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Review of Visual Complex Analysis, by Tristan Needham

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REVIEWS

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Visual Complex Analysis. By Tristan Needham. Oxford University Press, 1997, xxiii + 592 pp., $55.

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Tristan Needham's *Visual Complex Analysis* will show you the field of complex analysis in a way you almost certainly have not seen before. Drawing on historical sources and adding his own insights, Needham develops the subject from the ground up, drawing attractive pictures at every step of the way. If you have time for a year course, full of fascinating detours, this is the perfect text; by picking and choosing, you could use it for a variety of shorter courses. I am tempted to hide the book from my own students, in order to appear the more clever for popping up with crisp historical anecdotes, great exercises, and pictures that explain things like that mysterious $2\pi i$ that crops up in integrals. Whether you use *Visual Complex Analysis* as a text, a resource, or entertaining summer reading, I highly recommend it for your bookshelf.

Here in a nutshell is the basic approach of the book: a complex linear transformation of the plane to itself amounts to a dilation and a counterclockwise rotation, an "amplitwist" in the author's imaginative vocabulary; an analytic function is a mapping of the plane to itself that is infinitesimally an amplitwist. You can see from pictures when a function is analytic; you can use this property to reason about analytic functions, line integrals, and mathematical physics. It is possible that you have shared some version of this idea with your students. Indeed, every one of the five or so courses in complex variables I have taught has begun with the picture of how the complex squaring function deforms the plane, noting how the square grid of rectangular coordinates is mapped to a curiously regular network of curves that—surprise!—meet at right angles. But Needham's book centralizes this geometric approach, inviting the reader, for example, to differentiate the exponential function visually. The results are quite wonderful and, while the facts presented are essentially those that appear in standard texts, the inviting style and rich texture make this an attractive alternative.

The main problem with the book may also be its best feature: it develops the geometric discussion in such exhaustive detail that the definition of analytic function does not appear until page 197; complex line integrals are not introduced until page 383! This is wonderful if one wants to curl up before a fire with a fascinating story, off-putting if one needs to get one's engineers doing line integrals by the seventh week. As I outline some of the topics presented in Needham's book, I will raise questions about what we ask of our textbooks and ultimately recommend some features of this one as a model for any future text that wants to go beyond presentation toward intuitive explanation.

Introducing the Complex Numbers. As the author points out, the usual textbook motivation for entering the world of complex numbers, our need to solve the equation $x^2 + 1 = 0$, is rather undercut by the fact that graphing the equation does not show us any solutions. Needham provides a satisfying alternative in Bombelli's historical interpretation of Cardano's solution of the cubic equation using radicals, showing that it leads to a real solution that we can see, but only if we use the algebra of complex numbers. Your own attitude toward this example might be a litmus test for your enjoyment of the rest of this text: if you find it more efficient simply to tell your students the story about not being able to solve $x^2 + 1 = 0$ without complex numbers, you might prefer a blander, more straightforward text.

The rest of the first chapter, which introduces the reader to the geometry and algebra of complex numbers, is equally rich in interesting digressions, mathematical or historical, that develop appreciation of matters complex. Problems of trigonometry and elementary plane geometry (geometry that one would seldom see in a text on complex variables) are quickly solved once complex numbers are introduced. The idea needed for the later chapters is simply that complex affine functions amount to a combination of dilation, rotation, and translation, but on the way to this we visit such ideas as the theorem that every isometry of the plane is the product of three reflections.

Given what I've said so far, one might expect the second chapter to start with difference quotients and a definition of analytic functions: now that we see the geometry of complex linear transformations, perhaps it is time to use them as linear approximations to more general functions. But, again, we are treated to a longer journey through topics that are worth the digression. In particular, the mapping properties of the powers of z lead to discussion of general polynomials (and the lovely Cassinian curves associated with them), then to convergent power series and ultimately to the transcendental functions. I find this approach akin to one of the positive results of reforming the calculus: studying a variety of useful functions before developing derivatives in the abstract.

I am less enthusiastic about all the details of Möbius transformations in the sixty-seven pages that come between Chapter 2 and Chapter 4, where differentiation is finally introduced. It's not that I dislike Möbius transformations even a little bit; it's that I found myself champing at the bit, yearning for a book about analytic functions to give the definition of an analytic function. Still, Chapter 3 is full of marvels: the author is quite right that the correspondence between Möbius and Lorentz transformations, popularized among mathematical physicists by Penrose, deserves to be better known. The general properties of Möbius transformations are fully developed: their preservation of circles, their appearance on the Riemann sphere, their classification into parabolic, elliptic, hyperbolic, and loxodromic. The exercises that work through Steiner circles and "Soddy's Hexlet" are most welcome. Tom Banchoff once shared with me a view of teaching as leading students through a garden, where on each visit (or academic term) one might go by a slightly different path. Using this analogy, Needham's book begins to seem like a journey through the far north forty. I was delighted to go there, but may not take many of my students.

Defining Analytic Functions. In the Preface, Needham apologizes for a lack of rigor in the text. In most of the text, I respond with a hearty, "No apology necessary!" It seems to me that a course in real analysis is the appropriate place for a festival of epsilons and deltas, and once one has worked through that, one can easily make rigorous most of the arguments in *Visual Complex Analysis*. The following example, condensed from the text, shows the basic style of argument; it can easily be turned into a rigorous proof, but one wonders what that would add.

We compute the derivative of the function z^a, where a is an integer. At this point, we know that the derivative of an analytic function at a point is the complex number representing how much amplification and rotation is experienced by infinitesimal vectors emanating from that point. Figure 1 represents objects before the application of the mapping, with the open arrow representing an infinitesimal variation from z; since we know that the "amplitwist" is independent of the direction of this arrow, we make the clever choice of an arrow perpendicular to z. In Figure 2, after the mapping, we use what we know about the ath power function: the central angles are multiplied by a and the radii are raised to that power. Elementary geometry of circles leads to the formulas

$$\begin{aligned} \text{amplification} &= ar^{a-1} \\ \text{twist} &= (a-1)\theta \\ \Rightarrow \text{amplitwist} &= ar^{a-1}\, e^{i(a-1)\theta} = az^{a-1} \end{aligned}$$

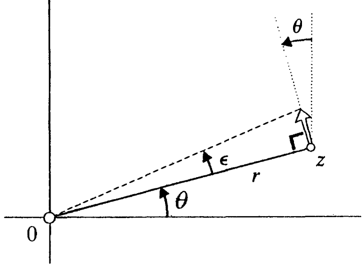

Figure 1

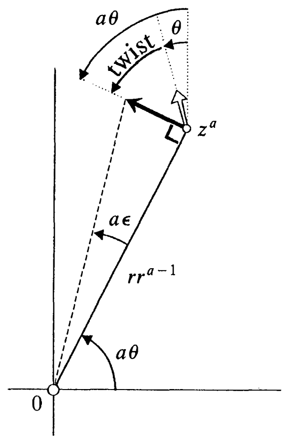

Figure 2

all without naming the length of the open arrow! Figures 1 and 2 are adapted from Figure 12, p. 230 of Needham. You can see this example and several others online in PDF format at http://www.usfca.edu/vca/. If you find such an argument appealing, as I do, then this book is for you, for there are many, many more arguments like it. And yet, for all my willingness to leave rigor for another time, one feature of this portion of the text disturbs me. The closest thing to a definition of analytic function is this:

> Analytic mappings are precisely those whose local effect is an amplitwist: all the infinitesimal complex numbers emanating from a single point are amplified and twisted the same amount (p. 197).

For a young mathematician reading this book in preparation for a career in mathematics, I find it almost a disservice to give such a non-traditional definition. I would be happier had Needham said that analytic functions are those for which limits of certain difference quotients exist and that it's possible to see that they all enjoy the property above. I am not talking about splitting hairs over pathological situations where the relevant limit exists only at a single point and the second partials are discontinuous there; I am concerned that a reader seeing this definition will experience a certain vertigo: "How can I check whether a function is analytic or not? How do I examine an infinite set of infinitesimal complex numbers?" It is an excellent idea to present ideas in an intuitive fashion, but for a definition, I'd like something I can check. Granted, the traditional approach refers to the concept of limit, which is frequently developed only intuitively in a book like this one, but at least the reader knows where to turn to begin a more rigorous inquiry. This is, I recognize, a book in which a function is "a rule" rather than a certain type of subset of a Cartesian product, but in the definition of analytic functions I wish something more rigorous were referenced. Of course, if one were using this book as a text, this could easily be remedied in class.

Still, the emphasis on geometry is so appealing as to override my objection. One can see (and therefore believe?) the differentiation formulas for the log and exponential functions. There is a very clever interpretation of the second derivative that seems to appear here for the first time. The curvature of a plane curve in the domain of an analytic mapping is altered by the mapping in two ways: the amplification reduces the curvature, just as dilating a circle makes it less curved, while the rate of change of the twist (a portion of the second derivative) can cause the image curve to curve more. An elegant formula!

Chapter 6, on Non-Euclidean geometry, reinforces the idea that what we have here is a pair of very interesting books woven together into one. With Möbius transformations as the principal tool and unifying idea, surprising connections are made among complex analysis, spherical geometry, and hyperbolic geometry. Some of these are well-known: that the group of isometries of the Poincaré disk consists of certain Möbius transformations. Others are less so: spatial rotations, as isometries of the Riemann sphere, have an easy presentation as Möbius transformations, which are then connected to quaternions. The new ideas and exercises bring together a body of information potentially invaluable to researchers in fields from topology to number theory. If students work through this material, it will probably not be in a first course in complex variables, but rather in the geometry course the author suggests as another possible use for his book as a text.

After this digression, this time in non-Euclidean geometry (wonderful in its use of the material developed on Möbius transformations, but distracting if one wants to get on to integration), Needham explains various big theorems: maximum modulus, Liouville's theorem, the Schwarz-Pick lemma, the argument principle. Whereas a rigorous development often leans on the Cauchy integral formula (often without proving Green's theorem, on which it is based), the approach here continues the intuitive, geometric style the reader will have come to enjoy by this point.

Line Integrals and Beyond. After a review of real Riemann sums, Needham shows us how to picture the analogous complex ones. The contour of integration is broken into pieces that are added up after being multiplied by representative values of the integrand. In the famous integral of $1/z$ around a circle centered at zero, a diagram (drawn for a midpoint rule approximation) makes it evident that the multiplied pieces are all vertical and add up to $2\pi i$. This is so much more appealing than the computation one typically uses to justify this fact, I had to wonder why I had never seen it before.

Perhaps you have seen it, but even so this is only the beginning of a long list of famous facts for which Needham offers attractive visual proofs, although "proof" is more my word than his—Needham modestly calls them "explanations" or "insights." Cauchy's theorem is a satisfying example: you can see the contribution to the integral from each infinitesimal square vanish before your eyes. Standard computational methods are certainly not ignored, but noting the appearance of parametric evaluation in the ninth of twelve sections in the chapter on complex integration shows where the intended emphasis lies.

The last three chapters deliver the payoff for physicists: vector fields, fluid flows, harmonic functions. All are treated with a nice combination of elegance and concreteness, accompanied by many computer-generated pictures. If you are not familiar with the Pólya vector field (for instance, from **[1]**), it is a pleasure to get to know: given a complex-valued function in the plane, $H(z)$, produce a vector field from the complex conjugate of H; this Pólya vector field is divergence-free and curl-free in a region if and only if $H(z)$ is analytic there.

Visual Complex Analysis ends, as many books do, with the Riemann mapping theorem. But I think few texts offer such vivid pictures, with meaningful connections to physics. The reader who has taken the entire journey Needham offers will be educated far beyond what one expects of a first course in complex variables.

An Alternative Visualization. Before closing, I would like to share an idea of my own that will interest those who enjoy a geometric approach to complex analysis. Needham, in discussing how we may picture complex functions, repeats what is often said about the possibility of graphing:

> In the case of a complex function this approach does not seem viable because to depict the pair of complex numbers $(z, f(z))$ we would need four dimensions (p. 56).

This is not so if we are willing to use color. Think of the complex plane as painted with colors similar to those in a traditional color wheel. We put red at the complex number 1, with green and blue at the other two cube roots of unity as shown in Figure 3. Hues are interpolated, giving secondary and tertiary colors. A continuous blending of hues would be possible, but here we use just twelve hues. Then we blend intensities toward white at the center, toward black going outwards.

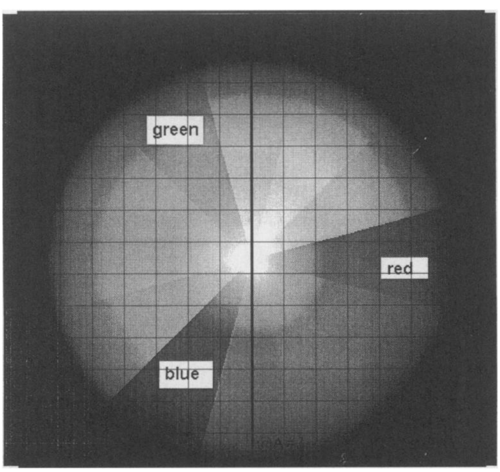

Figure 3. The color wheel

Thus, each complex number has a color associated to it. The association could be unique if we had a theoretically infinite palette; in practice, every complex number outside a certain radius looks black.

To visualize a complex-valued function in the plane we use what we call a domain coloring diagram: for each point in the domain of the function compute the color associated with the output value and use it to paint the point. Of course, in practice, we see only the pixels representing a grid of points in the plane. You can imagine that a domain coloring for the squaring function would show the colors cycling twice around the origin.

Figures 4 and 5 help us study the rational function $f(z) = (z^2 - i)/(z^2 + i)$. The first picture is the domain coloring for f, showing two zeroes and two poles, all simple. Note that the limit of the function as z approaches infinity is 1, which in terms of colors, is red. Notice also that the cyan color at the center of the diagram appears constant, leading us to believe that the derivative of this function is zero at the origin. In the second diagram, we show a domain coloring of $f(z) - f(0)$. Since the colors cycle around twice when we circle the origin, this function has a double zero and the derivative of the original function is zero at the origin. If you are dissatisfied seeing these pictures in grey-scale, with a few color labels, please visit my web page where these and other images appear in color: http://www-acc.scu.edu/~ffarris/complex.html.

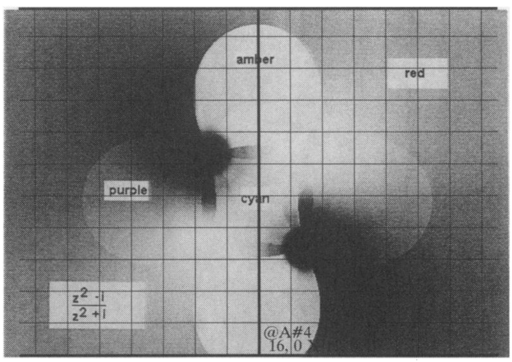

Figure 4. Domain coloring for a rational function

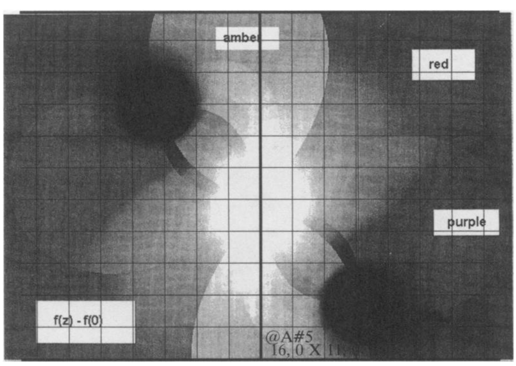

Figure 5. Domain coloring for $f(z) - f(0)$

Computers and Visualization. In Needham's approach and my own, computers are extremely useful for gaining insight and communicating ideas. My hope is that we have entered an era where these tools can be used without prohibitive start-up costs in time and frustration. Needham's text is a good example of a book that acknowledges the use of computing technology without letting it take center stage. As readers, we are invited to experiment for ourselves with Mathematica or a package called $f(z)$, but, if such software is not available, we can be just as happy reading about hypothetical experiments and enjoying images that the author has computed for us.

Conclusion. It seems to me that the typical mathematics book of a generation ago was written by an author whose personality was expected to play no role whatsoever in the text. Definitions were given and proofs presented with the utmost clarity, without human intervention. Needham's book strikes me as a healthy step away from that tradition; when we read this book, we have not only the clarity of the facts but the helpful voice of the author. Perhaps an author of another era would not have presumed to explain how to see something; Needham is so enthusiastic in sharing his vision that a book with perhaps three hundred pages of facts runs to almost six hundred pages. And yet, the extra length is not at all used to coddle the reader; this is not hand-holding but interesting advice for the vigorous traveler.

This is a book in which the author has been willing to make himself available as our teacher. His own voice enters in a rather charming way. When proving the standard theorem about how power series converge in disks, he engages us in speculation about the "ring of doubt" in which we have yet to discover the behavior of the series. A simple image of folding pie dough leads us toward the Cauchy-Riemann equations. The rigidity of analytic functions is poetically described as like "a crystal grown from a seed." And I would be happy should the word "amplitwist" become standard vocabulary.

So if you require a complex analysis book that tells the facts concisely without embroidery, you can easily pick up a used copy of something to fit that bill. It is also important to note that Needham's book does not address every possible topic: families of equicontinuous functions and Riemann surfaces are left for other books. Still I recommended *Visual Complex Analysis*, as something to read and enjoy, to share with students, and perhaps to inspire other books in which the voice of the author is vividly present to teach and explain.

REFERENCE

1. Bart Braden, Pólya's geometric picture of complex contour integrals, *Math. Mag.* 60 (1987) 321–327.

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