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On Enumeration of N-colored Trees

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Abstract

First, it is introduced that the number of labeled n-colored trees ($T\alpha_1\alpha_2\cdots\cdots\alpha_n$) with n distinctly-colored point-subsets is determined by

$$T\alpha_1\,\alpha_2\cdots\cdots\alpha_n=\alpha^{n-2}\prod_{i=1}^{n}(\alpha-\alpha_i)^{\alpha_i-1}$$

where $\alpha=\sum_{i=1}^{n}\alpha_i$. Next, several theorems and lemmata on these trees are proved and lastly their practical examples for electrical networks are described.

1. INTRODUCTION

An n-colord graph⁽¹⁾ is one of the most convenient concepts for social and technical matching-problems, and a tree being the subgraph in this graph is called the n-colored tree. Generally the concept of trees in a graph is the most fundamental one for graph theory and its applications, and also it is very important that the numbers of labeled n-colored graphs and trees are determined algebraically. Already the numbers of all the labeled n-colored graphs and all the labeled 2-colored trees have been introduced by F. Harary⁽¹⁾. Here are introduced the numeration of the labeled n-colorred trees, on which the several theorems are described with some applications of them for electrical network analysis.

2. N-COLORED GRAPH

A graph

$$G=(P,\ B) \tag{1}$$

with the *point* set P is a family B of pairings

$$\kappa=(a,\ b), \qquad a,\ b\in P \tag{2}$$

which indicates which points shall be considered to be connected. In keeping with the geometric image of a graph, each pairing defined in Eq. (2) shall be called a *branch* of the graph; the points a and b are called the *endpoints* of the branch κ. Especially in this paper, it is defined that *parallel branches* do not exist in a graph, unless otherwise provided. Here, the parallel branches are branches with same endpoints. In case that all the branches in a graph have the directions, the graph is called a *directed graph* and shortly a *digraph.*

An *n-colored graph*⁽¹⁾ (or *n-partite graph*)

$$G_{(n)}=(P,\ B) \tag{3}$$

consists of n sets of points

$$P=\{P_i\}, \qquad i=1,2,\cdots\cdots,n \tag{4}$$

together with a specified branch set

$$B=\{B_{ij}\}, \qquad i,\ j=1,\ 2,\cdots\cdots,\ n, \tag{5}$$

where each branch subset B_{ij} is a family of connected pairings

$$\kappa_{ij}=(a_i,\ a_j), \qquad a_i\in P_i,\ a_j\in P_j,\ i\neq j \tag{6}$$

and κ_{ij} $(i=j)$ does not exist.

Here, an n-colored graph with point subsets P_i consisting of α_i points is denoted by

$$G_{(n)}=G\alpha_1\alpha_2\cdots\cdots\alpha_n \tag{7}$$

and the point sets P_i are respectively painted in different colors. In Fig. 1, its examples are found.

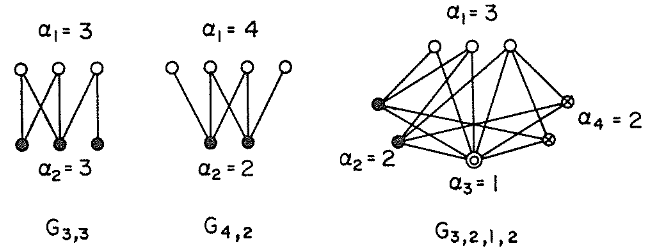

Fig. 1 N-colored graphs

Generally, the well-known graph of importance is the *complete graph* K_n (with n points) whose branches are all the pairs a and b ($\in P$). Similarly, a *complete n-colored graph* is the n-colored graph whese branches are all the pairs of possible associations for two different points a_i and a_j respectively in different point subsets P_i and P_j.

A complete n-colored graph is denoted by $K\alpha_1\alpha_2\cdots\cdots\alpha_n$ as shown in Fig. 2. Obviously, the complete n-colored graph $K_{1,1,\cdots\cdots,1}$ with n differently-colored points is equal to a complete graph K_n.

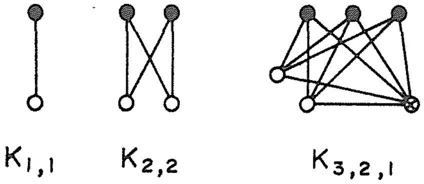

Fig. 2 Complete n-colored graphs

Denoting the series of i numbers m, m, ⋯, m by m^i, it follows that

$$\underbrace{K_{1,1,\cdots,1}}_{n} = K_1 n = K_n \tag{8}$$

Provided that neglecting all the branches in a subgraph G' ($\subset G$) is denoted by $G- G'$, the following relations are proved easily:

$$K_n - K_i = K_1{}^{n-i},\ i, \qquad n > i > 1 \tag{9}$$

And further, denoting j complete subgraphs K_i without common parts by jK_i, we have

$$K_n - jK_i = K_1{}^{n-ij},\ i^j \tag{10}$$

From the definition of the complete n-colored graph, we shall show:

THEOREM 1 "*The number of branches in a complete n-colored graph* $K\alpha_1\alpha_2\cdots\cdots\alpha_n$ *is*

$$b\,(K\alpha_1\alpha_2\cdots\cdots\alpha_n) = \sum_{i=1}^{n} \alpha_i\,\alpha_i\,/\,2 \tag{11}$$

where

$$\alpha = \sum_{i=1}^{n} \alpha_i \tag{12}$$

$$\widehat{\alpha}_i = \alpha - \alpha_i \quad \text{''} \tag{13}$$

From the above THEOREM 1, the well-known(1) LEMMA la is introduccd as follows : Putting that $\alpha_i = 1$, $\alpha_i = (n-1)$ in Eq. (11), we have :

LEMMA la "*The number of branches in a complete graph* K_n ($= K_1 n$) *is*

$$b\,(K_n) = n\,(n-1)\,/\,2 \text{ ''} \tag{14}$$

Denoting the number of Eq. (11) by b, the number of the n-colored graphs without parallel branches is determined by

$$N\ (G\alpha_1\,\alpha_2\cdots\cdots\alpha_n) = \sum_{i=0}^{b} \binom{b}{i} \tag{15}$$

Here, on the polynomial equation

$$(1+x)^b = \sum_{i=0}^{b} \binom{b}{i} x^i, \tag{16}$$

putting $x=1$, then

$$\sum_{i=0}^{b} \binom{b}{i} = 2^b \tag{17}$$

$$\therefore N(G\alpha_1\alpha_2\cdots\cdots\alpha_n) = 2^b \tag{18}$$

Therefore we have :

THEOREM 2 *"The number of n-colored graphs* $G\alpha_1\alpha_2\cdots\cdots\alpha_n$ *is determincd by*

$$N(G\alpha_1\alpha_2\cdots\cdots\alpha_n) = 2^{\sum_{i=1}^{n} \alpha_i \hat{\alpha}_i / 2} \text{ ''} \tag{19}$$

LEMMA 2a *"The number of n-graphs* G_n *of order n is determined by*

$$N(G_n) = 2^{n(n-1)/2} \text{ ''} \tag{20}$$

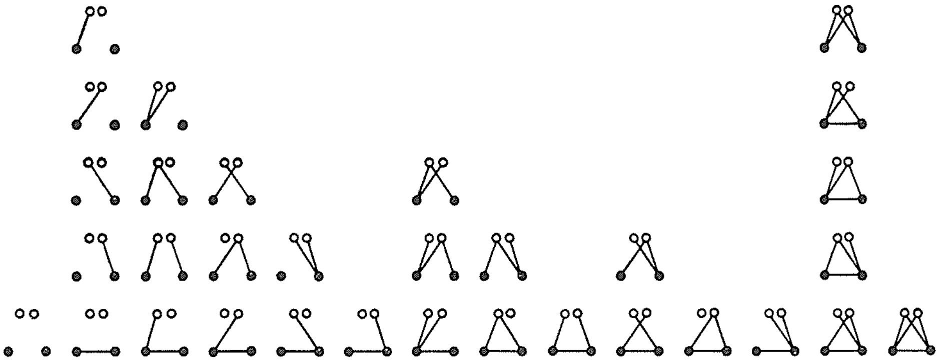

Fig. 3 3-colored graphs $G_{2,1,1}$

The 3-colored graphs $G_{2,1,1}$ are shown in Fig. 3, and the number of them is given by

$$N(G_{2,1,1}) = 2^{(2\cdot2+1\cdot3+1\cdot3)/2} = 2^5 = 32$$
$$= \binom{5}{0}+\binom{5}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4}+\binom{5}{5} = 1+5+10+10+5+1 = 32$$

The branch in a *digraph* is called a directed branch and shortly a *dibranch*. Since this dibranch can be given one of two directions, the number of n-colored digraphs $G\alpha_1\alpha_2\cdots\cdots\alpha_n$ is determined by

$$N(G\alpha_1\alpha_2\cdots\cdots\alpha_n) = \sum_{i=0}^{b} 2^i \binom{b}{i} \tag{21}$$

where $b = \sum_{i=1}^{n} \alpha_i \hat{\alpha}_i / 2$

Here, we know that

$$(1+x)^b = \sum_{i=0}^{b} \binom{b}{i} x^i$$

Supposing that $x=2$ in the above equation,

$$3^b = \sum_{i=0}^{b} 2^i \binom{b}{i} \tag{22}$$

Therefore, it follows :

LEMMA 2b *"The number of n-colored digraphs* $\tilde{G}\alpha_1\alpha_2\cdots\cdots\alpha_n$ *is determined by*

$$N\ (\tilde{G}\alpha_1\,\alpha_2\cdots\cdots\alpha_n) = 3^{\sum_{i=1}^{n} \alpha_i\,\bar{\alpha}_i\,/\,2}\ \text{''} \tag{23}$$

LEMMA 2c *"The number of digraphs* G_n *af order n is determined by*

$$N\ (\tilde{G}_n) = 3^{n\,(n-1)\,/\,2}\ \text{''} \tag{24}$$

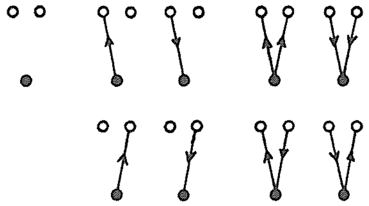

Fig. 4 2-colored digraphs $G_{2,1}$

For example, the 2-colored digraphs $\tilde{G}_{2,1}$ are shown in Fig. 4 , and the number of them is given by

$$N\ (\tilde{G}_{2,1}) = 3^{(2\cdot1+1\cdot2)\,/\,2} = 9$$
$$= 2^0\binom{2}{0} + 2^1\binom{2}{1} + 2^2\binom{2}{2} = 1+4+4=9$$

3. ENUMERATION OF TREES BY DETERMINANT METHOD

A *tree* is a connected graph that has no loop. The number of the spanning trees in a graph G is determined by (2)

$$T = \det(\mathbf{D}\ \mathbf{D}') = \det\ [D^{ab}] \tag{25}$$

where **D** : the connection matrix $[(\alpha-1)\times\beta-\text{matrix}]$

D′ : the transposed matrix of **D**

α : number of points in the graph G

β : number of branches in G

Here, element D^{ab} in the ath row and the bth column of T is shown as *the degree of point a (or the number of branches incident to point a), when a=b*

and as

the negative of the number of branches incident to pair a and b, when $a \neq b$.

For example, the determinant T namely the number of trees in graph G of Fig. 5 is given by

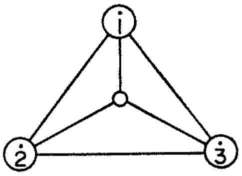

Fig. 5 Graph G

$$T = \begin{vmatrix} D^{\dot{1}\dot{1}} & D^{\dot{1}\dot{2}} & D^{\dot{1}\dot{3}} \\ D^{\dot{2}\dot{1}} & D^{\dot{2}\dot{2}} & D^{\dot{2}\dot{3}} \\ D^{\dot{3}\dot{1}} & D^{\dot{3}\dot{2}} & D^{\dot{3}\dot{3}} \end{vmatrix} = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 16$$

By the aforementioned calculation, we have the well-known theorem as follews :

THEOREM 3 "*The number of trees in a complete graph K_n of order n is given by*

$$T_n = \begin{vmatrix} n-1 & -1 & \cdots\cdots & -1 \\ -1 & n-1 & \cdots\cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots\cdots & n-1 \end{vmatrix} = n^{n-2} \text{"} \qquad (26)$$

This theorem already has been introduced by Cayley (1) and the several proofs of it have been given by Cayley, Prüfer, Harary, Kirchhoff (1) and Okada etc. (2) (3)

4. N-COLORED TREES

The trees in a n-colord graph are called *n-colored trees*, and the number of them in a complete one $K\alpha_1\,\alpha_2\cdots\cdots\alpha_n$ is given by

$$T\alpha_1\alpha_2\cdots\cdots\alpha_n=$$

$$\begin{array}{|c|c|c|c|c|l}
\begin{matrix}\widehat{\alpha}_1 & 0 & \cdots\cdots & 0\\ 0 & \widehat{\alpha}_1 & \cdots\cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots\cdots & \widehat{\alpha}_1\end{matrix} & \begin{matrix}-1 & -1 & \cdots\cdots & -1\\ -1 & -1 & \cdots\cdots & -1\\ \cdots & \cdots & \cdots & \cdots\\ -1 & -1 & \cdots\cdots & -1\end{matrix} & & [-1] & [-1] & \}\ \alpha_1\\
\hline
\begin{matrix}-1 & -1 & \cdots\cdots & -1\\ -1 & -1 & \cdots\cdots & -1\\ \cdots & \cdots & \cdots & \cdots\\ -1 & -1 & \cdots\cdots & -1\end{matrix} & \begin{matrix}\widehat{\alpha}_2 & 0 & \cdots\cdots & 0\\ 0 & \widehat{\alpha}_2 & \cdots\cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots\cdots & \widehat{\alpha}_2\end{matrix} & & [-1] & [-1] & \}\ \alpha_2\\
\hline
 & & \ddots & & & \\
\hline
[-1] & [-1] & & \begin{matrix}\widehat{\alpha}_{n-1} & 0 & \cdots\cdots & 0\\ 0 & \widehat{\alpha}_{n-1} & \cdots\cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots\cdots & \widehat{\alpha}_{n-1}\end{matrix} & \begin{matrix}-1 & -1 & \cdots\cdots & -1\\ -1 & -1 & \cdots\cdots & -1\\ \cdots & \cdots & \cdots & \cdots\\ -1 & -1 & \cdots\cdots & -1\end{matrix} & \}\ \alpha_{n-1}\\
\hline
[-1] & [-1] & & \begin{matrix}-1 & -1 & \cdots\cdots & -1\\ -1 & -1 & \cdots\cdots & -1\\ \cdots & \cdots & \cdots & \cdots\\ -1 & -1 & \cdots\cdots & -1\end{matrix} & \begin{matrix}\widehat{\alpha}_n & 0 & \cdots\cdots & 0\\ 0 & \widehat{\alpha}_n & \cdots\cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots\cdots & \widehat{\alpha}_n\end{matrix} & \}\ \alpha_n-1\\
\hline
\multicolumn{1}{c}{\underbrace{\hphantom{xxxxxxxx}}_{\alpha_1}} & \multicolumn{1}{c}{\underbrace{\hphantom{xxxxxxxx}}_{\alpha_2}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\underbrace{\hphantom{xxxxxxxx}}_{\alpha_{n-1}}} & \multicolumn{1}{c}{\underbrace{\hphantom{xxxxxxxx}}_{\alpha_n-1}} &
\end{array}$$

$$=\begin{array}{|c|c|c|c|c|c|}
\begin{matrix}1\\1\\ \vdots\\1\end{matrix} & \begin{matrix}0 & \cdots\cdots\cdots\cdots & 0\\ \widehat{\alpha}_1 & \cdots\cdots\cdots\cdots & 0\\ & \cdots\cdots\cdots\cdots\cdots\cdots & 0\\ 0 & \cdots\cdots\cdots\cdots & \widehat{\alpha}_1\end{matrix} & [-1] & & [-1] & [-1]\\
\hline
\begin{matrix}1\\1\\ \vdots\\1\end{matrix} & [-1] & \begin{matrix}\widehat{\alpha}_2 & 0 & \cdots\cdots\cdots & 0\\ 0 & \widehat{\alpha}_2 & \cdots\cdots\cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots\cdots\cdots & \widehat{\alpha}_2\end{matrix} & & [-1] & [-1]\\
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\begin{matrix}1\\1\\ \vdots\\1\end{matrix} & [-1] & [-1] & & \begin{matrix}\widehat{\alpha}_{n-1} & 0 & \cdots\cdots & 0\\ 0 & \widehat{\alpha}_{n-1} & \cdots\cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots\cdots & \widehat{\alpha}_{n-1}\end{matrix} & [-1]\\
\hline
\begin{matrix}0\\0\\ \vdots\\0\end{matrix} & [-1] & [-1] & & [-1] & \begin{matrix}\widehat{\alpha}_n & 0 & \cdots\cdots & 0\\ 0 & \widehat{\alpha}_n & \cdots\cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots\cdots & \widehat{\alpha}_n\end{matrix}\\
\hline
\multicolumn{1}{c}{} & \multicolumn{1}{c}{\underbrace{\hphantom{xxxxxxxx}}_{\alpha_1-1}} & \multicolumn{1}{c}{\underbrace{\hphantom{xxxxxxxx}}_{\alpha_2}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\underbrace{\hphantom{xxxxxxxx}}_{\alpha_{n-1}}} & \multicolumn{1}{c}{\underbrace{\hphantom{xxxxxxxx}}_{\alpha_n-1}}
\end{array}$$

$$
=\left|\begin{array}{c|c|c|c|c}
\begin{array}{c}1\\1\\\vdots\\1\end{array} &
\begin{array}{ccc}0&\cdots\cdots&0\\\widehat{\alpha}_1&\cdots\cdots&0\\\cdots&\cdots&\cdots\\0&\cdots\cdots&\widehat{\alpha}_1\end{array} &
\begin{array}{ccc}0&\cdots\cdots&0\\&&\\&&\\0&\cdots\cdots&0\end{array} & [0] & [0]\\
\hline
\begin{array}{c}1\\1\\\vdots\\1\end{array} & [-1] &
\begin{array}{cccc}\widehat{\alpha}_2+1&1&\cdots\cdots&1\\1&\widehat{\alpha}+1&\cdots\cdots&1\\\cdots&\cdots&\cdots&\cdots\\1&1&\cdots&\widehat{\alpha}_2+1\end{array} & [0] & [0]\\
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\begin{array}{c}1\\1\\\vdots\\1\end{array} & [-1] & [0] &
\begin{array}{cccc}\widehat{\alpha}_{n-1}+1&1&\cdots&1\\1&\widehat{\alpha}_{n-1}+1&\cdots&1\\\cdots&\cdots&\cdots&\cdots\\1&1&\cdots&\widehat{\alpha}_{n-1}+1\end{array} & [0]\\
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\begin{array}{c}0\\0\\\vdots\\0\end{array} & [-1] & [-1] & [-1] &
\begin{array}{cccc}\widehat{\alpha}_n&0&\cdots\cdots&0\\0&\widehat{\alpha}_n&\cdots\cdots&0\\\cdots&\cdots&\cdots&\cdots\\0&0&\cdots\cdots&\widehat{\alpha}_n\end{array}
\end{array}\right|
$$

$$
=1\cdot\widehat{\alpha}_1{}^{\alpha_1-1}\cdot
\underbrace{\left|\begin{array}{cccc}\widehat{\alpha}_2+1&1&\cdots&1\\1&\widehat{\alpha}_2+1&\cdots&1\\\cdots&\cdots&\cdots&\cdots\\1&1&\cdots&\widehat{\alpha}_2+1\end{array}\right|}_{\alpha_2}
\cdot\cdots\cdot
\underbrace{\left|\begin{array}{cccc}\widehat{\alpha}_{n-1}+1&1&\cdots&1\\1&\widehat{\alpha}_{n-1}+1&\cdots&1\\\cdots&\cdots&\cdots&\cdots\\1&1&\cdots&\widehat{\alpha}_{n-1}+1\end{array}\right|}_{\alpha_{n-1}}
\cdot
\underbrace{\left|\begin{array}{cccc}\widehat{\alpha}_n&0&\cdots\cdots&0\\0&\widehat{\alpha}_n&\cdots\cdots&0\\\cdots&\cdots&\cdots&\cdots\\0&0&\cdots\cdots&\widehat{\alpha}_n\end{array}\right|}_{\alpha_n-1}
$$

$$
=1\cdot\widehat{\alpha}_1{}^{\alpha_1-1}\cdot
\left|\begin{array}{cccc}\alpha&1&\cdots\cdots&1\\\alpha&\widehat{\alpha}_2+1&\cdots&1\\\cdots&\cdots&\cdots&\cdots\\\alpha&1&\cdots&\widehat{\alpha}_2+1\end{array}\right|
\cdot\cdots\cdot
\left|\begin{array}{cccc}\alpha&1&\cdots\cdots&1\\\sigma&\widehat{\alpha}_{n-1}+1&\cdots&1\\\cdots&\cdots&\cdots&\cdots\\\alpha&1&\cdots&\widehat{\alpha}_{n-1}+1\end{array}\right|
\cdot\widehat{\alpha}_n{}^{\alpha_n-1}
$$

$$
=1\cdot\widehat{\alpha}_1{}^{\alpha_1-1}\cdot\alpha
\left|\begin{array}{cccc}1&1&\cdots\cdots&1\\1&\widehat{\alpha}_2+1&\cdots&1\\\cdots&\cdots&\cdots&\cdots\\1&1&\cdots&\widehat{\alpha}_2+1\end{array}\right|
\cdot\cdots\cdot\alpha
\left|\begin{array}{cccc}1&1&\cdots\cdots&1\\1&\widehat{\alpha}_{n-1}+1&\cdots&1\\\cdots&\cdots&\cdots&\cdots\\1&1&\cdots&\widehat{\alpha}_{n-1}+1\end{array}\right|
\cdot\widehat{\alpha}_n{}^{\alpha_n-1}
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=1\cdot\widehat{\alpha}_1{}^{\alpha_1-1}\cdot\alpha
\left|\begin{array}{cccc}1&0&\cdots\cdots&0\\1&\widehat{\alpha}_2&\cdots\cdots&0\\\cdots&\cdots&\cdots&\cdots\\1&0&\cdots\cdots&\widehat{\alpha}_2\end{array}\right|
\cdot\cdots\cdot\alpha
\left|\begin{array}{cccc}1&0&\cdots\cdots&0\\1&\widehat{\alpha}_{n-1}&\cdots\cdots&0\\\cdots&\cdots&\cdots&\cdots\\1&0&\cdots&\widehat{\alpha}_{n-1}\end{array}\right|
\cdot\widehat{\alpha}_n{}^{\alpha_n-1}
$$

$$
=1\cdot\widehat{\alpha}_1{}^{\alpha_1-1}\cdot\alpha\widehat{\alpha}_2{}^{\alpha_2-1}\cdot\cdots\cdot\widehat{\alpha}_{n-1}{}^{\alpha_{n-1}-1}\cdot\widehat{\alpha}_n{}^{\alpha_n-1}
$$

$$=\alpha^{n-2}\,\widehat{\alpha}_1^{\,\alpha_1-1}\,\widehat{\alpha}_2^{\,\alpha_2-1}\cdots\cdots\widehat{\alpha}_n^{\,\alpha_n-1}$$

$$=\alpha^{n-2}(\alpha-\alpha_1)^{\alpha_1-1}(\alpha-\alpha_2)^{\alpha_2-1}\cdots\cdots(\alpha-\alpha_n)^{\alpha_n-1}$$

where $\alpha=\alpha_1+\alpha_2+\cdots\cdots+\alpha_n$

$\widehat{\alpha}_i=\alpha-\alpha_i$ (i=1, 2, ……, n)

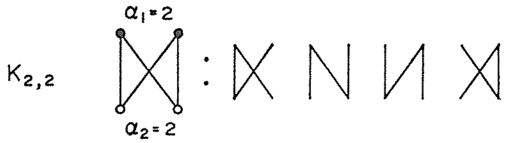
Then we have the following theorem :

THEOREM 4 *"The number of trees in a complete n-colored graph* $K\alpha_1\alpha_2\cdots\alpha_n$ *is given by*

$$T\alpha_1\alpha_2\cdots\alpha_n=\alpha^{n-2}\,\widehat{\alpha}_1^{\,\alpha_1-1}\,\widehat{\alpha}_2^{\,\alpha_2-1}\cdots\cdots\widehat{\alpha}_n^{\,\alpha_n-1}$$

$$=\alpha^{n-2}\prod_{i=1}^{n}(\alpha-\alpha_i)^{\alpha_i-1}\ , \tag{27}$$

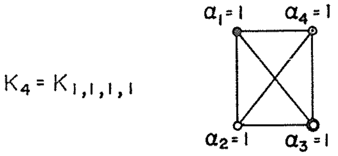
where $\alpha=\sum_{i=1}^{n}\alpha_i$ "



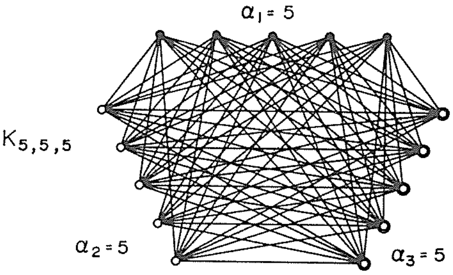
(a) $T_{2,2}=4^0\cdot 2^1\cdot 2^1=4$

(b) $T_{2,1,1}=4^1\cdot 2^1\cdot 3^0\cdot 3^0=8$

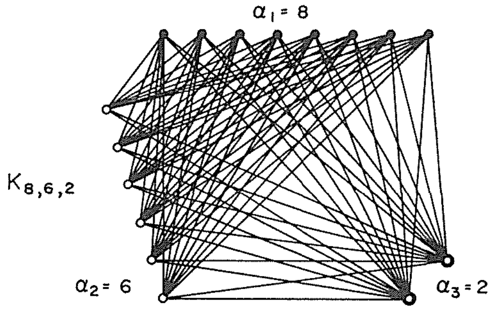
Fig. 6 Number of trees in complete n-colored graphs



(a) $T_4=T_{1,1,1,1}=4^2\cdot 3^0\cdot 3^0\cdot 3^0\cdot 3^0=4^2=16$



(b) $T_{5,5,5}=15^1\cdot 10^4\cdot 10^4\cdot 10^4=15\times 10^{12}$



(c) $T_{8,6,2} = 8^7 \cdot 10^5 \cdot 14^1 \cdot 16^1 = 469762048 \times 10^5$

Fig. 7 Number of trees in complete n-colored graphs

Provided that $\alpha_1 = 1$, theorem 3 is introduced. See examples in Fig. 6 and Fig, 7.

A comlete graph without the arbitrary branch is called a *semicomplete graph*, and is denoted by K'_n. Similarly a complete n-colored graph without the arditrary branch is called a *semicomplete n-colored graph*, and denoted by $K'\alpha_1\alpha_2\cdots\alpha_n$ • Especially the one without the branch spanning the points in last two point sets is denoted by $K\alpha_1\alpha_2\cdots\cdots\overline{\alpha_{n-1}\alpha_n}$ •

The numder of trees in the semicomplete n-coloreb graph $K\alpha_1\alpha_2\cdots\cdots\overline{\alpha_{n-1}\alpha_n}$ is written as $T\alpha_1\alpha_2\cdots\cdots\overline{\alpha_{n-1}\alpha_n}$, then we have $T\alpha_1\alpha_2\cdots\cdots\overline{\alpha_{n-1}\alpha_n}$

$$
= \begin{array}{|c|c|c|c|l}
\begin{matrix} \hat{\alpha}_1 & 0 & \cdots\cdots & 0 \\ 0 & \hat{\alpha}_1 & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & \hat{\alpha}_1 \end{matrix} & \begin{matrix} -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & -1 \end{matrix} & [-1] & [-1] & \Big\}\,\alpha_1 \\ \hline
\begin{matrix} -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & -1 \end{matrix} & \begin{matrix} \hat{\alpha}_2 & 0 & \cdots\cdots & 0 \\ 0 & \hat{\alpha}_2 & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & \hat{\alpha}_2 \end{matrix} & [-1] & [-1] & \Big\}\,\alpha_2 \\ \hline
 & & \ddots & & \\ \hline
[-1] & [-1] & \begin{matrix} \hat{\alpha}_{n-1} & 0 & \cdots & 0 & 0 \\ 0 & \hat{\alpha}_{n-1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \hat{\alpha}_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \alpha_{n-1}-1 \end{matrix} & \begin{matrix} -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 \end{matrix} & \Bigg\}\,\alpha_{n-1} \\ \hline
[-1] & [-1] & \begin{matrix} -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & \cdots & -1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & -1 & -1 \end{matrix} & \begin{matrix} \hat{\alpha}_n & 0 & \cdots & 0 \\ 0 & \hat{\alpha}_n & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \hat{\alpha}_n \end{matrix} & \Big\}\,\alpha_n-1 \\
\multicolumn{1}{c}{\underbrace{\qquad\qquad}_{\alpha_1}} & \multicolumn{1}{c}{\underbrace{\qquad\qquad}_{\alpha_2}} & \multicolumn{1}{c}{\underbrace{\qquad\qquad}_{\alpha_{n-1}}} & \multicolumn{1}{c}{\underbrace{\qquad\qquad}_{\alpha_n-1}} &
\end{array}
$$

$$= \hat{\alpha}_1^{\alpha_1-1}\,\hat{\alpha}_2^{\alpha_2-1}\cdots\cdots\hat{\alpha}_{n-2}^{\alpha_{n-2}-1}\,\hat{\alpha}_{n-1}^{\alpha_{n-1}-2}\,\hat{\alpha}_n^{\alpha_n-2}\cdot$$

$$\cdot\,\alpha^{n-3}\,\{\alpha(\hat{\alpha}_{n-1}\hat{\alpha}_n-\alpha+1)-\hat{\alpha}_{n-1}\hat{\alpha}_n+\alpha_n\alpha_{n-1}+\alpha-\alpha_{n-1}\alpha_n\}$$

$$= \prod_{i=1}^{n-2}(\hat{\alpha}_i^{\alpha_i-1})\cdot\hat{\alpha}_{n-1}^{\alpha_{n-1}-2}\,\alpha_n^{\alpha_n-2}\cdot\alpha^{n-3}\cdot\{\alpha^3-\alpha^2(\alpha_{n-1}+\alpha_n+2)$$

$$+\alpha\,(\alpha_{n-1}\alpha_n+\alpha_{n-1}+\alpha_n+2)-\alpha_{n-1}-\alpha_n\}$$

Therefore we have as follows :

THEOREM 5 *"The number of trees in the semicomplete n-colored graph* $K\alpha_1\alpha_2\cdots\overline{\alpha_{n-1}\alpha_n}$ *is given by*

$$T'\alpha_1\alpha_2\cdots\overline{\alpha_{n-1}\alpha_n} = \prod_{i=1}^{n-2}(\hat{\alpha}_i^{\alpha_i-1})\cdot\hat{\alpha}_{n-1}^{\alpha_{n-1}-2}\,\hat{\alpha}_n^{\alpha_n-2}\cdot\alpha^{n-3}\cdot$$

$$\cdot\,\{\alpha^3-\alpha^2\,(\alpha_{n-1}+\alpha_n+2)+\alpha\,(\alpha_{n-1}\alpha_n+\alpha_{n-1}+\alpha_n+2)-\alpha_{n-1}-\alpha_n\}$$

$$= (\alpha-\alpha_1)^{\alpha_1-1}\,(\alpha-\alpha_2)^{\alpha_2-1}\cdots\cdots(\alpha-\alpha_{n-2})^{\alpha_{n-2}-1}\cdot$$

$$\cdot\,(\alpha-\alpha_{n-1})^{\alpha_{n-1}-2}\,(\alpha-\alpha_n)^{\alpha_n-2}\cdot\alpha^{n-3}\cdot$$

$$\cdot\,\{\alpha^3-\alpha^2(\alpha_{n-1}+\alpha_n+2)+\alpha(\alpha_{n-1}\alpha_n+\alpha_{n-1}+\alpha_n+2)-\alpha_{n-1}-\alpha_n\},\qquad(28)$$

where $\alpha=\sum_{i=1}^{n}\alpha_i$ "

LEMMA 5a *"The number of trees in the semicomplete graph* K_n' $(=K_n-K_2=K_1'^{n-2},\overline{1,1})$ *is given by*

$$T_n' = (n-2)\,n^{n-3}\ \text{"}\qquad(29)$$

5. PRACTICAL FXAMPLFS

On electrical network theory, we know that the current response i^κ flowing in a voltage source e_κ is graphically given by[2]

$$i^\kappa = \frac{\Sigma\ \textit{cotree products of impedances in graph } G'}{\Sigma\ \textit{cotree products of impedances in graph } G}\,e_\kappa\qquad(30)$$

where G is the graph of the given electrical network, G′ is the graph G without the branch κ, and the cotree product is $z_{\kappa_1}\,z_{\kappa_2}\cdots\cdots z_{\kappa_k}$, which branch $\kappa_1\,\kappa_2\cdots\cdots\kappa_k$ consist of a cotree being the complement of a tree.

Provided that each impedance equals z, Eq. (30) is given as

$$i^\kappa = \frac{e_\kappa}{z}\,\frac{\textit{number of trees in } G'}{\textit{number of trees in } G} = \frac{e_\kappa}{z}\cdot\frac{T'}{T}\qquad(31)$$

See the following examples.

EXAMPLE 1 (Fig. 8) : When the voltage source e_κ exists in branch κ of a complete graph K_n with the same impedance z, the current i^κ in branch κ is shown as

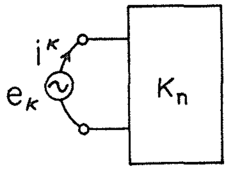


Fig. 8 Electrical complete network K_n

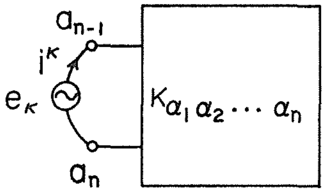


Fig. 9 Electrical network $K\alpha_1\alpha_2\cdots\alpha_n$

$$i^\kappa=\frac{e_\kappa}{z}\cdot\frac{T_n{}'}{T_n}=\frac{e_\kappa}{z}\frac{(n-2)n^{n-3}}{n^{n-2}}=\frac{e_\kappa}{z}\cdot\frac{(n-2)}{n} \tag{32}$$

EXAMPLE 2 (Fig. 9) : When the voltage source e_κ exists in branch κ of a complete n-colored graph $K\alpha_1\alpha_2\cdots\alpha_n$ with the same impedances z, the current i^κ in branch κ $[\kappa=(a_{n-1},\ a_n),\ a_{n-1}\in P_{n-1},\ a_n\in P_n]$ is determined as

$$\begin{aligned}i^\kappa&=\frac{e_\kappa}{z}\cdot\frac{T'\alpha_1\alpha_2\cdots\cdots\overline{\alpha_{n-1}\alpha_n}}{T\alpha_1\cdots\cdots\alpha_n}\\&=\frac{e_\kappa}{z}\cdot\frac{\alpha^{n-3}\prod_{i=1}^{n-2}(\alpha-\alpha_i)^{\alpha_i-1}\cdot(\alpha-\alpha_{n-1})^{\alpha_{n-1}-2}(\alpha-\alpha_n)^{\alpha_n-2}}{\alpha^{n-2}\prod_{i=1}^{n}(\alpha-\alpha_i)^{\alpha_i-1}}\cdot\\&\quad\cdot\{\alpha^3-\alpha^2(\alpha_{n-1}+\alpha_n+2)+\alpha(\alpha_{n-1}\alpha_n+\alpha_{n-1}+\alpha_n+2)-\alpha_{n-1}-\alpha_n\}\\&=\frac{e_\kappa}{z}\frac{\alpha^3-\alpha^2(\alpha_{n-1}+\alpha_n+2)+\alpha(\alpha_{n-1}\alpha_n+\alpha_{n-1}+\alpha_n+2)-\alpha_{n-1}-\alpha_n}{\alpha(\alpha-\alpha_{n-1})(\alpha-\alpha_n)}\end{aligned} \tag{33}$$

If $\alpha_i=\alpha_0$, it follows that

$$i^\kappa=\frac{e_\kappa}{z}\frac{n(n-1)\alpha_0{}^2-2n\alpha_0+2}{n(n-1)\alpha_0{}^2} \tag{34}$$

And further if n = 2, it follows that

$$i^\kappa=\frac{e_\kappa}{z}\frac{(\alpha_0-1)^2}{\alpha_0{}^2} \tag{35}$$

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ONODERA・SAISHU・ABE・KOMAMURA : On Enumeration of N-colored Trees

N色木の計数について

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異った色に塗られたN組の点集合（同集合同色）について，異色の点対にのみ枝を張ることを許したグラフを**N色グラフ**という。N色グラフが木なら**N色木**といい，それぞれ$\alpha_1\alpha_2\cdots\alpha_n$個の点が同色なら$T\alpha_1\alpha_2\cdots\alpha_n$と書く。本文では$T\alpha_1\alpha_2\cdots\alpha_n$の種類を数える一般公式

$$N(T\alpha_1\alpha_2\cdots\alpha_n)=\alpha^{n-2}\prod_{i=1}^{n}(\alpha-\alpha_i)^{\alpha_i-1}$$

と，これらに関するいくつかの公式を示し，その電気回路への二三の応用例を掲げる。しかも，これらの公式は一般のグラフG_nに関する公式を含んでいることを示した。