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Forbidden Symmetries

Frank A. Farris
Santa Clara University, ffarris@scu.edu

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Forbidden Symmetries

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The crystallographic restriction, as it applies to patterns in the plane, tells us that a pattern invariant under two linearly independent isometries cannot have 5-fold symmetry. And yet the pattern in Figure 1 seems to have translational symmetry in two directions as well as rotational symmetry through 72°. To see what I mean, start with the wheel shape at the point labeled A and notice translational symmetries along vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$; then rotate the image 72° about A, so that B goes to C, and see that the image is unchanged, apparently in violation of the crystallographic restriction. How can this be?

A moment's thought can break the illusion: If the pattern really enjoyed translational symmetry along vector $\overrightarrow{AB}$, then we could rotate 72° clockwise about B and move A to C. Alas, angle $\angle ABC$ is an unfortunate 54°, so B is not truly a translate of A.

The purpose of this paper is to show that an effort to construct functions known not to exist may on occasion produce interesting frauds. Our method produces a family of Harald Bohr's *quasiperiodic functions*, which may well remind readers of the quasicrystals that have been much in the news since Daniel Shechtman won the Nobel Prize in Chemistry in 2011 [1].

The term *quasicrystal* has an interesting history, as explained by Senechal in the *Notices* [6]. Diffraction patterns found by Shechtman in 1982 displayed 5-fold symmetry and so fell outside the mathematical categories commonly accepted as encompassing all possible crystalline structures. Our quasiperiodic functions are a different sort of object altogether. Unlike the structures studied by crystallographers, which are idealized as sets of isolated points in space, these are smooth *functions* that have honest 5-fold symmetry about a single point and come so close to having translational symmetry that they can easily fool the eye, provided we select a suitable translation.

The techniques developed to create these functions may offer more interest beyond creating fraudulent images such as Figure 1. We show how

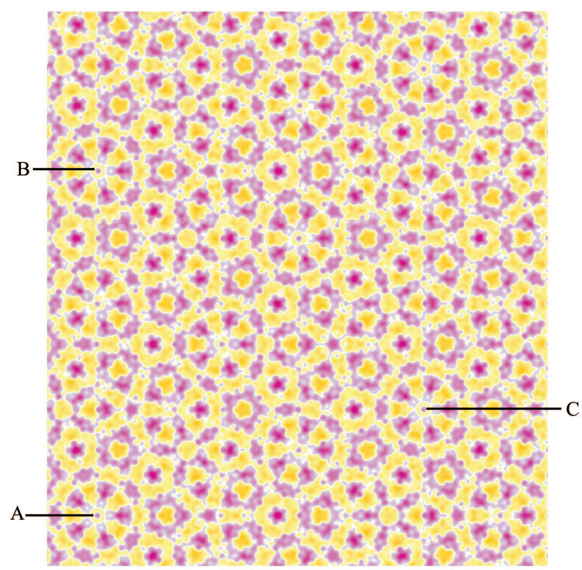

Figure 1. Wallpaper with 5-fold symmetry?

Frank A. Farris is associate professor of mathematics at Santa Clara University. His email address is `ffarris@scu.edu`.

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to construct functions with 3-fold symmetry and how the technique breaks down when we try to change 3 to 5.

Preliminaries

By *wallpaper group* we mean a group G of Euclidean isometries of the real plane whose translational subgroup is a lattice generated by two linearly independent translations. As is well known, there are seventeen isomorphism classes of such groups. Also well known is the fact that if ρ is a rotation in one of these groups, then its order is 2, 3, 4, or 6.

Unlike many discussions of patterns in the plane which refer to subsets, called *motifs*, being repeated without overlap [4, p. 204], we use analysis to develop the concept of pattern. For us, a pattern is given by a *wallpaper function*, which is a real- (or perhaps complex-) valued function on the real plane that is invariant under the action of one of the wallpaper groups [2]. In symbols, we require that

$$f(g\mathbf{x}) = f(\mathbf{x}) \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^2, \;\; g \in G.$$

For any given group G, it is easy to construct such functions by superimposing plane waves invariant with respect to the lattice of translations in the group. My paper "Wallpaper functions" [2] explains the construction and also covers functions with color-reversing symmetries. For wallpaper functions with 3-fold symmetry, an alternate method is possible, one that readers familiar with group representations may recognize. We present this method and try to generalize it to produce functions with 5-fold symmetry. In the generalization, we can see exactly why Figure 1 fails to enjoy honest translational symmetry.

Constructing Patterns with 3-fold Symmetry

To construct functions on the plane with 3-fold symmetry, we start with an unlikely object: cyclic permutation of *three* variables, considered as a linear transformation of $\mathbb{R}^3$. After all, this permutation, which we will call P, does have order 3. If f is any function of three variables, then $f(x,y,z)+f(z,x,y)+f(y,z,x)$ is a function invariant under P.

Since we seek periodic functions, let us suppose that the function $f(x,y,z)$ is periodic with respect to the integer lattice in $\mathbb{R}^3$. Let us limit ourselves to constructing continuously differentiable functions. Then we may assume that f is represented as the sum of its Fourier series in three variables:

$$f(x,y,z) = \sum_{\mathbf{n}\in\mathbb{Z}^3} a_{\mathbf{n}} e^{2\pi i \mathbf{n}\cdot\mathbf{x}}.$$

For 3-fold symmetry, we may either take such an f and average it over cyclic permutation of variables or require that

$$a_{\mathbf{n}} = a_{\mathbf{n}P} = a_{\mathbf{n}P^2},$$

since these coefficients will become equal after averaging. (Here, we think of the matrix P as acting on row vectors $\mathbf{n}$, the adjoint action.)

It remains to find planar functions somewhere in this 3-dimensional setup. The eigenspaces of the linear transformation P are easily found to be the line generated by $[1,1,1]$ and a plane Π, with basis $\mathbf{V}_1 = [1,-1,0]$ and $\mathbf{V}_2 = [0,1,-1]$, on which P acts as rotation through an angle $2\pi/3$.

If we restrict f to the plane Π, the resulting function has translational invariance with respect to the integer vectors $\mathbf{V}_1$ and $\mathbf{V}_2$, as well as rotational symmetry through an angle of $2\pi/3$. Of course, f is necessarily invariant with respect to all compositions of these symmetries and so is a wallpaper function with group at least as large as the one generated by the rotation and one translation. (Translation along $\mathbf{V}_2$ can be written by conjugating the other translation by the rotation.) We call this group $p3$, preferring the notation of the International Union of Crystallography to Conway's orbifold notation.

We can easily construct wallpaper functions with groups that contain $p3$ as a subgroup by including special groupings of terms in the Fourier expansion of the function of three variables before we restrict. To explain this in some detail, we begin with packets of waves invariant under 3-fold rotation:

$$W_{\mathbf{n}}(\mathbf{x}) = \frac{1}{3}(e^{2\pi i\mathbf{n}\cdot\mathbf{x}} + e^{2\pi i\mathbf{n}\cdot P\mathbf{x}} + e^{2\pi i\mathbf{n}\cdot P^2\mathbf{x}}),$$

where we divide by three simply so that $W_{\mathbf{n}}(0) = 1$. Notice that when restricted to the plane Π,

$$W_{\mathbf{n}+(j,j,j)} = W_{\mathbf{n}} \text{ for } j \in \mathbb{N}.$$

Setting $\mathbf{n} = (n_1, n_2, n_3)$ and representing a typical point in Π as $X\mathbf{V}_1 + Y\mathbf{V}_2$, we write these wave packets in the form

$$W_{n,m}(X,Y) = \frac{1}{3}(e^{2\pi i(nX+mY)} + e^{2\pi i(mX-(n+m)Y)} + e^{2\pi i(-(n+m)X+nY)}),$$

where

$$n = n_1 - n_2 \text{ and } m = n_2 - n_3.$$

If we wish to choose n and m first and find a vector $\mathbf{n}$ that gives rise to those frequencies, one choice is $(n+m, m, 0)$.

This is why we claim that every (sufficiently smooth) wallpaper function on the plane with 3-fold rotational symmetry can be exhibited as the restriction of a function of three variables that is periodic with respect to the integer lattice and invariant under cyclic permutation of variables: The functions $W_{n,m}(X, Y)$ form a basis for functions with the desired symmetry in the plane, and every $W_{n,m}$ is the restriction of some $W_{\mathbf{n}}$. We write

$$f(X, Y) = \sum_{n,m} a_{n,m} W_{n,m}(X, Y) \tag{1}$$

for the typical wallpaper function—a superposition of wallpaper waves.

To continue the discussion of functions with additional symmetries, we omit some messy details and present the reader with the fact that the map

$$\sigma_c(X, Y) = (Y, X)$$

is a mirror reflection in Π, so that any sum of the form (1) where

$$a_{n,m} = a_{m,n}$$

will represent a function with mirror symmetry. Every function of the form

$$f(X, Y) = \sum_{n,m} a_{n,m}(W_{n,m}(X, Y) + W_{m,n}(X, Y))$$

is invariant under the group p31m (or 3*3 for orbifold enthusiasts).

It is amusing to choose coefficients to create pleasing functions. One example of a function invariant under p31m is shown in Figure 2.

It is likewise amusing to work out recipes to create functions invariant under the groups p3m1, p6, and p6m, but that information appears elsewhere [2], along with the details about various ways to use color to depict a complex-valued function in the plane, so we move on to 5-fold symmetry.

Before we do, note this crucial fact about the construction: We were able to find a basis for

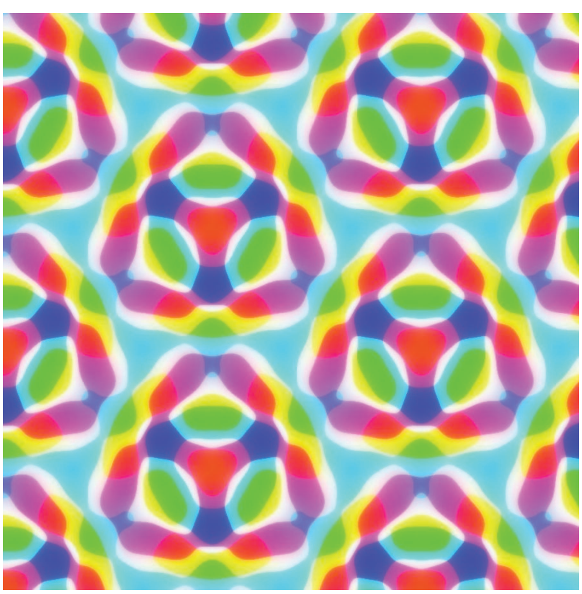

Figure 2. A function with symmetry group p31m.

Π consisting of vectors with integer coordinates. This is the only reason we could claim that the restricted function is periodic with respect to a lattice within Π, given only that it is periodic with respect to the integer lattice in $\mathbb{R}^3$.

What Happens If We Use 5 Variables?

The same construction can be attempted in $\mathbb{R}^5$. Let us review why we do not expect to create functions with 5-fold symmetry that are also periodic with respect to a lattice. The reason to include this well-known explanation is that it suggests something to look for in the images.

The Crystallographic Restriction

We prove a special case of the crystallographic restriction, showing that 5-fold rotations are never present in wallpaper groups: Suppose a wallpaper group G contains a translation T and a rotation R through $2\pi/5$ radians. In any wallpaper group, one may always find a shortest translation, so suppose further that T is a shortest translation in G. It is easy to check that $U := RTR^{-1}$ and $V := R^{-1}TR$ are translations along vectors at angles of $\pm 2\pi/5$ radians from the direction of T, as shown in Figure 3. If T translates the center of rotation O to the point A, then the composite translation UV translates O to X, producing a shorter translation and contradicting our assumptions. Therefore, no wallpaper group contains a rotation of order 5.

It is easy to compute that the ratio $\overrightarrow{OA}/\overrightarrow{OX}$ is the golden ratio, $(1 + \sqrt{5})/2$, which we denote ϕ. The reader may enjoy looking back at Figure 1, which seems to be invariant under a translation T and a rotation R through $2\pi/5$ radians. Where is the shorter translation that our computations have guaranteed?

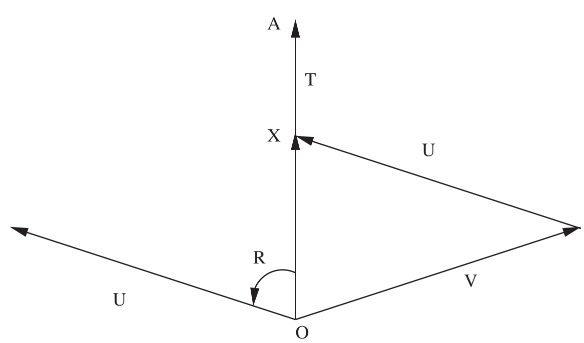

Figure 3. A translation and a rotation through 72° create a shorter translation: The composition of $V := R^{-1}TR$ with $U := RTR^{-1}$ produces a translation that is shorter than T by a factor of $1/\phi$.

Wallpaper with 5-fold Symmetry?

Let us imitate the procedure we used for 3-fold symmetry. Again we call P the linear transformation defined by cyclic permutation of the variables: $P(x, y, z, u, v) = (v, x, y, z, u)$.

Likewise, we take f to be any function periodic with respect to the integer lattice in $\mathbb{R}^5$ and symmetric under cyclic permutation. An easy first example, which we will carry through the remainder of this section, is

$$f(x, y, z, u, v) = \sin(2\pi x) + \sin(2\pi y) + \sin(2\pi z) + \sin(2\pi u) + \sin(2\pi v).$$

The eigenspaces of P are one line spanned by $[1, 1, 1, 1, 1]$ and two planes, with P acting as rotation through $2\pi/5$ in one plane and $4\pi/5$ in the other. We select the first of these planes to call Π. One basis for Π is

$$\mathbf{E}_1 = [1, \cos(2\pi//5), \cos(4\pi/5), \cos(6\pi/5), \cos(8\pi/5)],$$
$$\mathbf{E}_2 = [0, \sin(2\pi/5), \sin(4\pi/5), \sin(6\pi/5), \sin(8\pi/5)],$$

In our construction of functions with 3-fold symmetry we easily found an eigenbasis with integer entries. Some reorganization of $\mathbf{E}_1$ and $\mathbf{E}_2$ gives the basis

$$\mathbf{V}_1 = [\frac{1}{\phi}, \frac{1}{\phi^2}, -\frac{1}{\phi^2}, -\frac{1}{\phi}, 0],$$

$$\mathbf{V}_2 = [0, \frac{1}{\phi}, \frac{1}{\phi^2}, -\frac{1}{\phi^2}, -\frac{1}{\phi}] = P\mathbf{V}_1,$$

where ϕ is the golden ratio.

We can see that there is no way to obtain integer entries from these: If we clear a denominator of ϕ^2, we create a factor of ϕ in another entry. In fact, this plane is *irrational*, in the sense that it contains no rational vectors at all.

This throws a fly into the ointment if we are trying to construct wallpaper functions with 5-fold symmetry. However, it leads us to interesting pictures. Let us start with a function invariant under P and defined by a Fourier series relative to the integer lattice in $\mathbb{R}^5$. Then let us restrict f to Π by the equation

$$F(s, t) = f(s\mathbf{V}_1 + t\mathbf{V}_2).$$

When we do this with our example of the cyclic sum of sines, we obtain the function pictured in Figure 4. The origin of Π, near the lower left of the picture, is, in fact, a center of 5-fold symmetry. Toward the top there seems to be a translated appearance of the same pattern, with what looks like another 5-center. Another appears on a ray through the origin 72° from the vertical. Divide either of these apparent translation vectors by ϕ and notice that there is, well, not translational symmetry, but translational ballpark nearness of the pattern. Divide again.

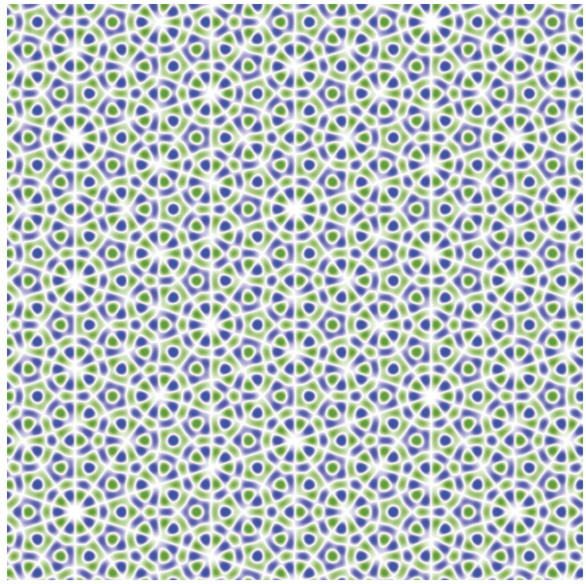

Figure 4. A cyclic sum of sine waves, restricted to the plane Π.

To move in the other direction, we recompute the image for the translated function $f((s+89)\mathbf{V}_1 + t\mathbf{V}_2)$. It looks exactly the same as Figure 4! We know that this function cannot have translational symmetry, but something interesting is going on.

Perhaps you have already guessed what, having recognized the number 89. Applying the Binet formula for F_n, the nth Fibonacci number,

$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}},$$

we find that, although there are no integer vectors in the plane Π, there are integer vectors arbitrarily close:

$$F_n\mathbf{V}_1 = [F_{n-1}, F_{n-2}, -F_{n-2}, -F_{n-1}, 0] + (-\phi)^{-n}[-1, 1, -1, 1, 0],$$

and likewise for Fibonacci multiples of $\mathbf{V}_2$. From this, it is an easy estimate to find that

$$|F(s + F_n, t) - F(s, t)| < 5/\phi^n.$$

Although this exemplary function f cannot have any translational symmetries (we know this from the crystallographic restriction), it does have what we will call *quasisymmetries*. As with Bohr's almost-periodic functions, given any $\epsilon > 0$, we can find a translational distance T so that

$$|F(s + T, t) - F(s, t)| < \epsilon.$$

Conclusion

The example is only one in a large space of functions with the same symmetry, or rather quasisymmetry. Just as in our construction of functions with 3-fold symmetry, we can start with any cyclically invariant waves in 5-space and

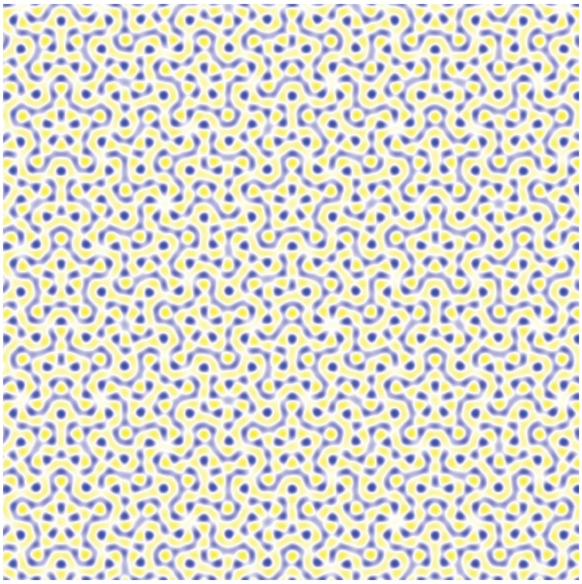

Figure 5. An unretouched depiction of our effort to create 5-fold symmetry.

restrict them to the plane Π. There are limited possibilities for mirror symmetry, though we will mention that the example that arose from the sine function had a mirror color-reversing symmetry. Superimposing only cosine waves does create functions with symmetry about five mirror axes, at least at one point. The opening example enjoyed this extra symmetry.

Though we have not violated the crystallographic restriction, we have found an interesting family of functions. They invite our eye to wander and enjoy the near-repeats. As for the attractive fraud that opened this paper: A bit of Photoshopping made it look rather more symmetrical than it is. Figure 5 gives an unaltered view of the quasisymmetry.

References

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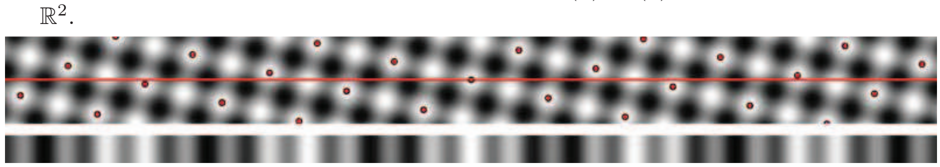
About the Cover

Irrational Symmetry

The cover image was produced by Frank Farris, and is a modification of Figure 4 in his article in this issue.

It records the values (through coloring and shading) of a fairly simple periodic function (specified in that article), restricted to one of the eigenplanes of the cyclic permutation in $\mathbb{R}^5$. What's important here, as he says in the article, is that although crystalline 5-fold symmetry is impossible, it can be very closely approximated. The basic idea behind what one sees is already apparent in two dimensions—the figure below shows in a similar way the restriction of $\cos(x)\cos(y)$ to an irrational line in $\mathbb{R}^2$.



But the details of what happens in Farris' figure are nonetheless striking, perhaps even hypnotic, and what one sees is certainly more striking in two dimensions than in one. Mathematically, the effect is due to lattice points in $\mathbb{Z}^5$ that are near to the plane, and the extraordinary accuracy with which patterns are repeated is presumably due to the accuracy with which the golden ratio is tightly approximated by rational numbers. And perhaps also something intrinsic to human visual perception.

We thank Frank Farris for the time and effort spent on this.

—*Bill Casselman*
Graphics Editor
(notices-covers@ams.org)