satisfies

$$\nabla X = \xi \wedge X, \quad \eta(X) = 0. \tag{1.9}$$

Then we have

Lemma 1. *Let X be a horizontal skew-symmetric Killing vector field. If we put $2l = \|X\|^2$, then we have the following properties:*
i) *$2l$ is an isoparametric function,*
ii) *grad $2l$ defines an infinitesimal concircular transformation and*
iii) *l is an eigenfunction of the Laplacian Δ.*

Also, we have

Lemma 2. *The above vector field X satisfies the following*

$$\nabla^3 X = 2(X^\flat \wedge \eta) \wedge dp,$$

i.e., by definition, X is a 2-exterior concurrent vector field, and

$$d(\mathcal{L}_X\Omega) = -2\eta \wedge \mathcal{L}_X\Omega,$$

i.e., by definition, X defines a relative almost cosymplectic transformation of Ω ($\mathcal{L}_X\Omega$ is exterior recurrent with -2η as recurrence form).

Proofs of the above lemmas will be given in the next section.

2 Main Result

We assume in this paper that a vector field X is a skew-symmetric Killing vector field having the Reeb vector field ξ as generative ([R2]), i.e.,

$$\nabla X = \xi \wedge X, \tag{2.1}$$

or, equivalently,

$$\nabla X = X^\flat \otimes \xi - \eta \otimes X. \tag{2.2}$$

Let $\mathcal{O} = \{e_A \,|\, A = 1, \ldots, 2m+1\}$ be a local field of orthonormal frame over M and let $\mathcal{O}^* = \{\omega^A\}$ be its associated coframe and we assume that $e_{2m+1} = \xi$ and $\omega^{2n+1} = \eta$.

We assume that X is a horizontal vector field ($\eta(X) = 0$). Then the vector field X is written as

$$X^\flat = \sum_{a=1}^{2m} X^a \omega^a \tag{2.3}$$

and

$$\nabla X = (dX^a + X^b \theta_b^a) \otimes e_a + X^\flat \otimes \xi, \quad a, b = 1, \ldots, 2m. \tag{2.4}$$

Hence, by (2.2), one obtains by a standard calculation

$$dX^a + X^b \theta_b^a = X^a \eta \tag{2.5}$$

and setting

$$2l = \|X\|^2, \tag{2.6}$$

one derives from (2.5)

$$dl = -2l\eta, \tag{2.7}$$

which is concordance with (1.8). Next, from (2.7), one has $grad\ l = -2l\xi$, which imply

$$\|grad\ l\|^2 = 4l^2, \tag{2.8}$$

and

$$div(grad\, l) = 4ml, \tag{2.9}$$

which say that the length $2l$ of the vector field X is an isoparametric function.

In addition, one has

$$g(\nabla_Z \; grad \; l, Z') = 2lg(Z, Z') \tag{2.10}$$

for any $Z, Z' \in \Gamma TM$. This means, by definition, that $grad\ l$ defines an *infinitesimal concircular transformation* of a vector field Z ([MRV]).

In the same order of ideas, one gets

$$\Delta l = 4ml, \tag{2.11}$$

i.e., l is eigenfunction of the Laplacian Δ.

In this way, Lemma 1 has been proved.

Next, since ∇ acts inductively, one derives

$$\nabla^2 X = X^\flat \wedge dp - 2(\eta \wedge X^\flat) \otimes \xi. \tag{2.12}$$

This means that the distinguished vector field X is a quasi-exterior concurrent vector field.

Further, one has

$$\nabla(\nabla^2 X) = \nabla^3 X = 2(X^\flat \wedge \eta) \wedge dp, \tag{2.13}$$

i.e., by definition, X is a 2-exterior concurrent vector field ([MRV]).

Finally, regarding the conformal cosymplectic form Ω, we define β

$$\beta = i_X \Omega = \sum_{a=1}^{n} (X^a \omega^{a^*} - X^{a^*} \omega^a). \tag{2.14}$$

Then, since

$$\mathcal{L}_X \Omega = d(i_X \Omega) + 2\eta \wedge i_X \Omega, \tag{2.15}$$

one may write

$$\mathcal{L}_X \Omega = d\beta + 2\eta \wedge \beta, \tag{2.16}$$

and, by exterior differentiation, one derives

$$d(\mathcal{L}_X \Omega) = -2\eta \wedge \mathcal{L}_X \Omega. \tag{2.17}$$

Then, the relation (2.17) affirms that the distinguished vector field X defines a *relative conformal cosymplectic transformation of* Ω (see [R1]).

In this way, Lemma 2 has been proved.

Summing up, and making use of Lemmas 1 and 2, we proved the following.

Theorem. *Let $M(\phi, \Omega, \xi, \eta, g)$ be a $(2m+1)$-dimensional locally conformal almost cosymplectic C^∞-manifold, with Reeb vector field ξ. Then, if M carries a horizontal vector field X such that X is a skew-symmetric Killing vector field, one has the properties:*

i) *$2l = ||X||^2$ is an isoparametric function; moreover, grad l is an infinitesimal concircular transformation and l is an eigenfunction of the Laplacian Δ;*

ii) *X is a closed vector field which is 2-exterior concurrent, i.e.,*

$$\nabla^3 X = 2(X^\flat \wedge \eta) \wedge dp;$$

iii) *X defines a relative conformal cosymplectic transformation of Ω, i.e.*

$$d(\mathcal{L}_X \Omega) = -2\eta \wedge \mathcal{L}_X \Omega.$$

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