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A Noneuclidean Universe

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Let us construct a hypothetical universe, if for no other reason than to challenge our preconceptions about space. We call it a noneuclidean universe because it contradicts some of the notions central to euclidean geometry, where, for instance, the angle measures in a triangle add up to 180 degrees. There are many noneuclidean universes; ours is of a type called *hyperbolic*.

Let us imagine ourselves as living outside this new universe, looking down upon it as if it were half a sheet of paper with a distinct edge. We see the inhabitants of this universe as 2-dimensional beings, confined to live on the paper, without even a fingernail popping up out of it, and unable to go past the edge. These beings have a strange property, one that defines their space: As they move about their world, they seem to shrink in size as they approach the edge of the paper. Since their rulers—indeed, all of their atoms!—shrink just as they do, they detect nothing strange as they move about and view themselves as remaining the same size, just as you do when you move about our world.

This hypothetical universe has been constructed before. It is often called the Poincaré Upper Halfplane, in honor of French mathematician Henri Poincaré (1854–1912). I admire Poincaré because he is said to have been the last person in the world who understood all the mathematics known in his day. In honor of Poincaré, let us call the inhabitants of our hypothetical universe the *Poincarites*.

FIGURE 1 shows a time-lapse photograph of a Poincarite moving with uniform speed directly toward the edge of the universe. Although it looks to us as if the Poincarite is changing size, life is proceeding normally from her point of view. Our pictures are taken at even time intervals. Note that the Poincarite appears to approach but can never reach the edge of the universe: It would take infinitely many copies of this Poincarite's shrinking ruler to reach the edge. There seems to be lots of room down near the edge of the universe, which is consistent with our idea that a universe—which ought to contain all the space that there is—should appear infinite in extent.

Our goal here is to understand the world of the Poincarites. In particular, we wish to develop formulas for the transformations that move them about their universe. Just as we can use translations

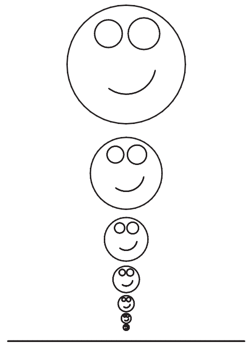

Figure 1: A walk toward the edge of the universe

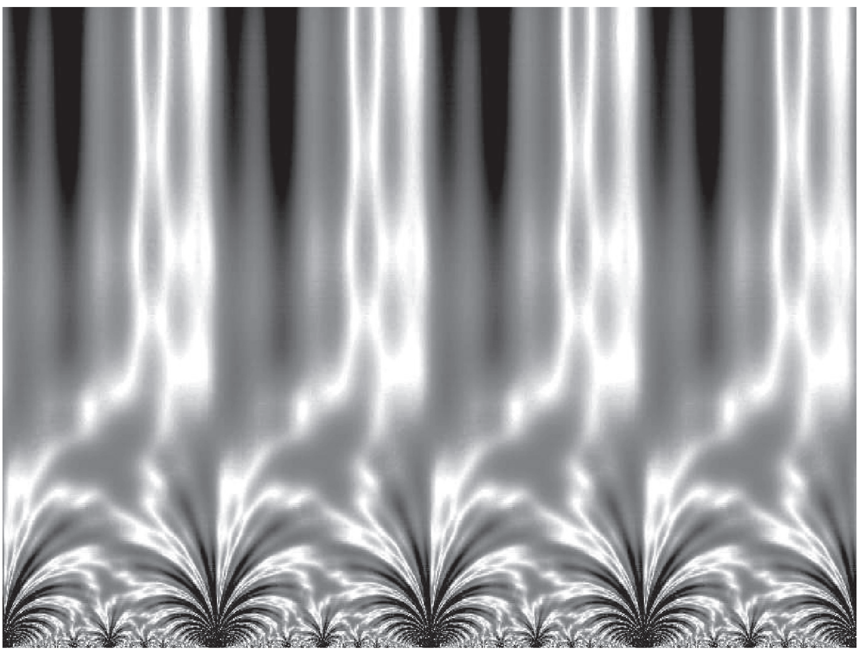

Figure 2: Wallpaper for the Poincarites

and rotations to move characters about a 2-dimensional space in a primitive video game, we can find the transformations that move the Poincarites about, always without changing their size or shape—from their point of view.

Knowing the transformations that move the Poincarites about our hypothetical universe allows us to say something about their physics: We can predict the straightest possible paths, the ones that a particle would follow if no external forces were acting upon it. Finally, we can wallpaper this universe with repeating patterns. In the same way that a checkerboard pattern fills out the ordinary Cartesian plane with infinite repetitions of the same picture, so our wallpaper patterns will fill out the Poincaré Upper Halfplane. An example is shown in FIGURE 2. Since there is so much extra room near the edge, these patterns get mind-bogglingly complex.

Moving Poincarites Around

Let us use ordinary Cartesian coordinates to describe the new universe. A point in the Poincaré Upper Halfplane is simply an ordered pair (x, y), with the restriction $y > 0$. (It will often make our computations more elegant to use complex notation and denote the same point by $z = x + iy$, where i is the famous symbol for the square root of -1. More on this later.) The line $y = 0$ is *not* part of the Upper Halfplane, but we do have occasion to refer to it, calling it the *ideal line* or *line at infinity*.

We need to be more specific about *how* the Poincarites shrink as they approach the edge of the universe. One way to do this is to explain how a Poincarite measures the length of a curve. Our explanations here become a bit technical. Feel free to skim them, as you can still get the main point of this article without this part.

Suppose a Poincarite's path is described by parametric equations

$$\alpha(t) = (x(t), y(t)), \quad \text{for } a \leq t \leq b.$$

We define the *length* of the path by the integral

$$\int_a^b \frac{\sqrt{(x'(t))^2 + (y'(t))^2}}{y(t)}\, dt. \tag{1}$$

The numerator of the integrand is the radical we would use, as euclidean observers, to measure the speed of motion; when we divide by $y(t)$, we make a unit velocity appear minuscule when y is large (and rulers are giant), while the same unit velocity would appear gigantic to the Poincarites if y were small—the region where rulers are tiny. This often seems backwards when one first hears of it. The key is that it takes many shrunken rulers to measure what appears to the outside observer as a unit distance.

The integral sign represents the theoretical limit obtained by breaking up the path into many small pieces and summing the lengths of the pieces. The reason we need such a complicated structure is that the Poincarites' rulers shrink continuously; when they move even a small amount toward the edge of the universe, they have shrunk just a bit. In fact, a ruler that is oriented with one end toward the edge of the universe is made of atoms that are smaller (always from the point of view of the outside observer) at the lower edge.

Transformation Geometry We take our strategy from Felix Klein (1849–1925), the German geometer who proposed that the best way to study any type of geometry is to study the transformations that leave that geometry unchanged. In Euclid's *Elements*, the monumental work from classical Greece, the idea of sliding one triangle on top of another to confirm congruence was present, though not explicitly mentioned. Klein would say that the essence of euclidean geometry is contained in the translations, rotations, and reflections that leave euclidean measurements unchanged. The modern term for any of these transformations is *isometry*, meaning "same measurement."

What are the analogous isometries of the Poincaré Upper Halfplane, the ones that experts call *hyperbolic isometries*? Given what we have said about the world of the shrinking ruler, it is easy to accept that the transformation

$$T_b(x, y) = (x + b, y),$$

which moves points b units to the right, is such an isometry. After all, we said that the ruler shrinks when we approach the edge of the universe and this transformation leaves us just as far from the edge as we were when we started.

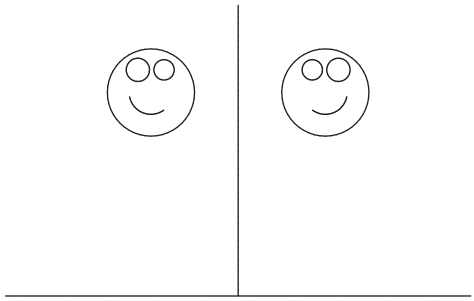

Figure 3: The reflection F is an indirect isometry

Similarly, the reflection about the y axis, given by

$$F(x, y) = (-x, y),$$

leaves all measurements the same. It does flip our Poincarite over, which cannot be done without leaving the universe—a somewhat suspect operation. However, we must admit that this reflection leaves measurements unchanged and hence fits our definition of an isometry. We call an isometry that reverses handedness—for instance changing a left-grinning Poincarite to a right-grinning one, as in FIGURE 3—an *indirect* isometry.

Referring back to (1), we can check that submitting the curve α to either the translation T_b or the reflection F does not change the length, confirming that each is an isometry.

A look back at FIGURE 1 might give insight into our next isometry. Although the Poincarite appears to be shrinking to an outside euclidean observer, in the universe of the shrinking ruler all copies are the same size. The figure was prepared using a *dilation*, which has the formula

$$D_R(x, y) = (Rx, Ry),$$

for various values of R. When $R < 1$, this transformation moves points toward the origin. It is a nice exercise to apply D_R to the curve α and check that the Rs cancel in (1); hence the length is unchanged and D_R is an isometry.

Poincarites Going Straight A beautiful result in euclidean geometry says that every isometry can be expressed as the composition of at most three reflections. Reflections are central to every study of transformation geometry. We are somewhat short on reflections and need to expand our repertoire—the reflection F is too special a case. This also gives us an excellent excuse to discuss the straightest possible paths of the Poincarites, the analogs of lines.

FIGURE 3 shows that every point on the y axis is left unchanged by the reflection F. This gives us a good reason to declare that this path is as straight as possible: If the path were actually not straight, then it would be curving either to the left or to the right; but, by symmetry, if it curves left, then it also curves right—a contradiction.

This vertical path is straight, and, since measurements are unchanged by horizontal translations left or right, *all* vertical paths are straight. Reflecting about any vertical line is easy; we will not need the formula for such a reflection, but the reader may enjoy deriving it.

Now for a more intriguing question: Suppose a Poincarite wishes to head in a horizontal direction, perhaps moved by the translation T_b, as b varies from 0 to 1. Is this path straight?

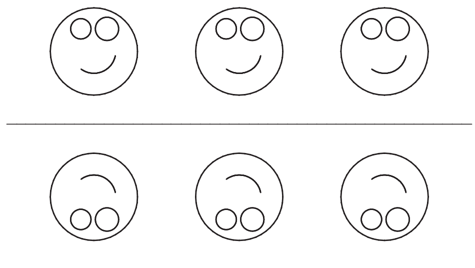

Figure 4: Reflecting about a horizontal path does not produce an isometry!

In FIGURE 4, we show a reflection about such a horizontal path. Poincarites in the top row are all the same size as one another, but the Poincarites in the bottom row, which look the same size to our euclidean eyes, are enormous by comparison! Their eyes alone would be humongous, measured with little tiny rulers down near the edge of the universe. Now imagine walking along the horizontal path from left to right, with your left hand near the upper row of Poincarites and your right near the lower row. Since your right hand travels much, much farther than your left, this path must be curving to the left!

What is the straightest path that heads in this initial direction? It turns out (and we will verify later) that the straightest possible path is a portion of a euclidean circle that meets the ideal line at right angles. (This is equivalent to the circle having its center on the ideal line—a property useful for constructing these paths.)

Consider the set of all of these straight paths, the vertical lines together with portions of circles that meet the ideal line at right angles. They share a startling number of properties with the lines of euclidean geometry: Given any two points, one and only one of these paths passes through them. Two such paths intersect in at most one point. Each path divides the plane into disjoint halves and a path joining points in different halves must intersect the dividing path. In fact, the first 28 of Euclid's propositions are true in this new interpretation (replacing the word *parallel* by *nonintersecting* in numbers 27 and 28) [1, p. 264].

The eerie similarity of these paths to euclidean lines was responsible for a remarkable shift in 19th century thought. I like to say that this situation "blew the collective mind of the 19th century." Here is why: If we call these paths "lines," they obey *all* of the same fundamental rules as euclidean lines, except for Euclid's Fifth Postulate, which can, roughly speaking, be stated as

> Given a line l and a point P not on it, there is exactly *one* line through P that does not intersect l.

Purists know this statement as *Playfair's Postulate*, but it is equivalent to Euclid's Fifth. This is a long story, which I will only touch on briefly, but FIGURE 5 shows a line l and a point P not on it, with two distinct lines through P that do not meet l. In fact, there are infinitely many such

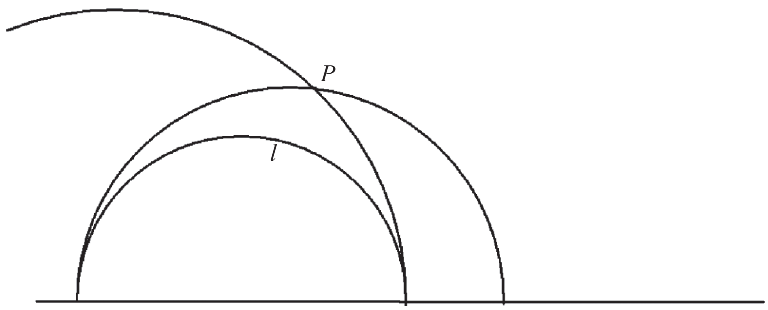

Figure 5: Given a line l and a point P not on it, there is *more than one* line through P that does not intersect l

nonintersecting lines; we have only shown the two limiting cases, where the lines meet on the ideal line, just outside the universe of the Poincarites.

From now on, the word *lines* (or *hyperbolic lines* for emphasis) will refer to portions of circles in the Upper Halfplane that meet the ideal line at right angles.

The Poincaré Upper Halfplane (and other geometric models like it) exposed a truth that the world resisted for centuries: Euclid's is not the only consistent theory of geometry. Indeed, although our experience seems to match euclidean geometry, we cannot really be sure that our own universe is euclidean. In fact, we cannot really be sure that the sum of the angle measures of a triangle in our own space really is 180 degrees; we only know that the angle sum is as close as we can measure. This should make the universe of the Poincarites seem less hypothetical.

More Hyperbolic Isometries It was easy to reflect the Poincaré Upper Halfplane across a vertical line. Surely there is a way to reflect across our new lines, these portions of circles with centers on the ideal line. In the interest of brevity, we reveal that a classical operation called *inversion in a circle* from euclidean geometry does the trick.

The following instructions for circle inversion use euclidean concepts: Suppose P is a point on the outside of circle C, with center O and radius R. To invert in circle C, locate the point P' on ray rOP with the property that

$$OP \cdot OP' = R^2,$$

where OP indicates the euclidean length of the segment from O to P. A moment's thought shows that P' must be inside the circle. This new point P' is called the *inversive image of P relative to the circle C*. Similarly, every point P' inside the circle (except for O) can be mapped to a point P outside the circle using this same formula. Points on the circle C are left fixed. (If you wish to send the center O somewhere, you must define a point called ∞ as its image; this is often done.) Coxeter [1, p. 77] explains how to construct P', given P and C, which is useful if you want to use something like *Geometer's Sketchpad* to investigate this type of geometry.

Let us use I to denote this inversion in the unit circle, restricted to points in the Upper Halfplane. We will not prove that I is really an isometry, but a brave reader can check, using (1) and the formula for I that we give later in (2). The fact that I is an isometry that fixes the circle proves that the circle is a straight path, justifying the claim we made earlier.

FIGURE 6 shows a row of Poincarites reflected through a typical nonvertical line. All six copies of the Poincarite are exactly the same size. Like the reflection in the vertical line, this reflection is

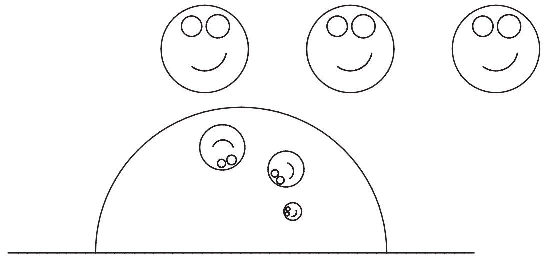

Figure 6: Reflecting a row of Poincarites about a typical hyperbolic line

an indirect isometry: Poincarites would have to temporarily leave the universe to end up with their features reversed, as in the picture.

What about rotations? So far, these have been neglected. One simple way to produce a rotation is to compose two reflections. For instance, if we reflect about the y-axis and then reflect (by circle inversion) across the hyperbolic line that corresponds to the unit circle, we will produce a rotation through 180 degrees about the point (0, 1), which can also be denoted i. Such a rotation is called a *halfturn*, because if you do it twice you have turned all the way around once. FIGURE 7 shows how our row of Poincarites looks after submitting them to this transformation, which we will call H.

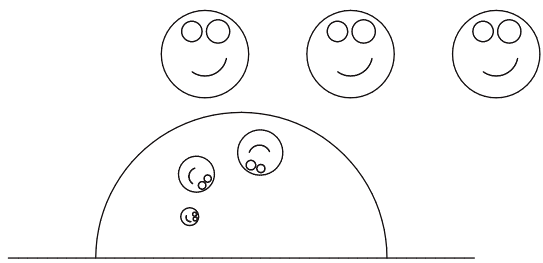

Figure 7: Rotating a row of Poincarites 180 degrees about i

Recall that the horizontal euclidean line through the top row of Poincarites—to be specific, say the line through the centers of their left eyes—is not a straight path in our hypothetical universe. However, it is a special type of curve and it has a nice name: This curve is called a *horocycle*. Since the shape of the curve is not changed by the isometry H, the turned Poincarites also line up along a horocycle. It turns out that horocycles are either horizontal lines or portions of euclidean circles that are tangent to the ideal line.

More general rotations can be obtained by reflecting through lines that meet at various angles, but this is not necessary here.

Complex Notation All of the transformations we have described so far take on elegant formulas when written in complex notation. Recall that this amounts to using the symbol $z = x + iy$ to denote the ordered pair (x, y). When we multiply two such expressions, we just use the distributive law and the convention that $i^2 = -1$. It is useful to use the notation $\bar{z}$ to mean $x - iy$, which is called the *complex conjugate* of z.

Since it negates the y coordinate of points, the operation of complex conjugation is just reflection across the x axis. For the Poincarites, this would mean leaving the universe, so we only use complex conjugates in conjunction with other operations. It is easy to compute that the isometry F, reflection about the y axis, can be written in complex notation as

$$F(z) = -\bar{z} = -(x - iy) = -x + iy,$$

which, in Cartesian coordinates is just $(-x, y)$ as before.

It is only a little harder to work out the formula for inversion through the unit circle. You can check that

$$I(z) = 1/\bar{z} = \frac{x}{x^2 + y^2} + i\frac{y}{x^2 + y^2} \tag{2}$$

matches the definition of the inversive image of z. Composing the two transformations gives a simple formula for the halfturn H:

$$H(z) = -1/z.$$

Check that $H(i) = -1/i = i$ and that $H(H(z)) = z$, so H is a halfturn with i as its unique fixed point, as claimed. This is all the complex notation we need to describe some wallpaper for our hypothetical universe.

The Wallpaper of the Poincarites

To introduce wallpaper patterns, we again draw on ideas from Felix Klein: We study the isometries that leave the pattern unchanged.

Start with a checkerboard pattern, which is something everyone can picture. If you had two copies of the pattern and slid the top one sideways, after sliding just the right distance, the two copies of the pattern would coincide, except at the edges. For a nice mathematical idealization, let's pretend that the pattern continues forever in all directions, so there are no edges. Any translation that causes the two copies of the pattern to coincide is an example of a *symmetry* of the pattern.

If we don't slide the top copy far enough, we might see a black square falling exactly on top of a red square. This is an example of an *anti-symmetry*. Symmetries and anti-symmetries are explained in the freely-available online article "Vibrating Wallpaper" [3], where you can also see animations of euclidean vibrating wallpaper drums.

Another way to bring the pattern into coincidence with itself is to rotate by 90 degrees about the center of any black square. (Rotating about a corner where two black squares meet two red ones produces another anti-symmetry.)

A principle that is obvious but nonetheless key is that when one performs one symmetry after another, the resulting transformation is also a symmetry. In other words, the composition of two symmetries is again a symmetry. In the case of the checkerboard, one can prove that *all* symmetries arise from compositions of a single horizontal translation and a single rotation. The formal language for this is that the *symmetry group* of the checkerboard (ignoring mirror symmetries) is *generated* by a translation and a 4-fold rotation. These are euclidean symmetries and the checkerboard is an example of euclidean wallpaper. For the Poincarites, we need symmetries that preserve their concepts of measurement. These are called *hyperbolic symmetries*.

A Particular Group of Hyperbolic Symmetries Suppose we would like to make a wallpaper pattern that stays the same when you translate it one unit to the right (using the transformation we called T_1). That much is easy, just construct a row of Poincarites, as in the top portions of FIGURES 6 and 7. Suppose that we also insist that our pattern should also be invariant under the halfturn H. FIGURE 7 gets us started, but the pattern is incomplete; it would need to have infinitely many translated copies of the original Poincarite, as well as inversive images of all those copies, as well as translated images of the inverted ones. Can you imagine these? Even they would not be enough, as we shall see.

We need to say something about the group generated by T_1 and H. Using the complex notation introduced in the last section, we see that we must include all transformations that can be produced by composing

$$T_1 = z + 1 \quad \text{and} \quad H = -1/z$$

in any combination. Certainly we need all powers of T_1. Note that $T_1^n = T_n$, so we can translate any whole number of units to the right. We also must have the inverse of any transformation, so we can translate any whole number of units to the left.

To see why the patterns are more interesting than a single inverted row of Poincarites and its translations, let us study the transformation

$$G = HT_{-1}H.$$

This transformation acts on our hypothetical universe by turning it about the point (0,1), then translating one unit to the *left*, and then turning back again. To understand this motion, let us look first at the turned row of Poincarites (the ones that lie along a horocycle) in the bottom of FIGURE 7. In the first halfturn, this row turns up to take the place of the top row, which is then shifted one unit to the left; in the second halfturn, the row returns to its position, but shifted over one Poincarite. Thus, G shifts the pattern one cycle along the lower horocycle.

This shifting is shown in several intermediate stages in FIGURE 8, with the action not quite completed in the last frame. Focus on the lower horocycle and notice that the Poincarites gradually shift along, so that in the last frame they have almost shifted one pattern to the right. Then focus on the upper horocycle and see that the Poincarites along it are moved to a new horocycle just next door to the first one.

For a real surprise, do the following thought experiment as you stare at FIGURE 8: You know that our pattern is supposed to be invariant under T_n for every integer n, so there must be a copy of the lower horocycle tangent to every ideal point of the form $(n, 0)$. Where do *they* go during this animation?

The answer is startling: They must *all* end up squished into the region between the original lower horocycle, along which we just shifted by one Poincarite, and the ending location of the top horocycle. There are infinitely many of them in there and, according to Poincarite measurements, they are all exactly the same size!

A little analysis can confirm the results of this thought experiment, especially if we consider what happens to ideal points, which we can consider as anchoring the horocycles in place. An easy computation shows that

$$G(z) = \frac{z}{z+1},$$

so $G(n) = n/(n+1)$, which will be between 0 and 1 on the ideal line, if $n > 0$.

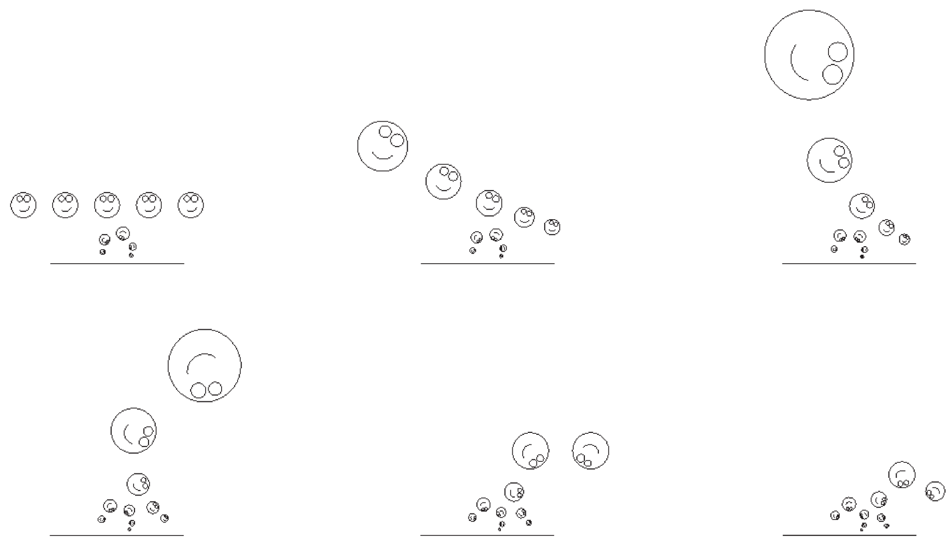

Figure 8: Sliding a row of Poincarites

Here is another way to look at FIGURE 8: Consider the horizontal row of Poincarites as lying along a horocycle anchored at ∞. The formula says that

$$\lim_{z\to\infty} G(z) = 1,$$

so, at the end of the sliding animation, the horizontal row moves to a horocycle anchored at 1.

One can show that

$$G^n(z) = \frac{z}{nz+1},$$

so after repeating the slide n times, that original horizontal row ends up along a horocycle anchored at $1/n$ (while remaining exactly the same size to the Poincarites the whole time).

Examine the pattern in FIGURE 9, which shows an x range of about -1.5 to about 1.7. The large fans suggest horocycles anchored at points -1, 0, and 1 on the ideal line. There is a congruent copy of the fan anchored at *every* rational point on the ideal line. I placed a black dot in the upper center of the figure to indicate the point i, which is the fixed point of the halfturn H.

Explaining the second black dot takes a little more work, but reveals something interesting. If we compose the halfturn with a unit translation to the left, we get the transformation

$$R(z) = T_{-1}(H(z)) = -\frac{1}{z} - 1 = -\frac{z+1}{z}.$$

We can easily check that $R^3(z) = z$, so that doing the transformation three times brings us back to where we started. We can also solve the equation $R(z) = z$ to find that the only fixed point of R in the Upper Halfplane is $-1/2 + i\sqrt{3}/2$, which is where I put the black dot. This means that R is a rotation through 120 degrees and that our pattern must have a center of 3-fold rotational symmetry. FIGURE 11 shows how a Poincarite moves under this isometry, staying exactly the same size and shape at each stage.

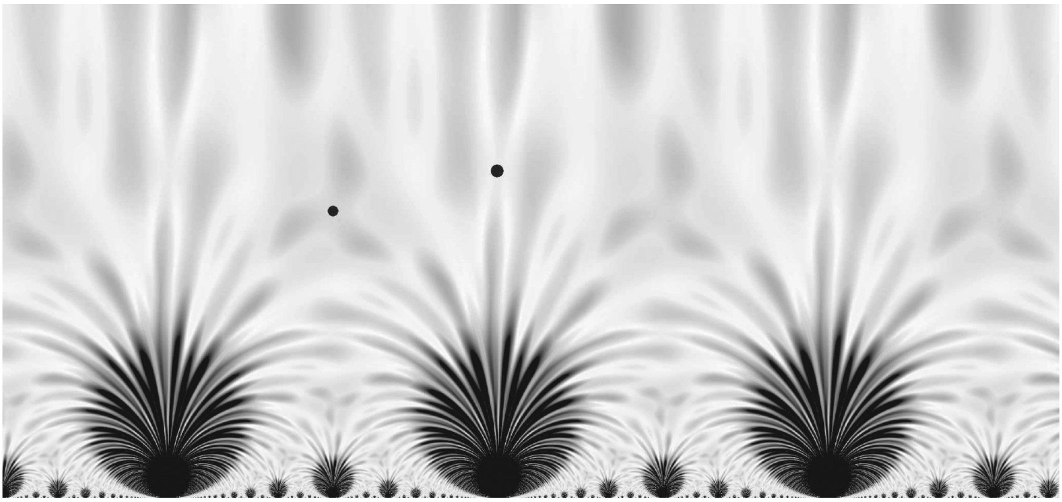

Figure 9: A swatch of hyperbolic wallpaper above the unit interval on the ideal line

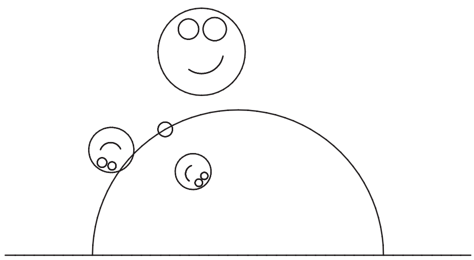

Figure 10: Images of a Poincarite under R, R^2, and R^3

Around the black dot, you may see something that looks a little like a lily with three petals. This is the center of 3-fold rotational symmetry and there are infinitely many echoes of it throughout the image.

The group of transformations generated by our halfturn and unit translation is well known. Roughly speaking, it consists of all transformations of the form

$$\frac{az+b}{cz+d}, \quad \text{where} \quad a, b, c, d \quad \text{are integers and} \quad ad - bc = 1.$$

To be precise, we need to acknowledge that replacing $a, b, c,$ and d by their negatives gives the same transformation, so every transformation has two expressions of this type. This contributes the word *projective* to the official name for this group: *the projective special linear group* $PSL(2, \mathbb{Z})$. Functions invariant under $PSL(2, \mathbb{Z})$ are called *modular functions* and all the images of hyperbolic wallpaper in this article were constructed by summing infinite series to produce various modular functions. The whole story is told in the article "The Edge of the Universe: Hyperbolic Wallpaper," [2] where the images appear in bright colors.

The most intriguing part of these images is the complicated part down near what we perceive as the edge of the universe. Just for fun, here is a magnified view of one more wallpaper—a particularly smooth one—showing that part in detail.

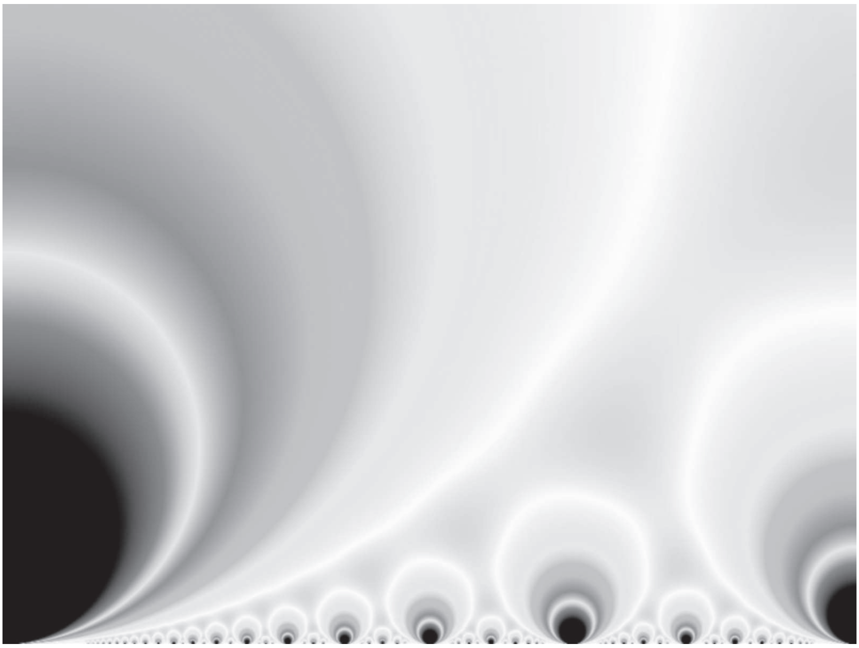

Figure 11: A close-up view of a wallpaper pattern over the interval (0, 1/2)

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Editor's note: Readers can order fabrics printed with these colorful designs at www.spoonflower.com/profiles/frankfarris.