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The Edge of the Universe
Noneuclidean Wallpaper

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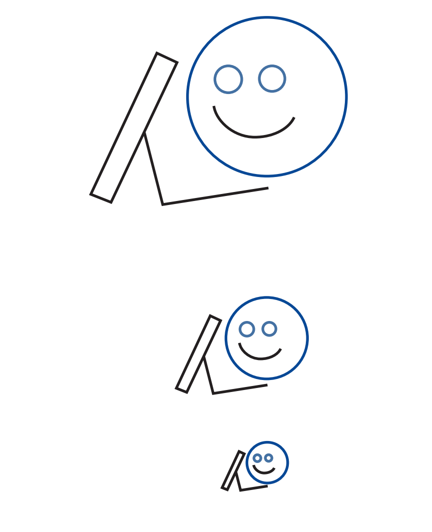
Perhaps, like me, you heard the following argument as a child on the playground: "The universe could not possibly have an edge, because if it did you could go there and put your hand through, and that new place would have to be part of the universe too."

If only I had known hyperbolic geometry, I might have refuted this seemingly unassailable argument, in the following manner: "What if you, and all the matter that makes up your measuring instruments, shrink as you approach the edge, and shrink in such a way that you could put your ruler end to end infinitely many times and still never reach the edge? After all, what evidence do you have that you do not change size as you move about the world?"

A simple mathematical model of a two-dimensional universe, called the *Poincaré Upper Halfplane*, illustrates the possibility of a universe with an unattainable edge. In this article, I describe this model —a famous example of a noneuclidean geometry—and explain how conversations with an analytic number theorist led me to create wallpaper patterns for its inhabitants. These are interesting not only for their high "Gee whiz!" factor, but also as a window for observing the features of this unusual geometry.



There is nothing below the line!

Figure 1

The World of the Shrinking Ruler

To describe the unusual universe we will study in this article, we must stand outside it. Imagine yourself looking down on the ordinary Cartesian plane; the model world consists of those points that lie above the x-axis, as if points on or below the x-axis have been declared off-limits to inhabitants of our model world, henceforth dubbed the Poincarites. In fact, for the Poincarites, that axis is infinitely far from any point. From our omniscient point of view, the inhabitants' rulers shrink in a particular way as they approach the x-axis.

I will later give a precise mathematical rule to describe how this shrinking occurs, but first let us consider how the inhabitants of this world travel if they attempt to move along a straight path. Because physical particles travel in the straightest way possible, in the absence of other forces, understanding how to move without curving is essential to life as a Poincarite.

Any vertical line in the plane, or at least the portion of it that belongs to our new universe, is straight for two reasons. First, the strange change in the size of matter occurs only when moving up and down, not side to side; therefore, if a Poincarite walks up this line with hands out on either side, each hand travels exactly the same distance—a reasonable criterion for straightness. Second, observe that transforming this universe by flipping the plane about that line does nothing to change the size of any measurements, and doing so leaves that line invariant: if the straightest path diverged to the left then, by symmetry, it would also have to diverge to the right; presumably, there is only one way to go straight in any given direction, so that vertical line must be straight.

Heading in a horizontal direction, what path would be the straightest? If a Poincarite maintains a fixed y-coordinate and walks to the left, the hand with the lower y-coordinate travels farther than the other hand,

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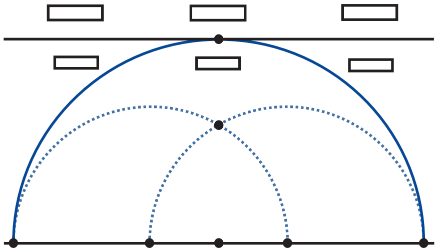


Figure 2

since that lower path is measured with shrunken rulers. Such a person is actually curving to the left! It turns out (we still have not given any rigorous definitions here) that the straightest path for the Poincarite who begins in this direction is a portion of a Euclidean circle whose center is on the x-axis, which is the edge of the universe.

The proof is beyond the scope of our discussion, but it turns out that these straightest paths also give the shortest way to connect any two points. Also, between any two points of the Poincarites' world, there is a unique hyperbolic line connecting them.

The space we have described is called the Poincaré Upper Halfplane or the hyperbolic plane. These straightest paths, whether vertical lines or portions of Euclidean circles that meet the boundary at right angles, are called hyperbolic lines. This conceptual universe played an important role in the historical development of noneuclidean geometry. All the axioms of Euclidean geometry are satisfied, except the crucial parallel postulate (can you see how the lines in Figure 2 contradict Euclidean assumptions?). Therefore, this model shows that postulate to be truly independent of the rest of Euclid's system. There are higher dimensional hyperbolic spaces, but we will stick with the two-dimensional version.

To construct our wallpaper it's crucial to understand the *isometries* of the Poincaré Upper Halfplane, that is, the transformations that leave all measurements unchanged. We assume an intuitive familiarity with the Euclidean isometries of translation, reflection, and rotation, and proceed to study the hyperbolic analogues of these.

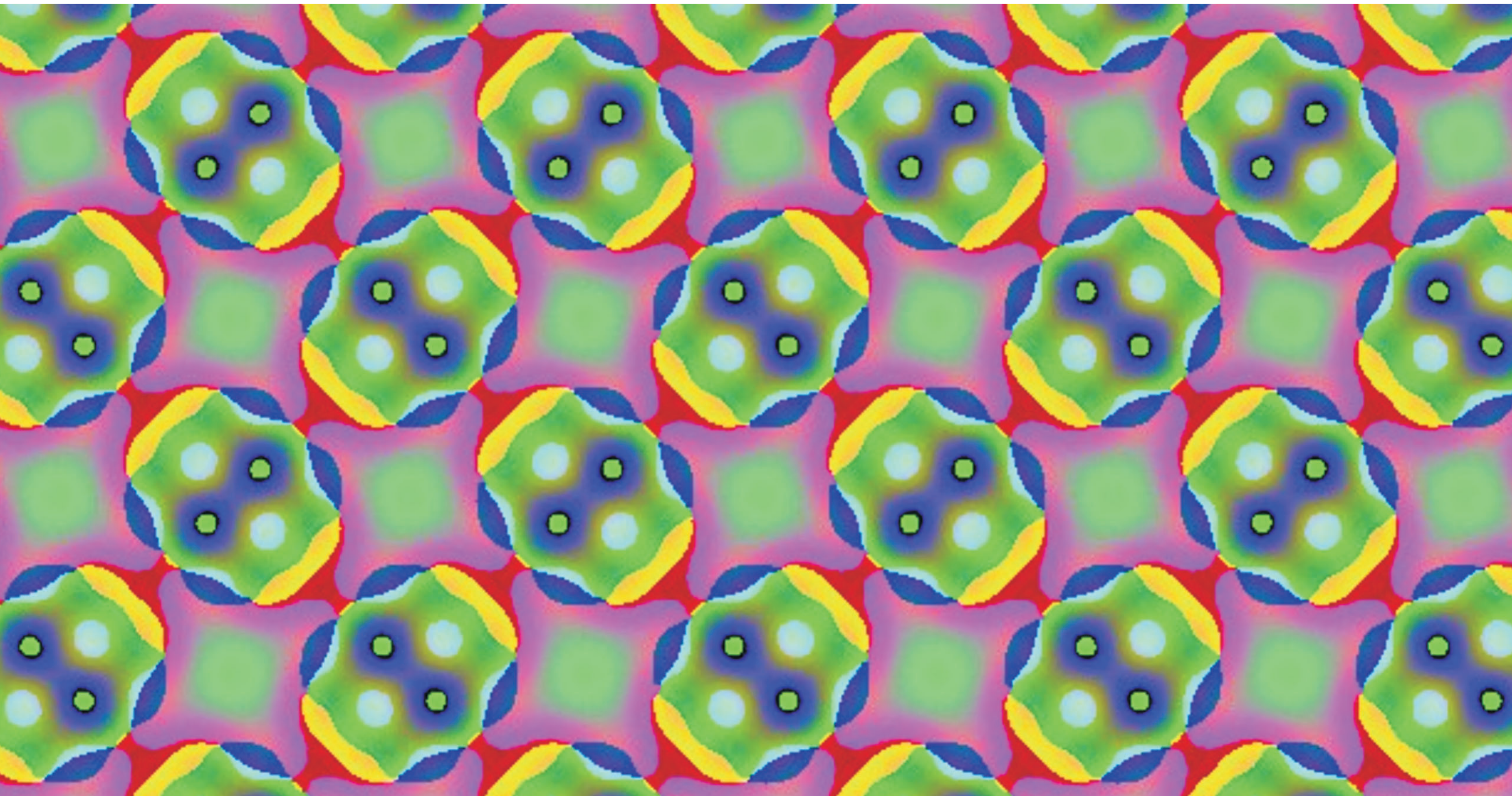
Reflections

The reflection noted above, flipping the plane about a vertical line, gives a simple example of a hyperbolic isometry. In Cartesian coordinates, assuming that the vertical line in the figure is the y-axis, it would be expressed by the equation $F(x, y) = (-x, y)$. Since this transformation does nothing to change any measurements of figures, this is an isometry of the hyperbolic plane.

What about reflections across other hyperbolic lines? In order to fit with physicists' ideas about empty space, we demand a property of *homogeneity*; space should look essentially the same everywhere, and therefore all hyperbolic lines should behave in the same way. In particular, the two sides of any hyperbolic line should be interchangeable.

Reflection about nonvertical hyperbolic lines is accomplished by a classical process called inversion in a circle, which you may have studied in other contexts. This is a beautiful way to interchange the inside and outside of a circle, leaving points on the circle fixed. If P is any point other than the center of the circle, which we'll call C, then the image of P is the point P' on the ray from C through P so that the product of the distances CP and CP' is the square of the radius of the circle. Figure 4 illustrates inversion in a

Figure 3. Euclidean wallpaper with many symmetries



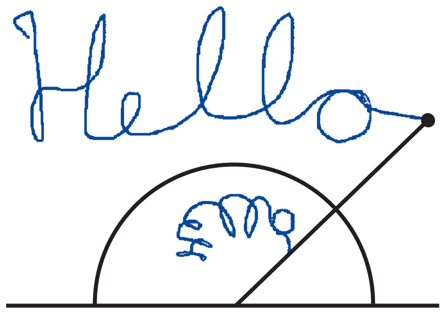


Figure 4

particularly simple semicircle (hyperbolic straight line). The formula for the illustrated inversion is

$$I(x,y)=\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2}\right).$$

Notice that in the figure, orientation has been reversed; for this reason, reflections are called *indirect isometries*. If we're willing to use complex coordinates for the Upper Halfplane, then the formula for $I(z)$ becomes appealingly simple

$$I(z)=1/\overline{z}.$$

If you want to look ahead to Figure 5, you will see a pattern invariant under reflections across certain vertical lines. It takes some imagination to see, but this one is also invariant under the inversion described above. It might help to know that the peacock fans that seem to be largest—for the Poincarites they are all exactly the same horizontal distance across—touch the x-axis at integer points.

Translations

There are two analogues of translations in the hyperbolic plane. The first is a rather obvious shift to the right or left, given in coordinates by

$$P(x, y) = (x + a, y) \;\text{ or }\; P(z) = z + a,$$

where a is any real number. Since we don't move anything up or down, P preserves all distances.

The next translation analogue is somewhat suprising, and brings me to say exactly what we mean by distances measured by a shrinking ruler. If we dilate the plane relative to any point on the x-axis, hyperbolic distances, the ones measured by shrinking rulers, are preserved. Suppose $\gamma(t) = (x(t), y(t))$, for $a \le t \le b$ is any parametric curve with $y(t) > 0$. Its hyperbolic length, that is, its length as measured by the Poincarites, is defined to be

$$L=\int_a^b \frac{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}}{y}\,dt.$$

This is quite similar to the usual formula for arc length, except that the factor of y in the denominator causes an apparently short piece of arc to count as large in the integral when it is close to the edge. This captures the essence of the shrinking ruler.

Now consider the transformation

$$H(x, y) = (rx, ry) \;\text{ or }\; H(z) = rz.$$

Apply this transformation to the curve $\gamma(t)$ and use the integral definition to compute the length of the new curve. It is easy to see that a factor of r cancels from numerator and denominator, leaving the length unchanged; thus, H is an isometry.

Why is H analogous to a translation? Note that the entire y-axis moves along itself under H. Of course, in a Euclidean translation, an entire family of parallel lines slides along itself, but that is not how things work in the hyperbolic plane. We must be content to slide along a single line at a time. While we are on this subject, it is interesting to note that the translations to the right and left, denoted by P above, are analogous to Euclidean translation in that they shift across a family of lines, in this case the family of vertical lines. Surprisingly, there is no line moved along itself by that type of translation.

Rotations and the Rest

To find an example of a rotation, simply compose the two reflections above. Check for yourself that following F by I gives

$$R(x,y)=\left(\frac{-x}{x^2+y^2},\frac{y}{x^2+y^2}\right) \text{ or } R(z)=-\frac{1}{z},$$

which is a hyperbolic rotation of 180 degrees about the point $(0, 1)$.

Again looking ahead to Figure 5, that image is invariant under this half-turn. The fixed point is not labelled, but you can find it in the pale pink area atop one of the large peacock fans. All the noneuclidean wallpapers shown also turn out to be invariant under the transformation

$$R_3(z)=\frac{z-1}{z},$$

which is a rotation through 120° about the point $z=\frac{1}{2}+i\frac{\sqrt{3}}{2}$. Figures 9 and 10 are good places to observe these rotations. In Figure 9, there are centers of two-fold rotation at the points where two lines cross, and centers of three-fold rotation at points where three lines cross.

The collection of isometries of the hyperbolic plane presented so far turns out to be representative of all possibilities. To investigate the totality of these isometries, let us focus on half of them, the set of direct isometries. If you have a feel for the operation of conjugation in the complex plane, you might guess that the formula for any direct isometry will involve only appearances of the variable z, with no $\overline{z}$s required. This is in fact the case.

Using the fact that the composition of two isometries is again an isometry, and looking at the types we have seen so far, it will

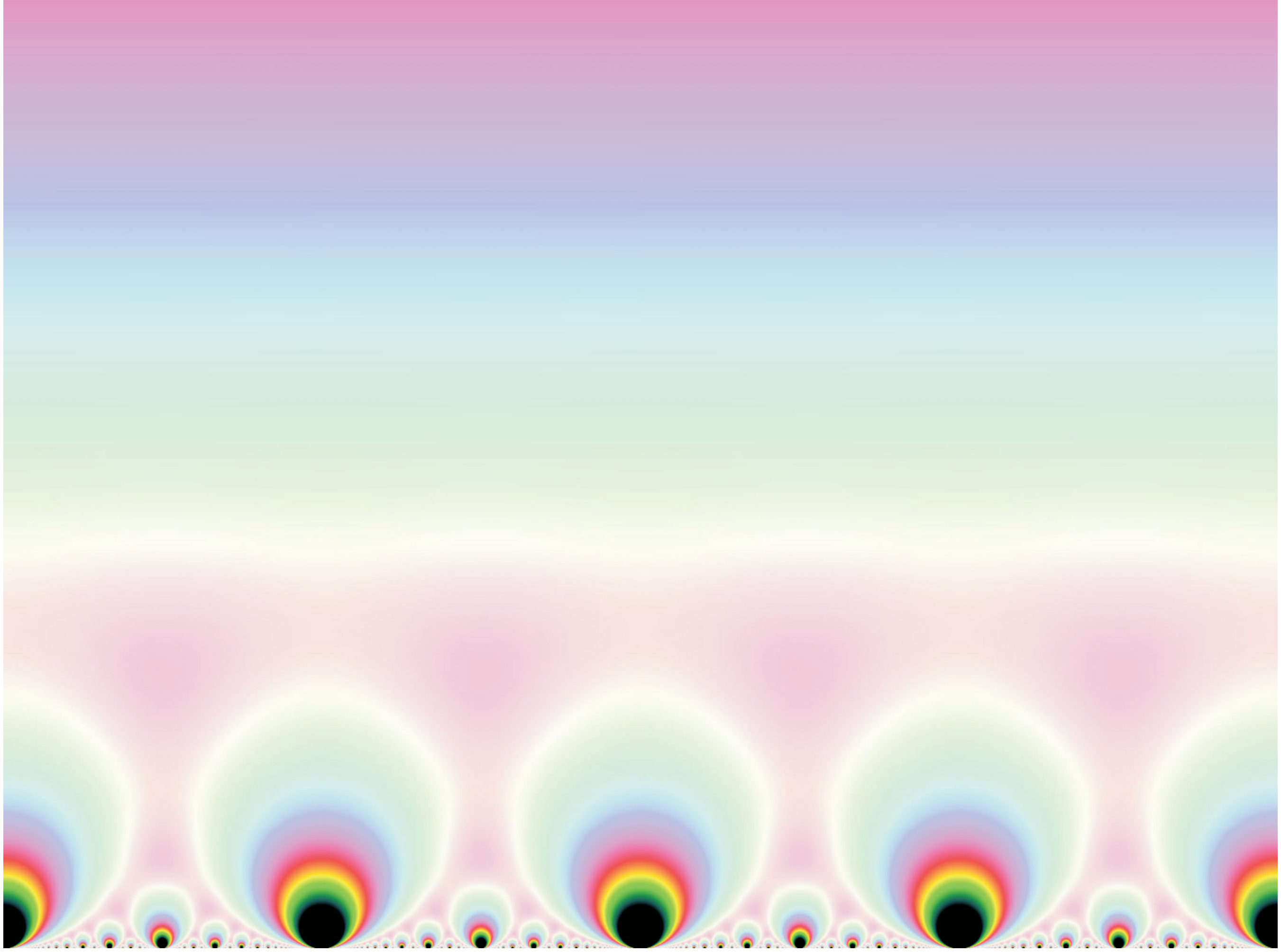


Figure 5

now be no surprise that the most general direct isometry of the hyperbolic plane looks like

$$\gamma(z)=\frac{az+b}{cz+d},$$

where *a, b, c,* and *d* are real numbers. These are called *fractional linear transformations*.

It takes some algebraic manipulation, but it is not too hard to show that this function takes points with $y > 0$ to other points in the Poincaré Upper Halfplane only when $ad - bc > 0$. Furthermore, note that if $ad - bc = 0$, the numerator and demoninator would have a common factor and γ would degenerate to a constant function, not a candidate for a transformation at all. Suppose we multiply all these coefficients by the same factor; it could be cancelled from numerator and denominator, resulting in the same transformation. Therefore, to avoid redundancies, we assume $ad - bc = 1$.

A helpful shorthand uses a 2×2 matrix to keep track of these fractional linear transformations:

$$\gamma(z)=\frac{az+b}{cz+d}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\circ z.$$

It should be considered a minor miracle that composition of functions corresponds exactly to multiplication of matrices. Try it!

We have now identified the set of direct isometries of the hyperbolic plane with a set of 2×2 matrices. If you know a little about groups, you can easily see that this set of matrices forms a group, which is usually called $SL(2,\mathbb{R})$. It is actually subtly different from the group of direct isometries of the hyperbolic plane, in that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\circ z=\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}\circ z,$$

so that two different matrices give the same transformation. We leave that distinction to the experts.

Creating Symmetry

As a basic example, consider a process for creating an even function of one variable, that is, a function $f(x)$ invariant with respect to reflection of the real line about the origin:

$$f(-x)=f(x).$$

Given any base function $g(x)$, we can *symmetrize* $g(x)$ to create a new function $f(x)$ by a process of averaging:

$$f(x)=\frac{g(x)+g(-x)}{2}.$$

Clearly, the new function is even. Of course, it is possible that the new function is identically zero, but it is indeed even.

For another easy example, suppose we wanted to create a function of two variables $f(x, y)$ that is invariant under rotation of 120 degrees about the origin. Again, a process of averaging

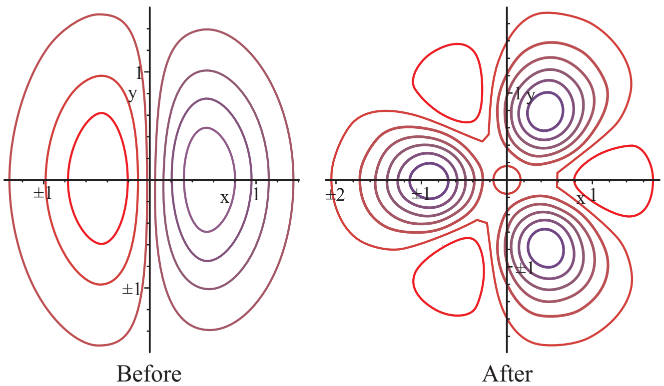


Figure 6

may be used, but this time there are three things to be averaged. Suppose $g(x, y)$ is any function of two variables, enjoying whatever properties of continuity or differentiability we wish to impose, and suppose ρ represents this rotation. Define a new symmetrized function by

$$f(x,y) = \frac{g(x,y) + g(\rho(x,y)) + g(\rho^2(x,y))}{3}.$$

Check for yourself that $f(\rho(x, y)) = f(x, y)$, so that f is indeed invariant under the desired rotation. The reason for dividing by three is to make the new function have values in about the same range as the old. Ponder the level curves in the "before and after" example in Figure 6, where $g(x,y) = (x^2 + 3x)e^{-(2x^2+y^2)}$.

This gives you some idea of a process called *averaging a function over a group action*. In the rotational example, the group consisted of three elements, e, ρ, ρ^2, where e is the identity transformation.

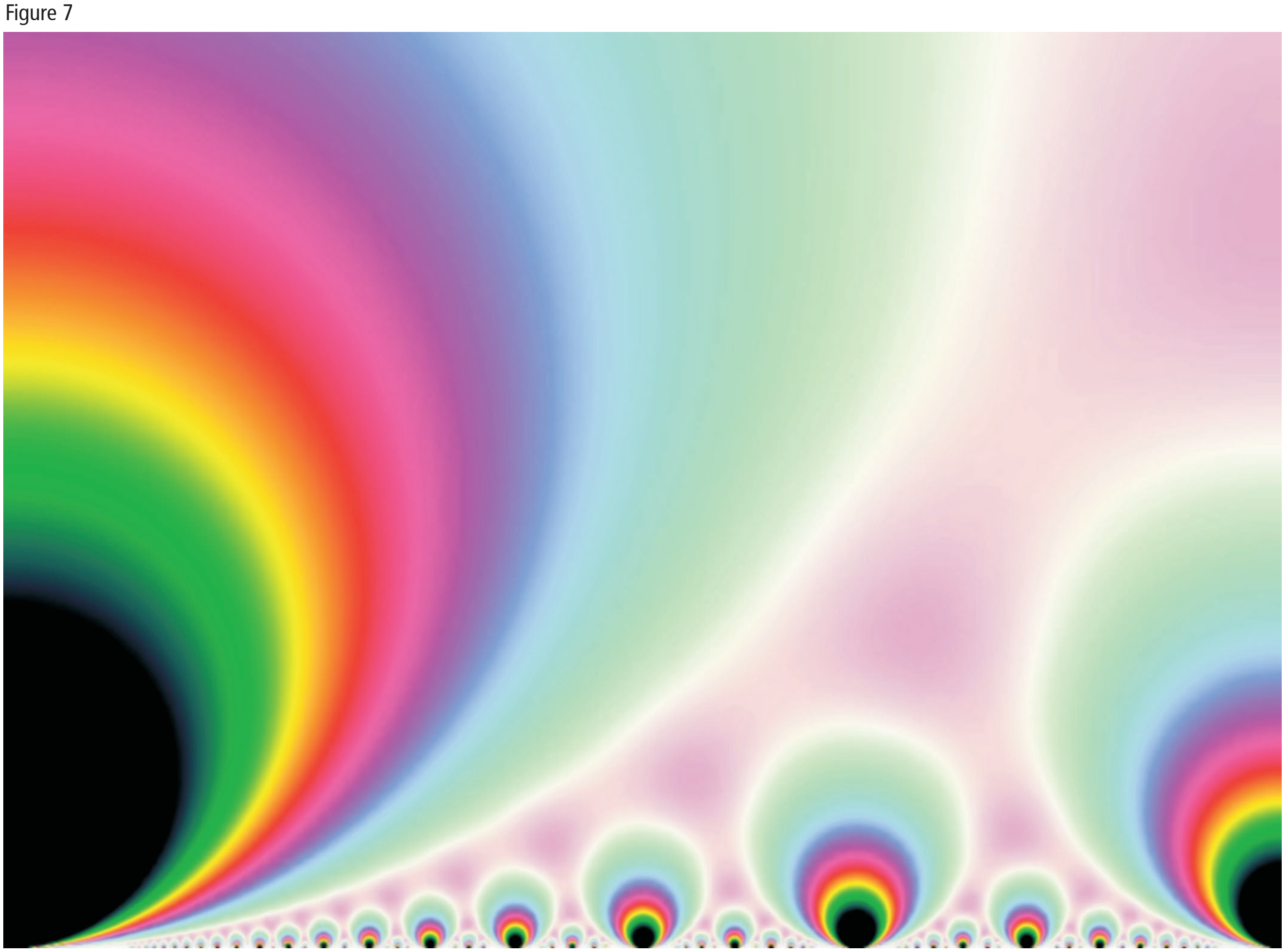
In making wallpaper for the Poincarites, my method was to look for functions that are invariant under a particular set of the isometries described above. Because it is a set much beloved of analytic number theorists, I chose the set of fractional linear transformations where all the coefficients are integers. This set, which also meets the requirements to be a group, is called

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ where } a,b,c,d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}.$$

Surprisingly, my next step amounted (almost) to taking a base function on the plane and averaging it over this infinite group!

But before I describe that averaging, let us imagine what we expect to see. In the swatch of Euclidean wallpaper shown in Figure 3, we can apply a large collection of Euclidean symmetries, reflections, rotations, and translations to the picture and find the pattern left unchanged; of course, we need to imagine that it continues infinitely in all directions, that what we are seeing is

Figure 7



a piece of an infinite pattern. For hyperbolic wallpaper of the type to be constructed, we should be able to apply any of the transformations in $SL(2, \mathbb{Z})$ and find the pattern unchanged.

In particular, consider two families of transformations. The simplest translations we discussed, $P(z) = z + b$, are indeed in $SL(2, \mathbb{Z})$ when b is an integer. (Use $a = d = 1$ and $c = 0$.) Thus, our picture should repeat itself with every unit translation to the right or left; that is reasonably easy to imagine. Call this collection of translations Γ.

This family of transformations has a sort of mirror image, in the set

$$\Gamma' = \left\{ \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \text{ where } n \in \mathbb{Z} \right\}.$$

To study a typical element, $\gamma(z) = \frac{1}{nz+1}$, we compute two limits:

$$\lim_{z \to \infty} \gamma(z) = 0$$

$$\lim_{z \to -\frac{1}{n}} \gamma(z) = \infty .$$

If the picture is supposed to look the same before and after we apply this transformation, then the behavior as z gets very large should look the same (to the Poincarites, of course) as when z approaches 0 and when z approaches $-1/n$. Of course, z cannot actually *be* 0, or ∞, or $-1/n$, as none of these is a point in our new universe, but the picture should look the same as you approach any of these points. With some modification, the same argument shows that the pattern must look the same as we approach any rational value on the x-axis. Before the first computed image appeared on my screen, I found it hard to imagine such a thing.

Constructing Wallpaper

The group $SL(2, \mathbb{Z})$ is actually too large to perform the averaging we described. To tell the real story, I need to use the language of *cosets* from group theory. If you want to skip this section, imagining the process as one of averaging is a good heuristic.

Recall that our goal is to take some base function $g(z)$ and, by some sort of averaging, produce a new function $f(z)$ which is invariant under every transformation of $SL(2, \mathbb{Z})$. Any function with all those symmetries is called a *modular function*. Modular functions and analogous objects called modular forms are well known to number theorists, so Jeffrey Hoffstein of Brown University was a natural person to ask for help. I told him that I had taken a stab at the construction using naive group averaging, and he showed me how to do it more cleverly. The result was a method I used to make the pictures in this article.

The naive idea would be to start with $g(z)$ and form

$$\sum_{\gamma \in SL(2,\mathbb{Z})} g(\gamma z)$$

hoping that the sum would converge. Unfortunately, it virtually never does. Instead, we start with a special choice of $g(z)$, one that is already invariant under the subgroup, Γ, of integer translations to the left and right. Such a g is easy to invent.

The key idea we need from group theory is that the big group $SL(2, \mathbb{Z})$ can be organized into cosets using the subgroup Γ, where any two elements of a coset differ by an element of Γ. It is a little like organizing the integers into three sets, $\{3n\}$, $\{3n+1\}$, and $\{3n+2\}$, the cosets of the subgroup of integers divisible by 3. To adapt this idea to $SL(2, \mathbb{Z})$, we need to remember that the group operation is matrix multiplication, so γ_1 and γ_2 differ by an element of Γ if $\gamma_1 \gamma_2^{-1} \in \Gamma$.

The following equation shows that any two elements of $SL(2, \mathbb{Z})$ that share the same bottom row belong to the same coset of Γ:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & a'b - b'a \\ 0 & 1 \end{pmatrix} \in \Gamma.$$

Figure 8

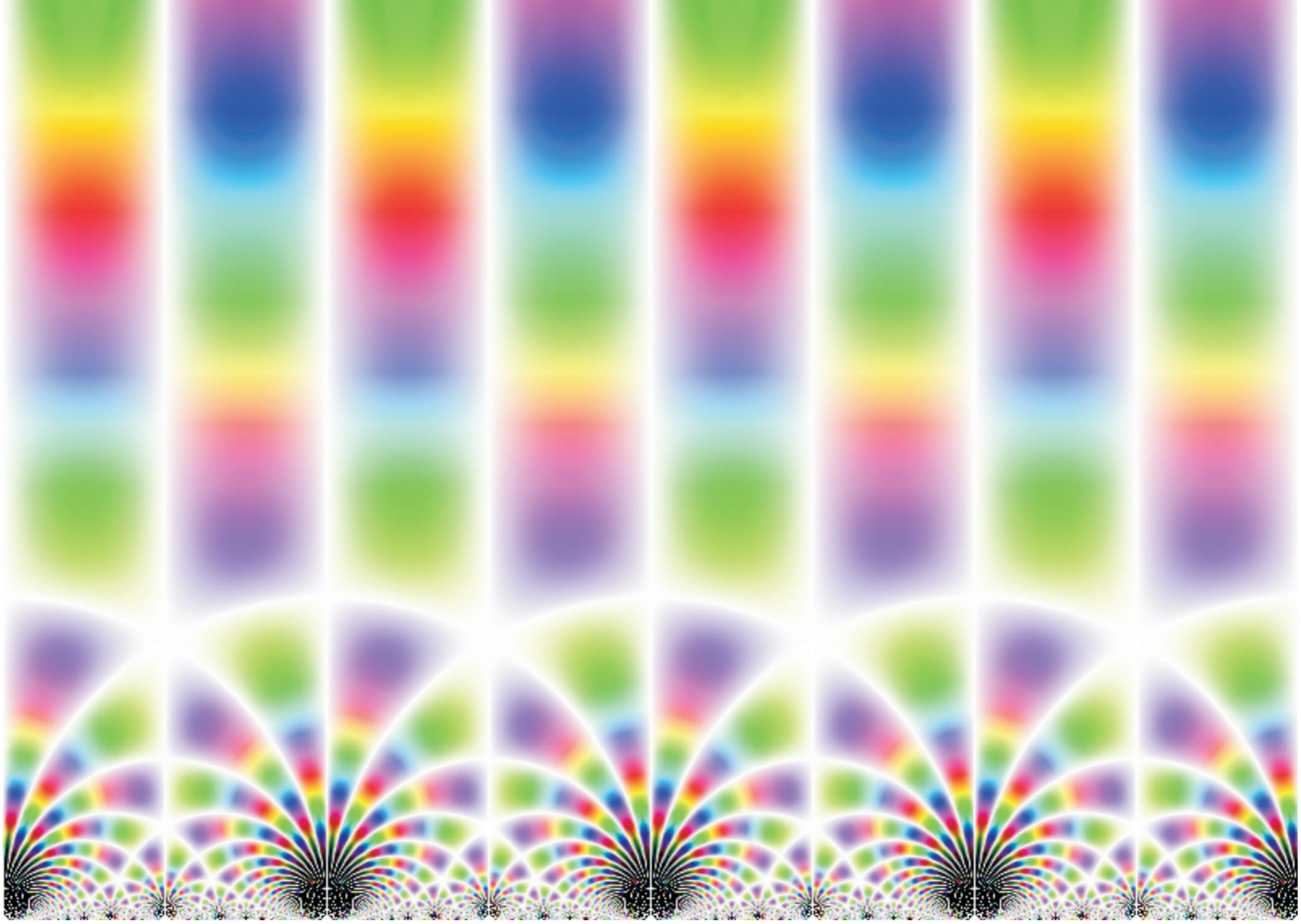


Figure 9

Conversely, multiplying any matrix on the right by an element of Γ does nothing to the bottom row. Thus, each coset of Γ corresponds to a pair of integers c and d. These must be relatively prime (denoted $(c, d) = 1$), as you can see from the equation $ad - bc = 1$.

We now should be able to perform some averaging, but there is one missing ingredient: given a pair of integers c and d, we need to find a top row for our matrix. The Euclidean algorithm comes to the rescue here. Siman Wong of the University of Massachusetts, Amherst, who was also at Brown University at the time, wrote a swatch of code to generate a list of relatively prime c, d pairs and one corresponding pair of values for a and b.

Putting all this together, I was ready to write a program to perform the average:

$$\sum_{(c,d)=1} g(\gamma z)$$

where γ is one of the matrices whose bottom row consists of c and d. Note that we are not dividing by the number of elements in the sum. In the first place, we are summing over an infinite number of elements, and in the second place, the sum converges nicely without doing so, provided we make the right choice for g.

Examples

We are now ready to choose some building blocks and carry out the averaging process to produce images. In light of the previous section, the function we choose as the building block for averaging must remain the same when you translate to the left and right by integer distances. One type of function that fits this bill is a function that does not depend on x at all. (Recall that we are using complex notation, where the point (x, y) corresponds to the complex number $z = x + iy$.) As an elementary building block, take

$$g(z) = y^s,$$

which is certainly invariant under the translations in Γ. An estimate using an integral test for convergence shows that the sum above will converge as long as s is any complex number with $Re(s) > 1$.

A favorite first example uses

$$g(z) = y^{1.5 + 5i} = y^{1.5}(\cos(5 \ln y) + i \sin(5 \ln y)).$$

This is pictured in Figure 5, with a close-up view of the enticingly complicated part of the image in Figure 7. Note that all the shapes that resemble peacock fans are exactly the same hyperbolic distance across, because any one can be taken into any other by one of our isometries, which the Poincarites see as leaving all distances unchanged. Furthermore, there is one of these fans tangent to the x-axis at every rational number. What a lot of room there is, down near the edge of the universe!

It may be startling to see that the function g, and hence the averaged function f, takes on complex values. How can these be pictured? Space constraints demands that I make a long story short. In the study that led to the article "Vibrating Wallpaper,"

Figure 10

I developed a way to visualize complex-valued functions in the plane using the artist's color wheel. (See `www.maa.org/pubs/amm_complements/complex.html`.) Every complex number receives a different color, with white at the center and black out toward infinity; hues are distributed in a circle (Figure 8). When you have a complex-valued function on a domain in the plane, you can color each point of that domain using the color corresponding to the output value for that point. For more information, follow links from my web page. Here, suffice it to say that the black portions of the image are places where the value of the function is very large in magnitude; white spots correspond to places where the function is near zero.

More exciting images are produced using $g(z) = y^s \sin(n\pi x)$, as in Figure 9, or $g(z) = y^s \cos(n\pi x)$. To achieve the required translational invariance, n must be an integer. In this picture, which uses the sine function as a building block, notice the white areas; since $\sin(n\pi) = 0$, this function is zero on a grid of hyperbolic lines. Since $\sin(-x) = -\sin(x)$, it also has an antisymmetry about the y-axis.

It opens a rather enjoyable can of worms to realize that one can superimpose these fundamental building blocks, to produce infinite variations on these pictures. Figure 10 uses a base function that superimposes functions like $y^s \sin(n\pi x)$ and $y^s \cos(n\pi x)$; any linear combination will do. I experimented until I was pleased with the result.

Where To Go From Here

Speedy computers, color monitors, the world wide web, all these give us tools for creating and sharing images that use color to illustrate mathematical ideas in a way not possible even ten years ago, when the computational power necessary to produce the images in this article was simply unavailable. With what you have seen in this article, I hope you will be inspired to create some images of your own. If you want to compute modular functions, and explore the endless variety of possibilities, I have outlined all the steps; there are also animations waiting to be made, showing these wallpapers in vibration. Alternatively, it would be great to see a video game where objects bounce along trajectories that follow hyperbolic lines; one thing to be overcome in that scenario is that a random walk in the hyperbolic plane almost certainly results in your getting lost in that expansive place that the external viewer sees as being down near the edge of the universe.

Tristan Needham's book *Visual Complex Analysis*, Clarendon Press, is an excellent place to learn about fractional linear transformations. To experience the Poincaré Upper Halfplane for yourself, you can use NonEuclid, Java simulation software developed by Joel Castellanos at Rice University. (See `cs.unm.edu/~joel/NonEuclid`.) With so many possibilities for visualization, the time to study this noneuclidean geometry and explore the edge of the universe is now.