Optimal Supply against Fluctuating Demand

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1 Introduction

Contrary to the common sense in economy, the optimal supply does not always agree with the average demand. This was pointed out by Sornette et al.(1999), who analyzed a bakery model where a daily demand fluctuates. They derived the formula of the optimal supply, assuming the fluctuations obey a uniform distribution. Although their result is reasonable and meaningful, it is not clear how the result will change if we consider a different distribution. In this note, we extend the model to general probability distributions. In particular, we calculate the optimal supply for Gaussian distribution, which is more realistic in a market.

2 Model and analysis of Sornette, Stauffer and Takayasu

In this section we review the model and analysis of Sornette et al. Let us consider a bakery shop where baked croissants are sold every day. A question is how many croissants should be baked a day to make the maximal profit. We define the variables as follows.

- x : the selling price of a croissant.
- y : its production cost.
- s : the production number of croissants per day (supply).
- n : the number of croissants requested by customers per day (demand).
- D: the average demand, i.e., $D \equiv < n >$.

The expectation of the total profit $L(s)$ is given by

$$L(s) \equiv < x \min(n, s) - ys > = x \int_0^s nP(n)dn + xs \int_s^\infty P(n)dn - ys, \tag{1}$$

where $P(n)$ is the probability distribution of n.

Sornette et al. assumed, for simplicity, a uniform distribution,

$$P_u(n) \equiv \begin{cases} 1/2\delta & \text{for} \quad D-\delta \leq n \leq D+\delta \\ 0 & \text{for} \quad n < D-\delta, \ D+\delta < n. \end{cases} \tag{2}$$

Then one can integrate (1) as

$$L(s) = -\frac{x}{4\delta}\left\{s - D - \delta\left(1-\frac{2y}{x}\right)\right\}^2 + (x-y)\left(D - \frac{\delta y}{x}\right). \tag{3}$$

$L(s)$ takes the maximam value when s takes

$$s_{\max} \equiv D + \delta\left(1 - \frac{2y}{x}\right). \tag{4}$$

This shows that, if the cost per price, y/x, is larger (smaller) than half, the optimal demand, $s_{\max}$, is smaller (larger) than the average demand, D.

3 Optimal supply for Gaussian distribution

Let us re-analyze (1) for general probability distributions. We do not have to integrate (1) directly, because what we want to know is the optimal supply $s_{\max}$, which is given by

$$\frac{dL}{ds}(s_{\max}) = x\int_{s_{\max}}^{\infty} P(n)dn - y = 0. \tag{5}$$

This equation is very simple but quite general; it gives the optimal supply for general probability distributions. For arbitrary distribution function $P(n)$, one can easily obtain the optimal supply $s_{\max}$ by integrating (5) numerically.

If we assume Gaussian distribution,

$$P_G(n) \equiv \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(n-D)^2}{2\sigma^2}\right], \tag{6}$$

we can integrate (5) analytically as follows. The integration in (5) is evaluated as

$$\int_{s_{\max}}^{\infty} P_G(n)dn = \frac{1}{2} - \frac{1}{2}\mathrm{Erf}\left(\frac{s_{\max}-D}{\sqrt{2}\sigma}\right), \tag{7}$$

where Erf is the error function, which is defined as

$$\mathrm{Erf}\ z \equiv \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt. \tag{8}$$

Then we arrive at the formula of the optimal supply for Gaussian distribution,

$$\begin{aligned}\frac{s_{\max}-D}{\sigma} &= \sqrt{2}\mathrm{Erf}^{-1}\left(1-\frac{2y}{x}\right)\\ &\fallingdotseq \sqrt{\frac{\pi}{2}}\left(1-\frac{2y}{x}\right)+\frac{\sqrt{2}\pi^{\frac{3}{2}}}{24}\left(1-\frac{2y}{x}\right)^3+O\left[\left(1-\frac{2y}{x}\right)^5\right] \quad \text{(Gaussian)}. \qquad (9)\end{aligned}$$

Because the cost is usually in the range $0.3x < y < 0.7x$, which reads $|1 - 2y/x| < 0.4$, the first-order approximation in (9) is sufficient in most cases.

For reference, we rewrite the result for the uniform distribution (4) . Because the variance of the uniform distribution (2) is evaluated as $\sigma^2 = \delta^2/3$, (4) is rewritten as

$$\frac{s_{\max}-D}{\sigma} = \sqrt{3}\left(1-\frac{2y}{x}\right) \quad \text{(uniform)}. \qquad (10)$$

We see that the difference between $\sqrt{\pi/2} \approx 1.25$ in (9) and $\sqrt{3} \approx 1.73$ in (10) is not negligible. Contrary to the speculation of Sornette *et al.*, however, the critical value of the cost-to-price ratio, $y/x = 1/2$, is unchanged. Because Gaussian distribution is more realistic, our simple formula (9) is useful in a real market to earn the largest income on average.

References

[1] SORNETTE, D., STAUFFER, D. and TAKAYASU, H. (1999). Market Fluctuations II:multiplicative and percolation models, size effects and predictions. In A. Bunde and H.-J. Schellnhuber (eds.). *Proceedings of the Workshop 'Facets of Universality: Cli-mate, Biodynamics and Stock Markets'*. Giessen University.(http://arXiv.org/abs/cond-mat?9909439).

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