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Bijections between topologies*

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Abstract

A homeomorphism induces a natural bijection between the family of open sets. We consider the inverse problem and obtain an answer and counter examples.

0 Introduction

Let X and Y be topological spaces, $f : X \to Y$ a homeomorphism. Then, f induces a natural bijection $f_* : \mathcal{O}(X) \to \mathcal{O}(Y)$, where $\mathcal{O}(\ )$ denotes the family of open sets of the topological space. Indeed, $f_*(U)$ is to be the image $f(U)$ for any open set U of X. Notice that the bijection f_* preserves the inclusion relation, that is, $U \subset V \Longleftrightarrow f_*(U) \subset f_*(V)$.

Now we consider the inverse problem : "Which bijection $\psi : \mathcal{O}(X) \to \mathcal{O}(Y)$ is induced by a homeomorphism ?"

Our answer is the following. "Assume the topologies $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ satisfy the T_1-axiom, that is, each one-point set is closed. If a bijection $\psi : \mathcal{O}(X) \to \mathcal{O}(Y)$ preserves the inclusion relation, then ψ is induced by a homeomorphism $f : X \to Y$."

Next, we will give countably many topologies on the set of natural numbers such that between any two of them, there is a bijection which preserves the inclusion relation but it is not induced by any homeomorphism.

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1 Main result

Theorem *Let* $(X, \mathcal{O}(X))$ *and* $(Y, \mathcal{O}(Y))$ *be* T_1*-topological spaces. Let* $\psi : \mathcal{O}(X) \to \mathcal{O}(Y)$ *be a bijection which preserves the inclusion relation. Then,* ψ *is induced by a homeomorphism* $f : X \to Y$*.*

Proof. Let $\psi : \mathcal{O}(X) \to \mathcal{O}(Y)$ be a bijection which preserves the inclusion relation. Then, we easily see the following equations :

$$\begin{aligned} &\psi(X) = Y, \quad \psi(U \cap V) = \psi(U) \cap \psi(V), \\ &\psi(\phi) = \phi, \quad \psi(\cup_\alpha U_\alpha) = \cup_\alpha \psi(U_\alpha). \end{aligned}$$

Moreover, we assume the topologies $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ satisfy the T_1-axiom. Then, we see that each $x \in X$ corresponds uniquely to an element $y \in Y$ satisfying the relation :

$$\psi(\{x\}^c) = \{y\}^c.$$

Here, $\{x\}^c$ denotes the complement of the one-point set $\{x\}$.

The above relation induces a mapping $f : X \to Y$ by $\psi(\{x\}^c) = \{f(x)\}^c$. By the assumption, we see f is a bijection. Moreover, we see the following :

$$x \in U \Longleftrightarrow f(x) \in \psi(U).$$

Hence we see

$$f(U) := \{f(x) \mid x \in U\} = \psi(U)$$

for any $U \in \mathcal{O}(X)$. The equation shows that f is an open map and

$$f^{-1}(V) = \psi^{-1}(V)$$

for any $V \in \mathcal{O}(Y)$. The last equation shows that f is continuous. And hence $f : X \to Y$ is a homeomorphism.

Therefore, we see $\psi = f_*$, that is, ψ is induced by the homeomorphism $f : X \to Y$. q.e.d.

2 Counter examples

Denote by $\mathbf{N}$ the set of all natural numbers. Put

$$U_t = \{n \in \mathbf{N} \mid n > t\}, \quad (t = 1, 2, 3, \cdots).$$

Define

$$\mathcal{O}_p = \{\mathbf{N},\ \phi\} \cup \{U_t \mid t \geq p\}, \quad (p = 1, 2, 3, \cdots).$$

Then we see $\mathcal{O}_p$ $(p = 1, 2, 3, \cdots)$ is a topology on $\mathbf{N}$.

Let p, q be natural numbers such that $p \leq q$. Then, the bijection $\psi : \mathcal{O}_p \to \mathcal{O}_q$ is defined by $\psi(U_t) = U_{t+q-p}$. Clearly, ψ preserves the inclusion relation.

On the other hand, there exists the smallest non-empty closed subset in the topological space $(\mathbf{N}, \mathcal{O}_p)$, which is the set $\{1, 2, \cdots, p\}$ of p elements. Therefore, the topological space $(\mathbf{N}, \mathcal{O}_p)$ is not homeomorphic to the topological space $(\mathbf{N}, \mathcal{O}_q)$ for any $p \neq q$, and hence the bijection $\psi : \mathcal{O}_p \to \mathcal{O}_q$ is not induced by any homeomorphism, if $p \neq q$.

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