2-2008

Crafty Counting

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Recommended Citation

FARRIS, Frank A. "Crafty Counting."Math Horizons, February 2008: 12-15. The article illustrates Burnside orbit-counting in an elementary way. It includes an homage to artist Sol LeWitt.

"A correct count calls for more craft: the notion of a group acting on a set."

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To count a set means to put it in one-to-one correspondence with a set of integers $\{1, 2, 3, \ldots, n\}$. Direct counting is nice, but in complicated situations it pays to be more crafty. A problem with patterns of colored tiles gives us a chance to illustrate a popular counting principle known by various names. We'll call it the Burnside-Cauchy-Frobenius formula. It is also popularly called the Burnside Orbit-Counting Lemma, though wags refer to it as "not Burnside," because it was known long before Burnside was born. Later, Pólya generalized the formula, so some readers may recognize this as Pólya Enumeration.

A case where counting is hard

You have a collection of identical tiles, each shaped like an isosceles right triangle. They are white on one side and royal blue on the other. Two together form a square, and eight of them fit nicely together to make a larger square. You discover that, by turning some blue-side-up and others white-side-up, you can make a variety of pretty patterns in the large square. How many patterns are possible?

For a naive answer, consider one quadrant of the larger square. It could be all blue or all white. If it is half blue and half white, then the diagonal could run from top left to bottom right or the other way. For each tilt of the diagonal, blue could be on top or on the bottom. This means that there are six different ways to tile that quadrant. Since there are four quadrants, the naive answer is that there are

$$6^4 = 1{,}296 \text{ ways.}$$

We have over-counted. Figure 1 shows four of the ways we counted, but I say this is just four copies of a single pattern. After all, if you turn your head 90°, the second image looks just like the first one, and so on. But all four appear in our list of 1,296.

With this in mind, let's refine our question:

> How many different patterns can you make by fitting together 8 isosceles triangles into a square, when each one is colored either blue or white, and when patterns are considered the same if you can rotate one to get the other?

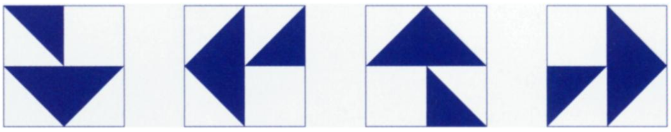

Figure 1. Four orientations of the same pattern.

Are you tempted to say, "Each pattern appears four times in the list of 1,296, so just divide by 4?" This would give

$$\frac{6^4}{4} = \frac{1296}{4} = 324 \text{ ways.}$$

Alas, this is just one more naive answer. The pinwheel in the top row of Figure 2 appears once, not four times, among the 1,296 patterns in our original list. The pattern in the bottom row appears twice. Can a pattern appear exactly three times? Why not? A correct count calls for more craft: the notion of a *group acting on a set*.

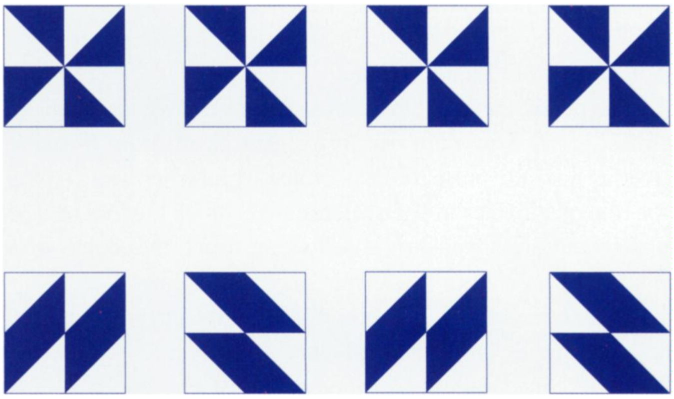

Figure 2. The top pattern occurs *once* in our list of 1,296, not four times; the bottom pattern accounts for two entries.

Counting with orbits

This problem presents a perfect opportunity to apply the Burnside-Cauchy-Frobenius (B-C-F) formula. It starts with a group of four rotations, the rotations through 0°, 90°, 180°, and 270°. Also, the group must *act* on a set, in this case by rotating configurations. Let us refer to our list of 1,296 ways of laying out the tiles as the set of *layouts*. The group acts on the set because any layout can be rotated to get another (possibly identical) layout.

When you take any layout and rotate it in each of four possible ways (a quarter turn, a half-turn, three-quarters of the way around, and all the way around, which is the same as not at all), you have formed the *orbit* of the layout. Figures 1 and 2 show three different types of orbits: those containing 4, 2, and 1 layouts.

Let us say that two layouts represent the same *pattern* when they belong to the same orbit, which is to say, you can rotate one to get the other. You can show that each pattern is an equivalence class of layouts.

Our question is now:

> Among the 1,296 possible layouts, how many different patterns are there? Equivalently, as the 4-element group of rotations acts on the layouts, how many orbits are there?

We'll count these orbits and then explain how the method is a special case of the B-C-F formula.

The count

Suppose that there are N different patterns (orbits) and take a huge collection that contains 4 copies of each one. Let's glue the tile patterns together so that we can toss them around, but let's agree, for now, not to turn them over. Arrange each set of four in a row, as pictured in the figures, where you progress across the columns by successively rotating the layout 90° clockwise.

First, observe that every one of the 1,296 layouts appears somewhere in this array, since every layout belongs to some orbit. In the vast majority of cases, a layout appears in a row with 3 other distinct layouts. But if a layout is fixed by a half-turn, then it appears twice in its row, along with a distinct layout obtained by turning it 90°. If a layout is fixed by a quarter-turn, then it appears four times. Since some layouts appear more than once, we see that $4N$ is larger than 1,296. To find out how much larger, let's imagine taking away layouts in a particular order:

Start by removing one copy of each of the 1,296 layouts, starting at the left-hand edge of each row. Most rows are now empty; let's try to count what's left.

Call ρ the rotation through 90°, so that our group consists of $\{\rho, \rho^2, \rho^3, \rho^4 = e\}$, where e represents no rotation at all. If a layout is fixed by ρ, then our start-from-the-left rule means that there will be a copy of that layout left in the second column. Such a pattern also has to be fixed by ρ^3, so there will be a copy in the last column as well. Remove those two and do this in every row like the top one of Figure 2.

What could remain after these removals? For each pattern invariant under ρ^2 but not ρ, like the bottom row of Figure 2, we would have a third and fourth column entry; there would also be a lone third column entry remaining for ρ-invariant patterns. Each of these represents a layout that is fixed by ρ^2. How can we count them? There must be 36, since the layout will be determined by a choice of one of 6 possibilities in the first quadrant and one of 6 possibilities in the second.

To summarize our count of this imagined $4 \times N$ array, let us proceed through the elements of the group:

- For the nonrotation, e, there are 1,296 layouts fixed by e and we removed all of these from our array.
- There are 6 layouts fixed by ρ and we removed copies from the second column in six rows to account for these.
- There are 6 copies of layouts fixed by ρ^3, which we removed.
- Finally, there are 36 layouts fixed by ρ^2 (including those 6 fixed by ρ). These were the ones left at the end of our process above.

This accounts for 4 copies of each pattern. Thus

$$4 \times N = 1{,}296 + 6 + 36 + 6,$$

and $N = 336$.

It is a short step from here to the general B-C-F formula, which we will state and then apply. Suppose a finite group G acts on a set X. For each element g of group G, call $\chi(g)$ the set of elements in X that are fixed by g. Let $|\chi(g)|$ be the size of that set and $|G|$ be the number of elements in G. The number of distinct orbits of X under the action of G is counted by

$$\frac{1}{|G|}\sum_{g \in G} |\chi(g)|.$$

An intuitive reason why this is so powerful is that, instead of counting how many things are different, we can count how many things are the same.

Other counts

Suppose we take a layout and turn over each triangular tile. Since the triangular pieces are blue on one side and white on the other, this trades blue for white and white for blue. Let us declare that two patterns are really the same if you get from one to the other by reversing all the colors, as if exchanging a photograph for its negative. How many patterns are there now, under this new concept of equivalence?

To apply the B-C-F formula, we need a group. Let's expand the 4-element group of rotations to include the operation of exchanging colors; call it ϕ. Does this group have 5 elements? No, if we want to say that two layouts represent the same pattern if you can either rotate or swap colors, we have to allow any combination of these operations. The group has 8 elements.

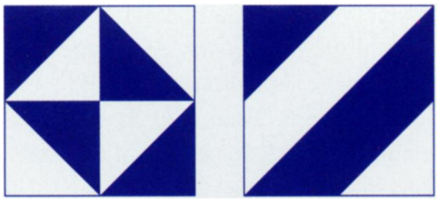

Figure 3. Examples of patterns fixed by $\rho\phi$ (left) and $\rho^2\phi$ (right).

Instead of going through a complicated process with 8 copies of each pattern, repeating the reasoning, let's just apply the formula. It starts in the same way as before, with 1,296 layouts fixed by e, 36 layouts fixed by ρ^2, and 6 layouts each fixed by ρ and ρ^3. What about the new group elements?

- For the color flip, ϕ, no layouts are fixed, because flipping the colors gives a different layout.
- There are 6 layouts fixed by $\rho\phi$. To count them, observe that the pattern in every quadrant is determined as the negative of the pattern in the quadrant 90° away. (An example appears on the left in Figure 3.)
- There are 36 layouts fixed by $\rho^2\phi$, because each quadrant has to be the negative of the one across from it diagonally. (An example appears on the right in Figure 3.)
- Finally, there are the 6 copies of layouts fixed by $\rho^3\phi$, the same ones fixed by $\rho\phi$.

The number of distinct orbits is therefore

$$= (1{,}296 + 6 + 36 + 6 + 0 + 6 + 36 + 6)/4,$$

and $N = 186$.

As an exercise, determine the number of distinct patterns if we decide that mirror images are the same (201) and the number of distinct patterns if we decide that mirror images *and* negatives are the same (108).

Listing rather than counting

Knowing that there are 336 different patterns (not counting mirrors or negatives) is not the same as being able to list them all. As an homage to minimalist artist Sol LeWitt, I wanted to make a picture of all the patterns, which is Figure 4.

To direct my computer (I used *Maple*) to draw this figure, I needed to put the patterns in some logical order. To do this, I

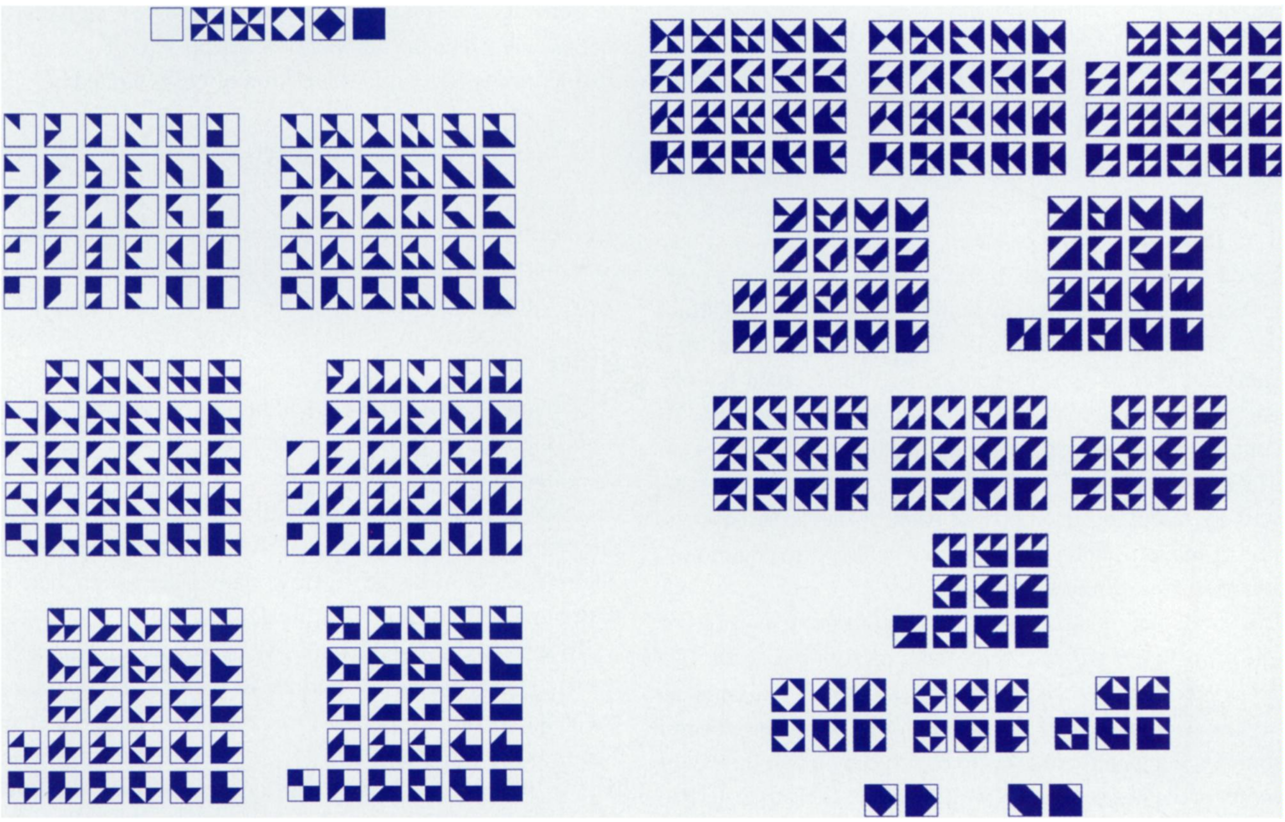

Figure 4. Homage to Sol LeWitt (1928–2007).

gave each layout a numeric name, which I called a *symbol*. First, I thought of the six ways to build the upper right quadrant and, like a computer scientist, numbered them 0, 1, 2, 3, 4, and 5, in the same order as they appear in the top left row of Figure 4. This means, for instance, that 0 refers to an all-white quarter-square and 5 to an all-blue one.

The first entry in the four-number symbol gives the coloring of the upper right quadrant. To encode the idea of rotation in the symbol, the second entry refers to the way the lower right quadrant is colored, but *rotated by* 90° and so on with the third and fourth. This is best explained with an example: The symbol for the pattern in the top row of Figure 2 is (1111), because each quadrant has a suitable rotation of the coloring I originally labeled as number 1. Figure 1 shows (0441) and its rotations, 1044, 4104, and 4410. The bottom row of Figure 2 shows (1414) and (4141).

With these symbols, which I listed in lexicographic order, it was not hard to direct *Maple* to draw all possible patterns. For instance, read down the first column in the block to find layouts (0001), (0002), (0003), (0004), and (0005). The program involved lots of nested loops; by stopping some of them early, I was able to eliminate the duplicates, so that (0441) appears (can you find it in lexicographic order?), but the rotated versions do not.

Is craft crucial?

With the picture of all possible configurations before us, we could simply count them. Another inelegant way to get the job done would be to prove that among the 1,296 layouts there are $1{,}260 = 6 \cdot 6 \cdot 5 \cdot 6 + 6 \cdot 6 \cdot 1 \cdot 5$ with no rotational symmetry, divide by 4 to get 315 asymmetric patterns, and then add in the 6 patterns with 4-fold symmetry and the 15 with 2-fold symmetry. This is not really hard, just messy.

If this reduces your enthusiasm for orbit-counting, you might try a larger problem, so large that no one would ever contemplate listing all possible patterns. For instance, if we generalize our original problem and use 18 triangular tiles to make a 9×9 square, there are 2,520,108 different patterns. (Try it!) Less artificial examples come up in combinatorial chemistry, where it is important to count the number of different compounds that can be assembled from given atoms.

Acknowledgment

Thanks to Mary Faye Zink, who brought this problem to our SCU math department lunch table. Her seventh graders found it challenging to ponder, and some found the correct answer! ■