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Woven Rope Friezes

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NOTES

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Introduction The rope mat shown in FIGURE 1 was woven by Nils Kristian Rossing following equations that produce frieze symmetry. To analyze the pattern, think of the curve the rope follows rather than the rope itself and recognize that, since rope knows more physics than mathematics, the rope does not quite follow the beautifully

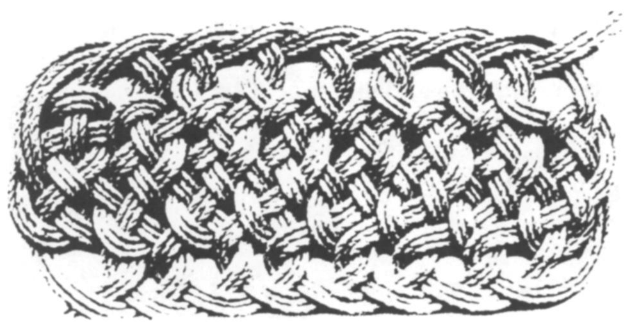

FIGURE 1
A woven rope mat with G_2 symmetry

symmetric curve shown. (See FIGURE 2.) This curve, meant to be interpreted as continuing indefinitely in both directions, has translational symmetry and rotational symmetry in the form of half-turns about the points indicated, leading one to identify its symmetry group as the *frieze group*, generally called G_2.

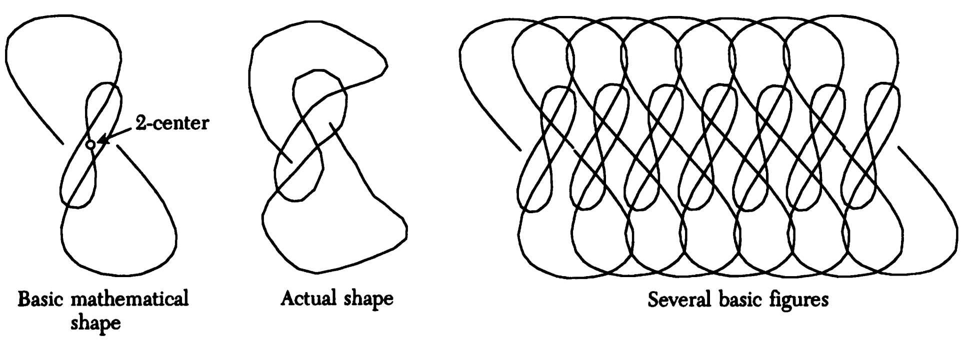

FIGURE 2

Here we present a complete set of recipes showing how to construct smooth curves with any desired frieze symmetry; we provide examples woven by Rossing for many of the pattern types, and invite readers to make others.

We review the concept of frieze symmetry, develop the formulas for parametric equations with given symmetries, and pose some open questions raised by our analysis. We also provide a Java applet to allow easy experimentation; this is now available on the World Wide Web along with high-resolution photographs of the ropes; please see `http://math.scu.edu/~ffarris/frieze.html` or `http://www.maa.org/pubs/mathmag.html`.

Frieze symmetry For us, a frieze (or frieze pattern) will be a set of points in the plane invariant under a translation, and hence an infinite cyclic group of translations. To relate this concept to the woven rope above, we idealize the path of the rope as a smooth curve, treating places where the rope passes over itself as self-intersections of the curve. We recognize that this treatment misses important features of the physical object pictured, such as whether the rope goes over or under itself at crossings; these considerations are expanded below and relegated to future discussion.

The set of all Euclidean motions, or isometries, that leave a frieze invariant is necessarily a group, called the symmetry group of the frieze. We say that two friezes have the same symmetry type if their groups are isomorphic. It is well-known (see, for instance, Cederberg **[1]**) that there are exactly seven isomorphism classes of symmetry groups containing a single infinite cyclic group of translations. Without going into detail, we assemble a few facts about these groups.

Suppose G is the invariant group of a frieze, which by definition contains an infinite cyclic subgroup of translations. Call the generator of the translation group τ, a translation of length K along a line l, so that K is the smallest distance one can translate the pattern and find that it falls into coincidence with itself. We find it convenient to think of l as the horizontal direction. The line l may not be uniquely defined if G has no elements other than multiples of τ, in which case we call the group G_1, or if the only other elements are reflections about lines perpendicular to l, when the group is called G_4.

Every other type of isometry possible in a frieze group leaves invariant a line parallel to the direction of translation, which we call the axis of the frieze, and name l. There are limited possibilities: The group G_2 contains a half-turn about a point of l, as well as all the half-turns generated by composing it with translations, but no other symmetries. We call the fixed point of a half-turn a 2-center of the frieze. G_3 contains no half-turns, but a reflection through l, which we will call a horizontal reflection. G_5 has half-turns as well as the horizontal reflection; since these together generate vertical reflections, this group has G_2, G_3, and G_4 as subgroups.

A glide reflection is the composition of a reflection and a translation. The group G_7 contains only a glide reflection along l in addition to its translation subgroup. G_6 has a glide reflection and half-turns. Examples of friezes with each type of symmetry are shown in FIGURE 3.

Frieze curves: general theory The rope frieze above was constructed by first finding a continuous curve in the plane, using a formula like $c(t)=(x(t), y(t))$, $-\infty < t < \infty$.

We suppose the curve is invariant under translation along the x-axis of a distance K, and that K is the smallest such distance. We assume that the curve is parametrized consistently with the translational invariance, so that

$$c(t+2L)=(x(t+2L), y(t+2L))=(x(t)+K, y(t)).$$

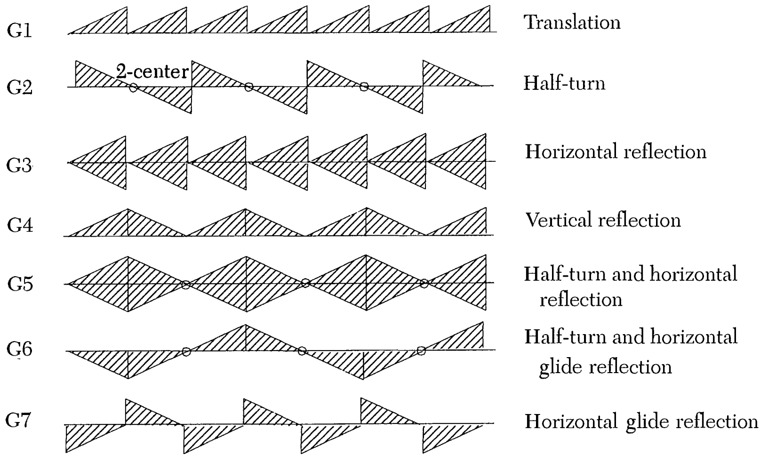

FIGURE 3

This can always be achieved by a reparametrization, using the equation above, recursively, to define a new parametrization outside a base interval $[-L, L]$. Note that we name the period $2L$ for a convenient match with the period 2π in most of our examples.

Before carrying out the Fourier analysis for such a curve, we make some observations about symmetry. There are several natural ways to decompose the functions $x(t)$ and $y(t)$. We define the even and odd parts of these two functions as usual, but also identify parts that we call the *glide-positive* and *glide-negative* parts:

$$x(t) = x_e(t) + x_o(t) = \frac{x(t) + x(-t)}{2} + \frac{x(t) - x(-t)}{2}$$

$$y(t) = y_e(t) + y_o(t) = \frac{y(t) + y(-t)}{2} + \frac{y(t) - y(-t)}{2}$$

$$x(t) = x_{g^+}(t) + x_{g^-}(t) = \frac{x(t) + x(t+L)}{2} + \frac{x(t) - x(t+L)}{2}$$

$$y(t) = y_{g^+}(t) + y_{g^-}(t) = \frac{y(t) + y(t+L)}{2} + \frac{y(t) - y(t+L)}{2}.$$

It turns out that every type of symmetry possible for curves of this type can be achieved by setting one or more parts of these decompositions to zero. To elaborate, we name some particular Euclidean motions of the plane and give their coordinate formulas.

TABLE 1. Some useful symmetries

Symbol	Description	Equation
τ	translation	$\tau(x, y) = (x + K, y)$
ρ	half-turn about origin	$\rho(x, y) = (-x, -y)$
σ_v	vertical mirror	$\sigma_v(x, y) = (-x, y)$
σ_h	horizontal mirror	$\sigma_h(x, y) = (x, -y)$
γ	glide reflection	$\gamma(x, y) = (x + \frac{K}{2}, -y)$

Note that our decompositions and choice of symmetries favor the origin. We comment on this later, but this is done without loss of generality.

The decomposition of $x(t)$ and $y(t)$ into even and odd parts makes identification of G_2 symmetry easy: if the even part of each function is zero, the curve $c(t)$ will be symmetric about the origin. Since it is also translation invariant, it will have G_2 symmetry.

G_4 symmetry is also easy to recognize in this set-up. If the even part of $x(t)$ and the odd part of $y(t)$ both vanish, G_4 symmetry will result. We elaborate this point with the equation

$$(x(-t), y(-t)) = (-x(t), y(t)) = \sigma_v(x(t), y(t)).$$

G_6 and G_7 are also simple cases. For G_7 simply require that x_{g^-} and y_{g^+} both vanish. Then

$$(x(t+L), y(t+L)) = \left(x(t) + \frac{K}{2}, -y(t)\right) = \gamma(x(t), y(t)).$$

G_6 symmetry results when the additional conditions for G_2 invariance are imposed.

A problem arises when we look for G_3 symmetry by this method. The desired equation would be:

$$(x(-t), y(-t)) = (x(t), -y(t)) = \sigma_h(x(t), y(t)),$$

and one might expect to satisfy this by choosing the odd part of $x(t)$ and the even part of $y(t)$ to be zero. However, a simple computation shows that $x(t)$ cannot be an even function; this is inconsistent with the translation equation. Intuitively, the curve cannot simultaneously move periodically to the right and appear equally on both sides of the horizontal axis.

We have solved this problem for woven ropes by using two strands to make a frieze. To create a pattern with a horizontal mirror, we introduce a second strand that mirrors the first. This can be done in several ways; to achieve G_3 symmetry, the original curve can have either G_1 or G_7 symmetry. Our classification of symmetries by the shape of the path the rope follows does not permit a distinction between these two types.

We arrive at G_5 realizing that a pair of curves will be necessary to create a frieze with a horizontal mirror. We would like the pattern created by both paths together to have G_5 symmetry, which means we need invariance under ρ and σ_v. Since $\rho = \sigma_v \sigma_h$ and $\sigma_v = \sigma_h \rho$, as long as we have one of these symmetries, we must have the other. Since there are two strands, we may construct either one to have one of these symmetries and the composite pattern will have G_5 symmetry. Thus G_5 patterns may be made from pairs of curves having G_2, G_4, or G_6 symmetry.

We summarize these results in Table 2.

It is natural to ask whether we have left something out. We have not considered every possible pair of conditions. For instance, what if $x_{g^-} \equiv 0$ and $y_{g^-} \equiv 0$? One can

TABLE 2. Symmetry recipes for frieze curves

Type	Conditions on components	min # strands
G_1	no components vanish	1
G_2	$x_e \equiv y_e \equiv 0$	1
G_3	G_1 or G_7 conditions in mirrored strands	2
G_4	$x_e \equiv 0$, $y_o \equiv 0$	1
G_5	G_2, G_4, or G_6 conditions in mirrored strands	2
G_6	$x_e \equiv y_e \equiv x_{g^-} \equiv y_{g^+} \equiv 0$	1
G_7	$x_{g^-} \equiv y_{g^+} \equiv 0$	1

easily check that such a function would have period L, rather than $2L$. The smallest translation of such a pattern would have length $\frac{K}{2}$. If one wanted to construct friezes of that period, presumably one would have started with that condition from the start. Also, some of the components cannot vanish. As noted above $x_o \equiv 0$ contradicts the translation equation. The same is true for x_{g^+}.

Frieze curves: Fourier analysis As a first step in the Fourier analysis, note that a curve satisfying the equation of translational symmetry can be thought of as having one component causing it to move to the right, and another, more interesting component causing decorative twists and turns. We sort out the interesting part by subtracting the known motion to the right (at a rate of K units of distance per $2L$ units of time), and define:

$$m(t) = (x(t), y(t)) - \left(\frac{Kt}{2L}, 0\right) = (h(t), v(t)),$$

where $h(t)$ and $v(t)$ indicate the horizontal and vertical components of the decorative portion of $c(t)$.

Now $m(t)$ is continuous and periodic of period $2L$, and so has a convergent Fourier series of the form

$$m(t) = \left(\sum_{n=0}^{\infty} a_n \sin\left(\frac{n\pi t}{L}\right) + b_n \cos\left(\frac{n\pi t}{L}\right), \sum_{m=0}^{\infty} c_m \sin\left(\frac{m\pi t}{L}\right) + d_m \cos\left(\frac{m\pi t}{L}\right)\right).$$

It becomes simple to translate the requirements above into conditions on the coefficients of the Fourier series. For example, $x(t)$ and $y(t)$ will both be odd functions if only sine terms appear; the recipe for constructing G_2 friezes is simply stated as $b_n = d_m = 0$.

The frieze in our first example has formula

$$c(t) = \left(\frac{13t}{2\pi} - \sin(2t) - 5\sin(3t) - 5\sin(4t), \sin(t) + 2\sin(2t)\right).$$

Here $L = \pi$ and $K = 13$. This was found by experimentation, once we knew that only sine terms could be included.

For symmetry type G_4, we see that one needs sines in the horizontal terms and cosines in the vertical terms. The rope frieze in FIGURE 4 was designed from this

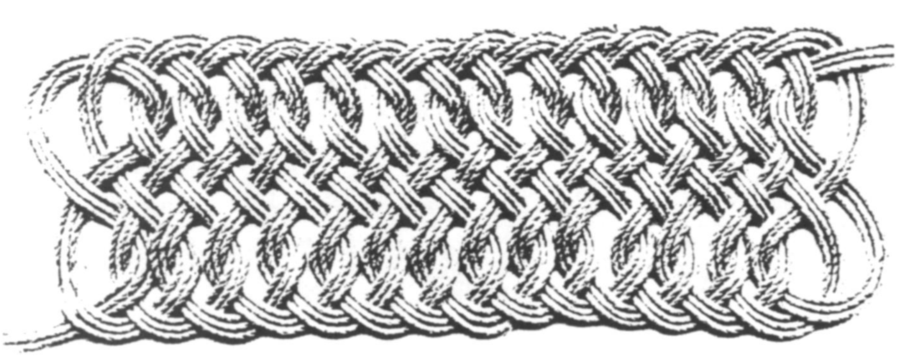

FIGURE 4
A woven G_4 frieze

recipe, using only one term of each Fourier series for simplicity. The equation is

$$c(t) = \left(\frac{0.8t}{2\pi} + \cos(2t), -0.8\sin(t) \right)$$

Types G_6 and G_7 are only slightly more tricky. One must use even values of n (the coefficients in the horizontal terms) and odd values of m (those in the vertical terms). Restricting both series to sine terms only will add the symmetry of the half-turn, yielding a frieze curve of type G_6.

One thing that may disturb the reader about G_6 patterns is that they have vertical mirrors without obeying the G_4 recipes. The vertical mirror in G_4 recipes was set up through the origin, through our privileged choice of coordinates. The vertical mirror in the G_6 patterns is about the line $x = \frac{L}{2}$.

FIGURE 5 shows a pattern woven from two ropes, with G_6 symmetry in each strand, producing G_5 symmetry overall. One thing to note at this point is that a formula for

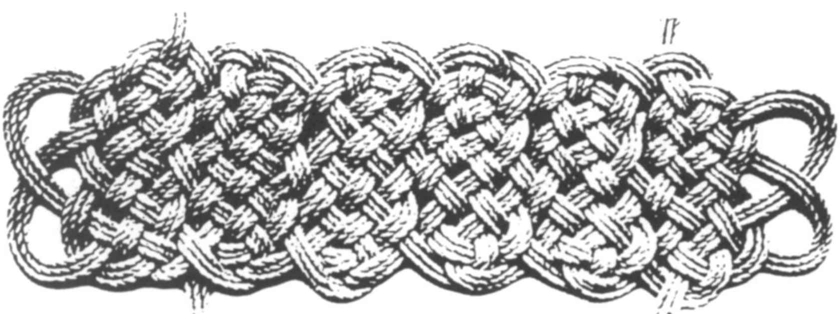

FIGURE 5
A G_6 pattern woven from two ropes

the pattern above could be detected by spectral analysis, using software like MATLAB. The equations used to produce a single strand are as follows:

$$x(t) = \frac{13t}{2\pi} - 1.5\sin(2t) - 3.5\sin(4t);$$

$$y(t) = -0.3\sin(t) - 3.2\sin(3t).$$

Table 3 gives a summary of all these recipes. We hope readers will experiment and make friezes of their own.

TABLE 3. Symmetry recipes for frieze curves

Type	Strands	Recipe and remarks
G_1	1	No additional requirements; use general parity
G_2	1	sine terms only
G_3	2	No additional requirements; use general parity
G_4	1	sines in $x(t)$, cosines in $y(t)$, general parity
G_5	2	G_2, G_4 or G_6 requirements in one strand, mirrored
G_6	1	n is even; m is odd; sines only
G_7	1	n is even; m is odd; both sines and cosines appear

Frieze ropes: further directions It may be unsatisfying to some readers that the instructions above only show how to construct curves with desired frieze symmetries. Surely the fact that these curves are realized physically as woven ropes is the most attractive thing about our illustrations. For empirically minded readers, we offer a way to experiment with these formulas in search of their own patterns to weave into ropes. For those interested in theory we expand a bit on the questions we have not yet answered.

To experiment making your own designs, you could use any computer algebra system, such as *Mathematica* or *DERIVE*, but we found this a bit cumbersome. Farris has written a JAVA applet allowing you to try out the formulas. An additional strand mirroring the first is available at a click of the mouse. To use this applet, direct your Java-enabled web browser to `http://math.edu/~ffarris/frieze.html` or `http://www.maa.org/pubs/mathmag.html`.

Probably the most interesting theoretical question involves the difference between curves and ropes. In all our examples we have simply assumed that the rope crosses itself in an alternate fashion, going over and then under every time. We call this the simple coding pattern. It is certainly possible to use more complex coding patterns, for instance, two over, one under, and so on. Such different coding patterns can be implemented very nicely in the simplest patterns, however for the friezes shown here it is difficult to attain handsome results.

On the other hand, almost every pattern we have encountered can be woven with the simple coding pattern. Experience teaches one to recognize patterns for which the simple coding pattern will not work. Perhaps a theoretical result is possible: the classification of curves for which the simple coding pattern is suitable.

There are further questions about the relationship between coding patterns and symmetry. In several examples, a half-turn has the effect of negating the coding pattern of the symmetry. A finer classification of frieze ropes might use the two-color frieze groups, outlined, for instance, in Grünbaum and Shephard **[3]**.

Finally, the techniques we use here are certainly adaptable for the construction of curves with wallpaper symmetry, at least if infinitely many strands are used. We would be delighted to see examples of wallpapers discovered using Fourier analysis and woven from rope.

How we came to write this note Our article had an interesting genesis. Rossing, a Norwegian radio engineer, responded to Farris's *Mathematics Magazine* article **[2]**, saying that he had already woven rotationally symmetric patterns like those described in the article. He had combined Fourier analysis with the old sailors' craft of weaving rosettes from rope, and found mathematical expressions for many existing rosettes. The tools Rossing developed had enabled him to produce new rosettes that he then wove out of rope **[4]**.

Farris proposed that they carry out the analysis to find the most general formula for patterns of each type of frieze symmetry, and see whether this would lead to the discovery of interesting designs suitable for rope work. Once the formulas were found, we experimented and found many examples for Rossing to weave with rope. As we have not actually met in person, we find this an Internet success story.

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