Inquiry: The University of Arkansas Undergraduate Research Journal

Volume 16 Article 6

Spring 2014

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EARLY INVESTIGATIONS IN CONFORMAL AND DIFFERENTIAL GEOMETRY

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ABSTRACT

The present article introduces fundamental notions of conformal and differential geometry, especially where such notions are useful in mathematical physics applications. Its primary achievement is a nontraditional proof of the classic result of Liouville that the only conformal transformations in Euclidean space of dimension greater than two are Möbius transformations. The proof is nontraditional in the sense that it uses the standard Dirac operator on Euclidean space and is based on a representation of Möbius transformations using 2x2 matrices over a Clifford algebra. Clifford algebras and the Dirac operator are important in other applications of pure mathematics and mathematical physics, such as the Atiyah-Singer Index Theorem and the Dirac equation in relativistic quantum mechanics. Therefore, after a brief introduction, the intuitive idea of a Clifford algebra is developed. The Clifford group, or Lipschitz group, is introduced and related to representations of orthogonal transformations composed with dilations; this exhausts Section 2. Differentiation and differentiable manifolds are discussed in Section 3. In Section 4 some points of differential geometry are reiterated, the Ahlfors-Vahlen representation of Möbius transformations using 2x2 matrices over a Clifford algebra is introduced, conformal mappings are explained, and the main result is proved.

1. Introduction

The present article introduces fundamental notions of conformal and differential geometry, especially where such notions are useful in mathematical physics applications. The author considers this article as part of a broader aim of bridging the gap of understanding between mathematicians and physicists. The present article does not strongly make a connection between the mathematics presented and its applications to mathematical physics. It is instead a preliminary work, in which the mathematical tools for making such a connection are introduced with a greater emphasis on conformal than differential geometry.

The selection and treatment of topics in this article reflects the author's interest in Clifford analysis, which is the study of Clifford algebras and Dirac operators. Indeed, Section 2 begins with basic notions of Clifford algebras. Section 4 uses these notions to define a special sort of 2x2 matrices over an appropriate Clifford algebra, called Ahlfors-Vahlen matrices, that are themselves used in a proof of Liouville's classic result that the only conformal mappings on Euclidean space of dimension greater than two are Möbius transformations. In Section 3, differential geometry is developed with the eventual goal of discussing curved (Riemannian) manifolds with spin structure; on such a manifold, one can construct a Dirac operator relevant to supersymmetric quantum mechanics. The present author hopes to consider this in future work.

2. Clifford Algebras and Related Topics

1. The Clifford Algebra over $\mathbb{R}^n$

The Clifford algebra over the Euclidean space $\mathbb{R}^n$, denoted $C\ell(\mathbb{R}^n)$ or $C\ell_n$, is first constructed. Although an abstract construction is possible by factoring the tensor algebra of $\mathbb{R}^n$ by an appropriate two-sided ideal, the present construction chooses a particular basis with a particular quadratic form. Some important definitions and mappings in $C\ell_n$ are then introduced.

Porteous and Garling both provide good discussions of these notions. Garling thoroughly and formally constructs Clifford algebras over general real vector spaces.

To this end, consider $\mathbb{R}^n$ with a particular choice of basis $\{\mathbf{e_1}, \ldots, \mathbf{e_n}\}$ and the familiar dot product. Then for arbitrary vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$,

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i .$$

Suppose further this basis is orthonormal, so $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ for every i and j in $\{1, \ldots, n\}$. Recall the Kronecker delta δ_{ij} is 1 if i equals j and is 0 otherwise. The Clifford algebra $C\ell_n$ is defined as the span

$$< 1, \mathbf{e}_1, \ldots, \mathbf{e}_n, \ldots, \mathbf{e}_{j_1} \ldots \mathbf{e}_{j_k}, \ldots, \mathbf{e}_1 \ldots \mathbf{e}_n >, \quad 1 \le j_1 < \ldots < j_k \le n ,$$

modulo the relation $\mathbf{e}_i \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i = -2\delta_{ij}$ for every i and j in $\{1, \ldots, n\}$. This condition suffices to define a multiplication on $C\ell_n$. Note that distinct indices anticommute if $n \ge 2$ and $\mathbf{e}_i^2 = -1$ for $1 \le i \le n$. By a simple combinatorial argument using the binomial theorem, dim $C\ell_n = 2^n$.

Alternatively, $C\ell_n$ is defined as the free algebra on the chosen orthonormal basis for $\mathbb{R}^n$ subject to the relation $\mathbf{e}_i \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i = -2\delta_{ij}$. The free algebra alone would be infinite-dimensional, but going modulo the relevant relation reduces the dimension to 2^n: any finite concatenation of the basis elements over the chosen basis for $\mathbb{R}^n$ is either the empty word or can be rearranged as a product $\mathbf{e}_{j_1} \ldots \mathbf{e}_{j_k}$, $1 \le j_1 < \ldots < j_k \le n$, and there are only 2^n such words (note the empty word corresponds to the identity 1).

If $\{\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_k}\}$ has k elements, then the product $\mathbf{e}_{j_1} \ldots \mathbf{e}_{j_k}$ is said to be a k-vector, and any linear combination of such k-vectors is also said to be a k-vector. The subspace consisting of all k-vectors is denoted $C\ell_n^k$, and $C\ell_n = C\ell_n^1 \oplus \ldots \oplus C\ell_n^n$. For $a \in C\ell_n$, the projection of a onto $C\ell_n^k$ is denoted $[a]_k$. Often 0-vectors, 1-vectors, and 2-vectors are respectively referred to as scalars, vectors, and bivectors. Since $\mathbb{R}^n \cong C\ell_n^1 \subset C\ell_n$, it is convenient to speak of a vector in $C\ell_n^1 \subset C\ell_n$ as being a vector in $\mathbb{R}^n$. Henceforth, unless otherwise specified, each reference to a vector in $\mathbb{R}^n$ should be considered a reference to a vector in $C\ell_n^1 \subset C\ell_n$ identified with a vector in $\mathbb{R}^n$. In general, for every vector $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x}^2 = -\mathbf{x} \cdot \mathbf{x}$ and orthogonal vectors anti-commute.

Throughout the present article are used three involutions on $C\ell_n$. These are mappings that, upon acting twice on an element of $C\ell_n$, return that element. Involutions on $C\ell_n$ are specified by their actions on vectors. For each $\mathbf{x} \in \mathbb{R}^n$ and a and b in $C\ell_n$, these involutions and their actions are the following.

1. The main (principal) automorphism

$$\mathbf{x}' \to -\mathbf{x},\ (ab)' = a'b' \Rightarrow (\mathbf{e}_{j_1} \ldots \mathbf{e}_{j_k})' = (-\mathbf{e}_{j_1}) \ldots (-\mathbf{e}_{j_k}) = \pm \mathbf{e}_{j_1} \ldots \mathbf{e}_{j_k}$$

2. The main (principal) anti-automorphism

$$\overline{\mathbf{x}} \to -\mathbf{x}, \overline{(ab)} = \overline{b}\,\overline{a} \Rightarrow \overline{(\mathbf{e}_{j_1} \ldots \mathbf{e}_{j_k})} = (-\mathbf{e}_{j_k}) \ldots (-\mathbf{e}_{j_1}) = \pm \mathbf{e}_{j_1} \ldots \mathbf{e}_{j_k}$$

3. The reversion anti-automorphism

$$\mathbf{x}^* \to \mathbf{x},\ (ab)^* = b^* a^* \Rightarrow (\mathbf{e}_{j_1} \ldots \mathbf{e}_{j_k})^* = \mathbf{e}_{j_k} \ldots \mathbf{e}_{j_1} = \pm \mathbf{e}_{j_1} \ldots \mathbf{e}_{j_k}$$

Each involution can be expressed in terms of the other two. For example, $\overline{a} = (a')^* = (a^*)'$. Since $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x} = \overline{\mathbf{x}}\mathbf{x} = \mathbf{x}\overline{\mathbf{x}}$, every nonzero $\mathbf{x}$ has a well-defined multiplicative inverse $\mathbf{x}^{-1} = \overline{\mathbf{x}} / |\mathbf{x}|^2$.

2. The Lipschitz Group

The Lipschitz group over $\mathbb{R}^n$ is now introduced; it is denoted $\Gamma(n)$ and sometimes called the Clifford group. It is the subset of $C\ell_n$ generated by all invertible vectors (equivalently, all non zero vectors) in $\mathbb{R}^n$. Hence, the Lipschitz group:

$$\Gamma(n) = \{a \in C\ell_n : \exists \mathbf{x}_1, \mathbf{x}_d \in \mathbb{R}^n/\{0\}, d \in \mathbb{N} : a = \mathrm{x}_1 \ldots \mathrm{x}_d\}.$$

Orthogonal transformations can be represented using particular elements of the Lipschitz group. Indeed, let $\mathbf{y} \in S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : | \mathrm{x} | = 1\}$. Consider the decomposition $\mathbf{x} = \mathbf{x}^{\mathsf{P}} + \mathbf{x}^{\perp}$, where $\mathbf{x}^{\perp}$ is a vector orthogonal to $\mathbf{y}$ (so $\mathbf{x}^{\perp}$ and $\mathbf{y}$ anticommute) and $\mathbf{x}^{\mathsf{P}}$ is a vector parallel to $\mathbf{y}$ (so $\mathbf{x}^{\mathsf{P}}$ and $\mathbf{y}$ commute). This implies $\mathbf{y}\mathbf{x}\mathbf{y} = \mathbf{y}(\mathbf{x}^{\mathsf{P}} + \mathbf{x}^{\perp})\mathbf{y} = -\mathbf{x}^{\mathsf{P}} + \mathbf{x}^{\perp}$, and $\mathbf{y}\mathbf{x}\mathbf{y}$ is a reflection of $\mathbf{x}$ in the $\mathbf{y}$-direction — say $R_{\mathrm{y}} \in O(n)$, where $O(n)$ is the orthogonal group on $\mathbb{R}^n$. Since $\mathbf{y} \in S^{n-1}$ was arbitrary, this shows there exists a reflection $R_{\mathrm{y}} \in O(n)$ for every such $\mathbf{y}$.

It is convenient to define the so-called Pin group:

$$Pin(n) = \{a \in C\ell_n : \exists \mathbf{x}_1, \mathrm{K}, \mathbf{x}_d \in \mathrm{S}^{n-1}, d \in \mathbb{N} : a = \mathbf{x}_1 \ldots \mathbf{x}_d\}.$$

The Pin group $Pin(n)$ is a subgroup of $\Gamma(n)$. By repeated application of the above argument for a single reflection, for $\mathbf{a} \in Pin(n)$, $\mathbf{a}\mathbf{x}\mathbf{a}^*$ represents a (finite) sequence of reflections. By the Cartan-Dieudonne Theorem, every orthogonal transformation in $O(n)$ can be represented as such a sequence. Thus there is a surjective group homomorphism between $Pin(n)$ and $O(n)$. Note that $(-\mathbf{a})\mathbf{x}(-\mathbf{a}^*) = \mathbf{a}\mathbf{x}\mathbf{a}^*$, so $\{\pm 1\}$ is part of (in fact, it is) the kernel of this homomorphism. This shows $Pin(n)$ is a double-cover of $O(n)$. Moreover, this shows orthogonal transformations can be represented by particular elements of the Lipschitz group.

The above arguments generalize for representing orthogonal transformations composed with dilations. Let $\beta \in \Gamma(n)$, so $\beta = (\beta/\sqrt{\beta\bar{\beta}})\sqrt{\beta\bar{\beta}} \in Pin(n) \otimes \mathbb{R}^{n}.^{+}$. Thus $\beta \mathbf{x} \beta^{*}$ represents an element of $O(n) \otimes \mathbb{R}^{n+}$. Again using the Cartan-Dieudonne Theorem, one readily shows $\Gamma(n)$ is a double cover of $O(n) \otimes \mathbb{R}^{n+}$. Hence one can represent an orthogonal transformation with a dilation using elements of the Lipschitz group. An equivalent representation is $\nu(\beta)(\mathbf{x}) = sig(\beta)\beta\mathbf{x}\bar{\beta}$, where the "sign" function is defined as $sig(\beta) = \bar{\beta}\beta' / (\beta\bar{\beta}) \in \{\pm 1\}$ and takes the value +1 or -1 in accordance with $sig(\beta)\bar{\beta} = \beta^{*}$. This equivalent representation is used by Cnops and appears below in the proof of Liouville's Theorem.

3. Differentiation on Topological Manifolds

1. Differentiation

This treatment of differentiation in several variables follows after Rudin's, by considering several cases of functions of the form $f: \Omega \to \mathbb{R}^m$ with $f: \Omega \subset \mathbb{R}^n$ an open set. It works toward the definitions stated in Aubin's treatment of differentiation. Througout the entirety of this section, the Euclidean space $\mathbb{R}^n$ (respectively, $\mathbb{R}^m$) is treated in the standard way and not identified with $Cl_n^1 \subset Cl_n$ (respectively, $Cl_m^1 \subset Cl_m$).

n =1 and m =1

Let f be a real-valued function with domain $(a, b) \subset \mathbb{R}$; that is, f: $(a, b) \to \mathbb{R}$. For $x_0 \in (a,b)$, the derivative of f at x_0 is the real number defined by

$$f'(x_0) = \lim_{y \to 0} \frac{f(x_0 + y) - f(x_0)}{y}, \; x_0 + y \in (a,b),$$

provided this limit exists. This implies

$$f(x_0 + y) - f(x_0) = f'(x_0)y + |y|\,\omega(x_0, y)$$

for some remainder function ω: $\mathbb{R}$ x $\mathbb{R} \to \mathbb{R}^m$ satisfying $\lim_{y \to 0} \omega(x_0, y) = 0$. This expresses the difference $f(x_0 + y) - f(x_0)$ as a linear operator mapping y to $f'(x_0)y$ plus a small remainder. Using a natural bijective correspondence that exists between $\mathbb{R}$ and $L(\mathbb{R})$, one may regard the derivative of f at x_0 as a linear operator (not a real number) that maps y to $f'(x_0)y$. [The set of all linear transformations from $\mathbb{R}^n$ to itself is denoted $L(\mathbb{R})$. More generally, for real vector spaces V and W, $L(V,W)$ is the set of all linear transformations from V and W; if $V = W$, then $L(V,W) = L(V)$. For any fixed real number, we may regard multiplication by that real number as a linear operator on $\mathbb{R}$; conversely, any linear operator from $\mathbb{R}$ to itself is multiplication by some real number.]

n =1 and m ≥1

Let f be a vector-valued function with domain $(a, b) \subset \mathbb{R}^n$; that is, $f : (a, b) \to \mathbb{R}^m$. For $x_0 \in (a, b)$, the derivative of f at x_0 is the real vector defined by

$$f'(x_0) = \lim_{y \to 0} \frac{f(x_0 + y) - f(x_0)}{y}, \; x_0 + y \in (a, b),$$

provided this limit exists. This implies

$$f(x_0 + y) - f(x_0) = f'(x_0)y + |y| \omega(x_0, y)$$

for some remainder functions ω: $\mathbb{R}$x$\mathbb{R} \to \mathbb{R}^m$ satisfying $\lim_{y \to 0} \omega(x_0, y) = 0$. Note that y and $(f^1(x_0)y) \in \mathbb{R}^m$, so associated with each real number y $\in \mathbb{R}$ is a real vector $(f^1(x_o)y) \in \mathbb{R}^m$. This identifies $f'(x_0)$ as a linear operator from $\mathbb{R}$ to $\mathbb{R}^m$, that is, as a member of $L(\mathbb{R}, \mathbb{R}^m)$. The only difference between the present case and the n = m = 1 case is the real number values of functions are replaced by real vector values.

Thus, for $x_0 \in (a, b) \subset \mathbb{R}$ and differentiable mapping f: $(a, b) \to \mathbb{R}^m$ with m ≥1, the derivative of f at x_0 is the linear transformation $A \in L\,(\mathbb{R}, \mathbb{R}^m)$ satisfying

$$f'(x_0) = \lim_{y \to 0} \frac{|\, f(x_0 + y) - f(x_0) - Ay\,|}{|\, y\,|}.$$

$n \geq 1$ and $m \geq 1$

In this case, a transition is made from Rudin's treatment to Aubin's.

Definition 3.1.1. Let $\Omega \subset \mathbb{R}^n$ be an open set and let f: $\Omega \to \mathbb{R}^m$. Then f is **differentiable at** $x_0 \in \mathbb{R}^n$ if there exists a linear mapping $A \in L\,(\mathbb{R}^n, \mathbb{R}^m)$ such that for all $y \in \mathbb{R}^n$ satisfying $x_0 + y \in \Omega$,

$$f(x_0 + y) - f(x_0) = Ay + |\, y\,|\, \omega(x_0, y),$$

where $\omega\, \mathbb{R}^n$ x $\mathbb{R}^{n \to}\, \mathbb{R}^m$ is a remainder function satisfying $\lim_{y \to 0} \omega(x_0, y) = 0$. The linear mapping A is called the **differential of** f **at** x_0 and, if f is differentiable for all $x \in \Omega$, then f **is differentiable on** Ω. As A is a function of $x \in \Omega$, we may write $A = f'(x)$. If Ω Â $x \to f'(x) \in L\,(\mathbb{R}^n, \mathbb{R}^m)$ is a continuous map, then f is continuously differentiable on Ω $(f \in C^1(\Omega))$.[1]

The above definition raises a concern about uniqueness of the differential of f at x_0. This differential is unique: if A_1 and A_2 are both differentials of f at x_0, then one can readily show that $B = A_1 - A_2$ is identically zero.

Consider now a function (not necessarily differentiable) f: $\Omega \to \mathbb{R}^m$, where $\Omega \subset \mathbb{R}^n$ is an open set. Let $\{e_1, \ldots, e_n\}$ and $\{u_1, \ldots, u_m\}$ be standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. The components of f are real-valued functions $f_1, \ldots, f_m$ such that $f(x) = \sum_{i=1}^{m} f_i(x) u_i$, for all $x \in \Omega$. For $x \in \Omega$, $1 \leq i \leq m$, $1 \leq j \leq n$, the partial derivative $D_j f_i$ is defined to be the limit

[1] Here continuity is understood in the usual sense for metric spaces (pre-images of open sets are open) and *$L(\mathbb{R}^n, \mathbb{R}^m)$* is endowed with the norm defined by $|||\, A\,||| = sup\{|Ax\,|: x \in \mathbb{R}^n, |x| \leq 1\}$.

$$(D_j f_i)(x) = \lim_{t \to 0} \frac{f_i(x + te_j) - f_i(x)}{t},$$

provided this limit exists. This is also the derivative of f_i with respect to x_j, so $D_j f_i$ is often denoted $\frac{\partial f_i}{\partial x_j}$. If f is differentiable, then the various $D_j f_i$ exist and

$$f'(x)e_j = \sum_{i=1}^{m} (D_j f_i)(x) u_i, \ 1 \le j \le n.$$

This last expression suggests the matrix representation of $f'(x)$ with respect to the standard basis (which exists since $f'(x)$ is a linear operator), given by the $m \times n$ matrix

$$[f'(x)] = \begin{bmatrix} D_1 f_1 & \cdots & D_n f_1 \\ \vdots & \ddots & \vdots \\ D_1 f_m & \cdots & D_n f_m \end{bmatrix}.$$

In the case that $m = n$, then when f is differentiable at a point $x \in \Omega$, the Jacobian of f at x is defined as the determinant of the linear operator $f'(x)$, which may be considered as the determinant of the matrix $[f'(x)]$ in any particular basis of $\mathbb{R}^n$:

$$\frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)} = J_f(x) = \det f'(x) = \det [f'(x)].$$

It is a fact that f is C^1 on Ω if and only if all the various partial derivatives $D_j f_i$ exist and are continuous everywhere on Ω. A function f is said to be C^2 on Ω $(f \in C^2(\Omega))$ if and only if $\Omega \ni x \to f'(x) \in L(\mathbb{R}^n, \mathbb{R}^m)$ is a C^1 map of Ω; more concretely, f is C^2 on Ω if and only if all the various second-order derivatives $D_i D_j f$ exist everywhere on Ω and are continuous there. Functions that are k-times continuously differentiable, or C^k maps ($k \in \mathbb{N}$), are defined by induction (the above function f is said to be C^3 on Ω $(f \in C^3(\Omega))$ if and only if $\Omega \ni x \to f'(x)$

$\in L(\mathbb{R}^n, \mathbb{R}^m)$ is C^2 on Ω, and so on). Maps that are C^k for all $k \in \mathbb{N}$ are called C^∞ maps.

2. Manifolds and Tangent Spaces

The general notion of a topologial manifold is first introduced. The notion of local charts allows such manifolds to be treated in the familiar setting of $\mathbb{R}^n$, by mapping open neighborhoods of such manifolds to open neighborhoods of $\mathbb{R}^n$. Earlier differentiability results on $\mathbb{R}^n$ apply to these latter neighborhoods. This treatment follows after Aubin's.

Definition 3.2.1. A **manifold** M **of dimension** n is a Hausdorff topological space[2] such that each point P of M has a neighborhood Ω homeomorphic[3] to $\mathbb{R}^n$ (equivalently, to an open neighborhood of $\mathbb{R}^n$).

Definition 3.2.2. A local chart on a manifold M is a pair (Ω, φ), where Ω is an open set of M and φ a homeomorphism of Ω onto an open set of $\mathbb{R}^n$. An **atlas** is a collection of $(\Omega_i, \varphi_i)_{i\in I}$ of local charts such that $\bigcup_{i\in I} \Omega_i = M$, where I is some nonempty indexing set. The coordinates of $P \in \Omega$ related to the local chart (Ω, φ) are the coordinates of the point $\varphi(P)$ in $\mathbb{R}^n$.

Definition 3.2.3. Let $(\Omega_\alpha, \varphi_\alpha)$ and $(\Omega_\beta, \varphi_\beta)$ be local charts on M such that $\Omega_\alpha \cap \Omega_\beta \neq \varnothing$. The map $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(\Omega_\alpha \cap \Omega_\beta) \to \varphi_\alpha(\Omega_\alpha \cap \Omega_\beta)$ is a **change of charts**. An **atlas of class** C^k (resp. C^∞) on M is an atlas for which every change of charts is C^k (resp. C^∞). This notion allows one to consider equivalence classes on C^k atlases (resp. C^∞), where two atlases $(U_i, \varphi_i)_{i\in I}$ and

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[2] A topological space, in general, is a set X together with a collection of subsets T of X such that (1) $\varnothing \in T$, (2) $X \in T$, (3) T is closed under finite intersection, and (4) T is closed under arbitrary unions. The sets in T are called open sets and T is a topology on X. A topological space X is Hausdorff (also called T_2, to use the terminology of separation axioms) if, for any distinct points x and y in X, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

[3] Two topological spaces X and Y are homeomorphic if there exists a homeomorphism $f : X \to Y$ that is a continuous bijection with a continuous inverse function.

$(W_\alpha, \Psi_\alpha)_{\alpha \in \Lambda}$ of class C^k are equivalent if their union is an atlas of class C^k (that is, $\varphi_i \circ \Psi_\alpha^{-1}$ is C^k on $\Psi_\alpha(U_i \cap W_\alpha)$ when $U_i \cap W_\alpha \neq \varnothing$). A manifold together with an equivalence class of C^k atlases is a **differentiable manifold of class** C^k.

The remainder of this discussion is restricted to a well-behaved class of manifolds, what are called paracompact manifolds.

Definition 3.2.4. A topological space X is said to **paracompact** if every open cover of X has an open refinement that is locally finite. More explicitly, for every collection of open sets $\{\Omega_i\}_{i \in I}$ (where I is a nonempty index set) such that $X \subset \bigcup_{i \in I} \Omega_i$, there exists a collection of open sets $\{\Theta_i\}_{i \in I}$ such that $\Theta_i \subset \Omega_i$ for all $i \in I$ and $X \subset \bigcup_{i \in I} \Theta_i$ (there exists an open refinement) and for every point $x \in X$ there exists a neighborhood W of x such that $\{i \in I : W \cap \Theta_i \neq \varnothing\}$ is finite (locally finite).

A related notion is a topological space that is **countable at infinity**, in which there exists a collection of compact sets $\{K_l\}_{l=1}^{\infty}$ such that $K_l \subset \operatorname{int} K_{l+1}$ for all $l \in \mathbb{N}$ and $E = \bigcup_{l=1}^{\infty} K_l$.

Theorem 3.2.5. A paracompact manifold is the union of a family of connected manifolds that are countable at infinity.

Proof This is precisely Theorem 1.1 in Aubin's book, where a proof is given.

The notion of a **partition of unity** is introduced. Let $\{\Omega_i\}_{i \in I}$ be an open cover of a topological manifold M, where I is some indexing set. A family of real-valued functions $\{\alpha_i\}_{i \in I}$ is a **partition of unity** of M if (1) any point x has a neighborhood U such that $\{i \in I : U \cap \operatorname{supp} \alpha_i \neq \varnothing\}$ is finite and (2) $0 \leq \alpha_i \leq 1$ (for all $i \in I$), $\sum_{i \in I} \alpha_i = 1$. This partition of unity is **subordinate to the covering** $\{\Omega_i\}_{i\in I}$ if $\operatorname{supp}\alpha_i \subset \Omega_i$ for all $i \in I$). If each α_i is C^k, then $\{\alpha_i\}_{i\in I}$ is said to be a C^k partition of unity.

Theorem 3.2.6. On a paracompact differentiable manifold of class C^k (resp. C^∞), there exists a C^k (resp. C^∞) partition of unity subordinated to some covering.

Proof This same theorem is stated and proved as Theorem 1.12 in Aubin's book.

Definition 3.2.7. Let γ_i $(i = 1, 2)$ be differentiable maps of a neighborhood of $0 \in \mathbb{R}$ into an n-dimensional manifold M such that $\gamma_i(0) = P \in M$. Let (Ω, φ) be a local chart at P. Define a relation *R* by $\gamma_1 \sim \gamma_2$ if $\varphi \circ \gamma_1$ and $\varphi \circ \gamma_2$ have the same differential at zero. The relation *R* is an equivalence relation, and a **tangent vector** X **at** P **to** M is an equivalence class for *R*.

A differentiable real-valued function f on a neighborhood of P (as in the above definition) is **flat** at P if $d(f\circ\varphi^{-1})$ is zero at $\varphi(P)$, where $d(f\circ\varphi^{-1})$ is the differential of $f\circ\varphi^{-1}$ defined on $\varphi(\Omega)$. The definition is independent of any particular chart: if Ω is as above and (Θ, ϕ) is another local chart at P, then on $\Omega \cap \Theta$,

$$d(f\circ\phi^{-1}) = d(f\circ\phi^{-1})\circ d(\phi\circ\varphi^{-1}).$$

This allows one to introduce an alternative definition of a tangent vector, which can be shown to be equivalent to the earlier definition in terms of equivalence classes.

Definition 3.2.8. A tangent vector at $P \in M$ is a map X: $f \to X(f) \in \mathbb{R}$ defined on the set of the differentiable functions in a neighborhood of P, where X satisfies the following two conditions:

(a) If λ and μ are real numbers, then $X(\lambda f + \mu g) = \lambda X(f) + \mu X(g)$.

(b) $X(f) = 0$ if f is flat at P.

In the above definition, conditions (a) and (b) imply that $X(fg) = f(P)X(g) + g(P)X(f)$.

This follows because

$$X(fg) = X\Big(\big[(f - f(P)) + f(P)\big]\big[(g - g(P)) + g(P)\big]\Big)$$

$$= X\big((f - f(P))(g - g(P))\big) + f(P)X(g) + g(P)X(f) + X\big(-2f(P)g(P)\big),$$ the

constant function $-2f(P)g(P)$ is flat for all P, and $(f - f(P))(g - g(P))$ is flat at P. [For differentiable functions h and k, $d((hk)\circ\varphi^{-1})|_{\varphi(P)} = d[(h\circ\varphi^{-1})]d[(k\circ\varphi^{-1})]|_{\varphi(P)} = 0$ if h and k are zero at P.]

Definition 3.2.9. The tangent space $T_P(M)$ at $P \in M$ is the set of tangent vectors at the point P on the n -dimensional manifold *M*.

To see that the two definitions are equivalent, one may first show that $T_P(M)$ has an n -dimensional vector space structure with the n -partial derivatives evaluated at P as a basis. This allows one to construct a bijective mapping from the equivalence classes of R (recall that these classes are families of curves) to tangent vectors in $T_P(M)$. (Aubin 45-46) A general vector space structure is evident by defining, for an arbitrary differentiable function f on *M*, addition and scalar multiplication by

$$(X + Y)(f) = X(f) + Y(f) \text{ and } (\lambda X)(f) = \lambda X(f).$$

As for dimension, let $\{x^i\}$ be the coordinate system for a local chart (Ω, φ) of *P*. Consider then the vector $(\partial/\partial x^i)_P$ in $T_P(M)$ defined by

$$\left(\frac{\partial}{\partial x^i}\right)_P (f) = \left(\frac{\partial(f\circ\varphi^{-1})}{\partial x^i}\right)_{\varphi(P)},$$

which satisfies (a) and (b) in the second definition for a tangent vector. The vectors $(\partial/\partial x^i)_P$ $(1 \le i \le n)$ are independent by the orthogonality condition $(\partial/\partial x^i)_P(x^j) = \delta_{ij}$. Moreover, these vectors span $T_P(M)$, since $(f - \sum_{i=1}^{n} (\partial / \partial x^i)_P x^i)$ (where f differentiable) is flat, which implies $X(f) - \sum_{i=1}^{n} (\partial / \partial x^i)_P X(x^i)) = 0$; so $X(f) = \sum_{i=1}^{n} (\partial / \partial x^i)_P X(x^i)$. It follows that $\{\partial / \partial x_P^i\}$ is a basis, and the various $X^i = X(x^i)$ are the components of $X \in T_P(M)$ in this basis. Now let $\gamma(t)$ be a map in the equivalence class $\tilde{\gamma}$ ($\gamma(0) = P \in M$) and f a real-valued function in a neighborhood of *P*. Consider the map $X : f \to [\partial(f \circ \gamma)/\partial t]_{t=0}$ (a tangent vector in the second definition). One may then define a map $\Phi : \tilde{\gamma} \mapsto X$, since every γ_1 and γ_2 in $\tilde{\gamma}$ have the same differential at P and so $[\partial(f \circ \gamma_1)/\partial t]_{t=0} = [\partial(f \circ \gamma_2)/\partial t]_{t=0}$. It can be shown that Φ is bijective. The equivalence of the two definitions of a tangent vector follows.

The notions of smooth topological manifolds and tangent spaces at points on such manifolds are well demonstrated in classical mechanics. See the relevant discussion in the book by Takhtajan.

Definition 3.2.10. The **tangent bundle** $T(M)$ is $\bigcup_{P \in M} T_P(M)$. If $T_P^*(M)$ is the dual space of $T_P(M)$, then the **cotangent bundle** $T^*(M)$ is $\bigcup_{P \in M} T_P^*(M)$.

4. A Clifford Algebraic Approach to Möbius Transformations and Liouville's Theorem

This section introduces a Clifford algebraic approach to Möbius transformations and Liouville's Theorem. A Möbius transformation on Euclidean space is a finite composition of translations, orthogonal transformations, dilations, and inversions on that space. Such transformations are conformal; that is, they preserve measure and orientation of angles. In 1850, the French mathematician Joseph Liouville proved that, in Euclidean space of dimension greater than two, all conformal maps are Möbius transformations. In the 1930s, the Dutch mathematician

Johannes Haantjes generalized Liouville's result to nondefinite pseudo-Euclidean space (see the cited Haantjes paper). In the late 19th-century, the English geometer William Kingdon Clifford introduced what are now called Clifford algebras. In 1902, the German mathematician K.T. Vahlen introduced a representation of Möbius transformations using 2x2 matrices with entries in a Clifford algebra. This representation was largely forgotten until the late Finnish mathematician Lars Valerian Ahlfors revived it in the 1980s. The proof of Liouville's Theorem presented below uses this so-called Ahlfors-Vahlen representation, and adapts a proof by Jan Cnops. Indeed, the last two parts of this section provide a refined version of the treatment by Cnops, though restricted to the positive definite case.

1. Manifolds Revisited

The present part of Section 4 recalls the calculus on manifolds necessary to prove our main results. It also characterizes the tangent space at an arbitrary point in the Lipschitz group when considered as a manifold embedded in $C\ell_n$. General notions are discussed as in Warner's treatment rather than Aubin's. Results concerning the Lipschitz group are introduced by Cnops. Starting in Section 4.2, the remainder of this article identifies $\mathbb{R}^n$ with Cl_n^1.

Definition 4.1.1. A (smooth, parametrized) curve γ on $\mathbb{R}^n$ is an infinitely continuously differentiable (C^∞) mapping $\gamma : D \to \mathbb{R}^n$, where the domain D is an open interval of $\mathbb{R}^n$. If $0 \in D$, then γ is said to start at $\gamma(0)$ or to have a starting point at $\gamma(0)$. If $\gamma(D) \subset M$, where M is a manifold embedded in $\mathbb{R}^n$, then γ is a curve on *M*. In the present context, the tangent space $T_{\mathbf{a}}M$ at $\mathbf{a} \in M$ consists of all vectors $\mathbf{x}$ such that $\mathbf{x} = \partial_t \gamma(0)$ for some curve γ on M with starting point $\mathbf{a}$.

Definition 4.1.2. Let $f : M \to N$ be a function between two manifolds M and N embedded in $\mathbb{R}^n$. The differential of f in a point $\mathbf{a} \in M$ is the function $df_{\mathbf{a}} : T_{\mathbf{a}}M \to T_{f(\mathbf{a})}N$ such that, if the curve γ starts at $\mathbf{a}$, then $df_{\mathbf{a}}(\partial_t \gamma(0)) = \partial_t(f \circ \gamma)(0)$; note $df_{\mathbf{a}}$ is a linear map.

It is indicated without proof that $\Gamma(n)$ is a manifold embedded in $C\ell_n$. From this follows $\Gamma(n)$ is a Lie group. The tangent space of $\Gamma(n)$ at 1 is the span of Cl_n^0 and Cl_n^2 (scalars and bivectors). That is, $T_1\Gamma(n) = \mathrm{Cl}_n^0 \oplus \mathrm{Cl}_n^2$. Cnops provides a derivation of these facts (32-38). An immediate consequence is the following theorem.

Theorem 4.1.3. For arbitrary $\beta \in \Gamma(n)$, the tangent space of $\Gamma(n)$ at β is given by $T_\beta\Gamma(n) = (\mathrm{Cl}_n^0 \oplus \mathrm{Cl}_n^2)\beta = \beta(\mathrm{Cl}_n^0 \oplus \mathrm{Cl}_n^2)$.

Proof An arbitrary curve γ starting at β can be expressed in terms of a curve γ_β starting at 1, defined by $\gamma(t)\beta^{-1} = \gamma_\beta(t)$. The element $\beta \in \Gamma(n)$ is fixed, so $\partial_t\gamma(t) = \partial_t(\gamma_\beta(t))\beta$. By the preceding fact, $\partial_t\gamma(0) \in (\mathrm{Cl}_n^0 \oplus \mathrm{Cl}_n^2)\beta$. Since $\Gamma(n)$ is a manifold, $T_\beta\Gamma(n)$ has the same dimension as $T_1\Gamma(n)$. By considering an arbitrary basis of $T_1\Gamma(n)$, it follows that $T_\beta\Gamma(n) = (\mathrm{Cl}_n^0 \oplus \mathrm{Cl}_n^2)\beta$. Using $\beta^{-1}\gamma(t)$ rather than $\gamma(t)\beta^{-1}$ shows that $T_\beta\Gamma(n) = \beta(\mathrm{Cl}_n^0 \oplus \mathrm{Cl}_n^2)$, and the result follows.

2. Möbius Transformations

It turns out that, similar to the more familiar case in elementary complex analysis, Möbius transformations of $\mathbb{R}^n$ can be represented by matrices in the space of $C\ell_n$ -valued 2×2 matrices (2×2 matrices whose entries are elements of $C\ell_n$), $M(2, \mathrm{Cl}_n)$. The best geometrical motivation for the definitions in this part is given in cited paper by Ahlfors. Lounesto gives an excellent treatment of Ahlfors-Vahlen matrices in nondefinite (pseudo-Euclidean) space.

Definition 4.2.1. A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2, \mathrm{Cl}_n)$ is an **Ahlfors-Vahlen matrix** if it fullfills the following three conditions:

(a) a, b, c, and d are products of vectors in $\mathbb{R}^n$ (equivalently, $a, b, c, d \in \Gamma(n) \cup \{0\}$)

(b) bd^*, ac^*, a^*b,$c^*d \in \mathbb{R}^n$

(c) A has nonzero real pseudodeterminant $ad^* - bc^* \in \mathbb{R}^n \{0\}$.

This may be compared to Definition 2.1 in the Ahlfors paper.

Theorem 4.2.2. Any Möbius transformation of $\mathbb{R}^n$ is given by a map $g : \mathbf{x} \to (a\mathbf{x} + b)(c\mathbf{x} + d)^{-1}$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an Ahlfors-Vahlen matrix.

Proof Ahlfors states and proves an equivalent proposition as Theorem A in his paper.

The set of all Ahlfors-Vahlen matrices is a group under matrix multiplication. This indicates the representation of some classical subgroups of the group of Möbius transformations. Note that the main automorphism on $C\ell_n$ is used in representing orthogonal transformations.

1. Translations: $T = \{T_{\mathbf{v}} = \begin{pmatrix} 1 & \mathbf{v} \\ 0 & 1 \end{pmatrix} : \mathbf{v} \in \mathbb{R}^n\}$; $T_{\mathbf{u}} T_{\mathbf{v}} = T_{\mathbf{u}+\mathbf{v}}$

2. Dilations: $D = \{D_{\mathbf{v}} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{R} / \{0\}\}$; $D_\lambda D_\mu = D_{\lambda\mu}$

3. Orthogonal Transformations: $R = \{R_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix} : \alpha \in Pin(n)\}$; $R_\alpha R_\beta = R_{\alpha\beta}$

An inversion is given by any nonzero scalar multiple of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

3. Conformal Map

Here a conformal map is defined in Euclidean space. An expression for the differential in an arbitrary point of $\mathbb{R}^n$ is obtained, using the fact an orthogonal transformation composed with a dilation can be represented by elements of the Lipschitz group.

Definition 4.3.1. Let M and N be manifolds embedded in $\mathbb{R}^n$. Then an injective differentiable map ϕ from a domain Ω of M to a domain Π of N is said to be a conformal map if the differential $d\phi_{\mathbf{a}}$ in an arbitrary point $\mathbf{a} \in \Omega$ is, up to a nonzero factor, an isometry between the respective tangent spaces.

Any finite-dimensional real vector space is, in a natural way, a differentiable manifold, and the tangent space at any point of that space can be naturally identified with that space itself (Warner 86). Since a and $f(a)$ are in $\mathbb{R}^n$, this implies that both $T_{\mathbf{a}}\Omega$ and $T_{f(\mathbf{a})}\Pi$ are isomorphic to the real quadratic space $\mathbb{R}^n$, and $d\phi_{\mathbf{a}} : T_{\mathbf{a}}\Omega \to T_{f(\mathbf{a})}\Pi$ is an orthogonal mapping up to some nonzero factor. In the above definition of a conformal map, the relation between the respective tangent spaces becomes, for every $\mathbf{x}$ and $\mathbf{y}$ in $T_{\mathbf{a}}\Omega$,

$$\mathbf{x} \cdot \mathbf{y} = \mu(\mathbf{a})(d\phi_{\mathbf{a}}(\mathbf{x}) \cdot d\phi_{\mathbf{a}}(\mathbf{y})) = \mu(\mathbf{a})(\partial_t(\phi \circ \gamma)(0) \cdot \partial_t(\phi \circ \lambda)(0)) \quad (4.3.1)$$

where γ and λ are arbitrary curves starting at $f(a)$ for which $\mathbf{x} = \partial_t \gamma(0)$ and $\mathbf{y} = \partial_t \lambda(0)$ and $\mu(\mathbf{a})$ is some nonzero factor. The local contraction factor $\mu(\mathbf{a})$ generally depends on $\mathbf{a}$.

This characterization of conformal maps in terms of distances can be rendered in terms of angles. In Euclidean space, one may define the angle θ between two vectors $\mathbf{x}$ and $\mathbf{y}$ by $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}|\cos\theta$, where $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$ and $|\mathbf{y}| = \sqrt{\mathbf{y} \cdot \mathbf{y}}$. Comparing with $(4.3.1)$ above, the magnitude (not necessarily the sign) of θ is preserved.

Now recall from Section 1.2 that an orthogonal transformation composed with a dilation on $\mathbb{R}^n$ can be expressed using the Lipschitz group $\Gamma(n)$. Then, choosing a point $\mathbf{x} \in \Omega$, for some function $\beta : \Omega \to \Gamma(n)$ (a Lipschitz-valued function defined on Ω), $(4.3.1)$ becomes

$$d\phi_{\mathbf{x}} X = sig(\beta(\mathbf{x}))\beta(\mathbf{x}) X \overline{\beta(\mathbf{x})} \quad (4.3.2)$$

for the function $sig(\beta) = \bar{\beta}\beta' / (\beta\bar{\beta}) \in \{+1, -1\}$ and each X in $T_{\mathbf{x}}\Omega$.

Note that X is being acted on by a member of $O(n)\oplus\mathbb{R}^+$. Then it is not clear that β is continuous. (More explicitly, there are always at least two choices for the value of β at each $\mathbf{x} \in \Omega$. The Lipschitz group is not connected, but consists of two separated components. Conceivably, a Lipschitz-valued function could satisfy $(4.3.2)$, but for a point infinitesimally close to another point in $\mathbb{R}^n$ would be mapped to the other component of the Lipschitz group, and this Lipschitz-valued function would not be continuous. So only Lipschitz-valued functions taking values in exactly one component of $\Gamma(n)$ are considered.) Nonetheless one may choose β that is at least locally continuous. Moreover, considering a sufficiently small domain $\Omega \in \mathbb{R}^n$, one may always choose β continuous on Ω.

4. Liouville's Theorem

This final part of Section 4 develops the main results of the present article. Two lemmas are used throughout the proofs of these results.

Lemma 4.4.1. (Braid Lemma) If a vector-valued function k in three variables is symmetric in its first two variables and anti-symmetric in its last two variables, then that function is identically zero.

Proof It suffices to show that $k(x, y, z) = -k(x, y, z)$, which follows from six transpositions of the variables:

$$k(x,\ y,\ z) = -k(x,\ z,\ y) = -k(z,\ x,\ y) = k(z,\ y,\ x)$$

$$= k(y,\ z,\ x) = -k(y,\ x,\ z) = -k(x,\ y,\ z).$$

Lemma 4.4.2. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$ identfied with Cl^1_n. Then $1+\mathbf{x}\mathbf{y}$ is a product of vectors.

Proof If $\mathbf{x}$ or $\mathbf{y}$ is invertible (nonzero), then $1+\mathbf{x}\mathbf{y} = \mathbf{x}(\mathbf{x}^{-1}+\mathbf{y})$ or $1+\mathbf{x}\mathbf{y} = (\mathbf{y}^{-1}+\mathbf{x})\mathbf{y}$. If $\mathbf{x}$ or $\mathbf{y}$ is zero, then $1+\mathbf{x}\mathbf{y} = 1$, which is the product of any invertible vector with its inverse.

The following discussion determines for a continuously differentiable Lipschitz-valued function $\beta:\Omega\to\Gamma(n)$ sufficient conditions for the existence of a conformal map ϕ whose differential satisfies $(4.3.2)$. The notation $\beta^{-1}(\mathbf{x})$ is used for $1/\beta(\mathbf{x})$. Also used is the Dirac operator on $\mathbb{R}^n$ applied to a scalar function, say *f*, which coincides with the familiar gradient operator: $D_X f(\mathbf{x}) = -X\cdot Df(\mathbf{x})$. In an orthonormal basis $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ of $\mathbb{R}^n$, $D=\sum_{k=1}^{n}\mathbf{e}_k(\partial/\partial x_k)$.

Theorem 4.4.3. Let Ω and Π be domains in $\mathbb{R}^n$. Suppose $\beta:\Omega\to\Gamma(n)$ is a continuously differentiable Lipschitz-valued function. Then there exists a conformal map $\phi:\Omega\to\Pi$ such that, for all $\mathbf{x}\in\mathbb{R}^n$,

$$d\phi_{\mathbf{x}}X = sig(\beta(\mathbf{x}))\beta(\mathbf{x})X\overline{\beta(\mathbf{x})},\ sig(\beta) = \overline{\beta}\beta'/(\beta\overline{\beta})\in\{+1,-1\},\ \forall X\in T_{\mathbf{x}}\Omega \quad (4.4.1)$$

only if there exists a vector-valued function $\mathbf{v}$: $\mathbf{\Omega}\to\mathbb{R}^n$ such that

$$(D_X\beta^{-1}(\mathbf{x}))\beta(\mathbf{x}) = X\mathbf{v}(\mathbf{x}),\ \mathbf{v} = \frac{1}{2}D(\beta\overline{\beta})/(\beta\overline{\beta}) \quad (4.4.2).$$

Proof It is convenient to adopt a standard orthogonal basis $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$. By linearity, if the result holds for each $\mathbf{e}_i$ $(1\le i\le n)$, then it holds for all $X\in T_{\mathrm{X}}\mathbf{\Omega} = \mathbb{R}^n$. Choose such i. By the earlier result concerning the tangent space at an arbitrary point in $\Gamma(n)$, $\partial_i\beta^{-1} = B_i\beta^{-1}$ for some

$B_i \in \mathrm{Cl}_n^0 \oplus \mathrm{Cl}_n^2$. Using $\beta\beta^{-1} = 1$, $\partial_i(\beta\beta^{-1}) = 0$. By the product rule, $B_i = (\partial_i \beta^{-1})\beta = -\beta^{-1}(\partial_i \beta)$.

By explicit calculation from the definition of the differential of a differentiable function ϕ for $X = \mathbf{e}_i$ and supposing $(4.4.1)$ holds, it follows $\partial_i \phi = sig(\beta)\beta\mathbf{e}_i\overline{\beta}$. As β is continuously differentiable, $\partial_j \partial_i \phi$ exists and is continuous. As i and j are arbitrary, this implies the integrability condition ϕ given by $\partial_i \partial_j \phi = \partial_j \partial_i \phi$ for arbitrary indices i and j. If ϕ has a differential of the form appearing in $(4.4.1)$, then the integrability condition becomes,

$$\partial_j(sig(\beta)\beta\mathbf{e}_i\overline{\beta}) = \partial_i(sig(\beta)\beta\mathbf{e}_j\overline{\beta}) \quad (4.4.3).$$

The function $sig(\beta) \in \{+1,-1\}$ is also continuously differentiable since β and $\overline{\beta}$ are, so any partial derivatives of $sig(\beta)$ for given $\mathbf{x} \in \Omega$ must be zero. Then $sig(\beta)$ drops out. Together with the product rule, this implies

$$(\partial_j \beta)\mathbf{e}_i\overline{\beta} + \beta\mathbf{e}_i(\partial_j\overline{\beta}) = (\partial_i \beta)\mathbf{e}_j\overline{\beta} + \beta\mathbf{e}_j(\partial_i\overline{\beta}) \quad (4.4.4).$$

Multiplying by $-\beta^{-1}$ on the left and $\overline{\beta}^{-1}$ on the right,

$$-\beta^{-1}(\partial_j\beta)\mathbf{e}_i - \mathbf{e}_i(\partial_j\overline{\beta})\overline{\beta}^{-1} = -\beta^{-1}(\partial_i\beta)\mathbf{e}_j - \mathbf{e}_j(\partial_i\overline{\beta})\overline{\beta}^{-1} \quad (4.4.5).$$

Using $\overline{\beta}^{-1} = \beta/(\beta\overline{\beta}) = \overline{\overline{\beta}/(\beta\overline{\beta})} = \overline{\beta^{-1}}$ and $\overline{\partial_i\beta} = \partial_i\overline{\beta}$, this gives $\overline{B_i} = \overline{-\beta^{-1}(\partial_i\beta)} = -(\partial_i\overline{\beta})\overline{\beta^{-1}}$. Substituting this into $(4.4.5)$,

$$B_j\mathbf{e}_i + \mathbf{e}_i\overline{B_j} = B_i\mathbf{e}_j + \mathbf{e}_j\overline{B_i} \quad (4.4.6).$$

As $B_i \in \mathrm{Cl}_n^0 \oplus \mathrm{Cl}_n^2$, one may decompose B_i into its scalar and bivector parts

$$B_i = [B_k]_0 + \sum_{j,k,j\neq k} T^i_{jk}\mathbf{e}_j\mathbf{e}_k = \sum_{j,k} T^i_{jk}\mathbf{e}_j\mathbf{e}_k \text{ and } \overline{B_i} = [B_k]_0 - \sum_{j,k,j\neq k} T^i_{jk}\mathbf{e}_j\mathbf{e}_k = \sum_{j,k} T^i_{jk}\mathbf{e}_k\mathbf{e}_j \quad (4.4.7),$$

where $T^i_{jk} = -T^i_{kj}$, $j \neq k$, and $T^i_{kk} = \frac{1}{2}(-\mathbf{e}_k^2)[B_i]_0$. This used $\mathbf{e}_k^4 = (\mathbf{e}_k^2)^2 = 1$ (equivalently, $\mathbf{e}_k^2 = \mathbf{e}_k^{-2}$)

and the anti-commutativity of all distinct $\mathbf{e}_j$ and $\mathbf{e}_k$ in a standard orthogonal basis. Using $(4.4.7)$, $(4.4.6)$ may be written as

$$4(-\mathbf{e}_i^2)\sum_k T_{ik}^{\ j}\mathbf{e}_k = 4(-\mathbf{e}_j^2)\sum_k T_{jk}^{\ i}\mathbf{e}_k \quad (4.4.8).$$

For distinct i, j, and k (if $n \le 2$, then one may proceed to the case of at least two equal indices), $T_{jk}^{\ i}$ has been defined as anti-symmetric in its lower two indices. Using $\mathbf{e}_i^2 = \mathbf{e}_j^2 = -1$ and considering $(4.4.8)$ component-wise, one sees $T_{ik}^{\ j}$ is symmetric in the upper and lower left indices. By the Braid Lemma, such $T_{ik}^{\ j}$ is identically zero. Then the only remaining terms in B_i and $(4.4.8)$ (considered component-wise) include at least two equal indices: $B_i = -2\mathbf{e}_i\sum_k T_{ki}^{\ i}\mathbf{e}_k$ and $\mathbf{e}_k^2 T_{ki}^{\ i} = \mathbf{e}_i^2 T_{ii}^{\ k}$. Hence

$$B_i = -2\mathbf{e}_i(\sum_k T_{ki}^{\ i}\mathbf{e}_k) = -2\mathbf{e}_i(\sum_k (\mathbf{e}_i^2 T_{ii}^{\ k}\mathbf{e}_k^2)\mathbf{e}_k = -2\mathbf{e}_i(\sum_k (\mathbf{e}_i^2(\frac{1}{2}(-\mathbf{e}_i^2)[B_k]_0)\mathbf{e}_k^2)\mathbf{e}_k$$

$$= \mathbf{e}_i(\sum_k (\mathbf{e}_i^4[B_k]_0\mathbf{e}_k^3) = \mathbf{e}_i(\sum_k [B_k]_0\mathbf{e}_k^{-1})$$

To complete the proof, it suffices to show that $[B_k]_0 = -\frac{1}{2}\partial_k(\beta\bar{\beta})/(\beta\bar{\beta})$, from which it will follow that $B_i = (\partial_k\beta^{-1}(\mathbf{x}))\beta(\mathbf{x}) = -\mathbf{e}_i\sum_k(\partial_k(\beta\bar{\beta})/(\beta\bar{\beta}))\mathbf{e}_k^{-1} = \mathbf{e}_i D(\beta\bar{\beta})/(\beta\bar{\beta}) = \mathbf{e}_i\mathbf{v}$, where $\mathbf{v}$ has the same meaning as in the statement of the theorem. This is straightforward:

$$\partial_k(\beta\bar{\beta}) = (\partial_k\beta)\bar{\beta} + \beta(\partial_k\bar{\beta}) = -\beta B_k\bar{\beta} - \beta\overline{B_k}\bar{\beta} = -2\beta(B_k + \overline{B_k})\bar{\beta} = -2\beta[B_k]_0\bar{\beta}.$$

The above used $B_k = -\beta^{-1}(\partial_k\beta)$ implies $-\beta B_k = (\partial_k\beta)$ implies $-\overline{\beta B_k} = \partial_k\bar{\beta}$, and the earlier expressions $B_i = [B_k]_0 + \sum_{j,k,j\neq k} T_{jk}^{\ i}\mathbf{e}_j\mathbf{e}_k =$ and $\overline{B_i} = [B_k]_0 - \sum_{j,k,j\neq k} T_{jk}^{\ i}\mathbf{e}_j\mathbf{e}_k$.

For Euclidean space of dimension greater than two, there is a rather restrictive condition that β^{-1} is linear in $\mathbf{x} \in \mathbb{R}^n$ if $(4.4.1)$ in the statement of the previous theorem holds.

Corollary 4.4.4. Suppose that the hypotheses of the preceding theorem hold. Suppose further $n > 2$ and ϕ is three times continuously differentiable. Then β^{-1} has the form $\mathbf{x}\gamma + \delta$, where γ and δ are constants in the Lipschitz group.

Proof Since ϕ is at least three times continuously differentiable, using $(4.4.1)$ gives β is at least two times continuously differentiable. Noting that $\partial_k \beta^{-1} = -\beta^{-1}(\partial_k \beta)\beta^{-1}$, it is also true β^{-1} is at least two times continuously differentiable. By Theorem 4.4.3, $\partial_k \beta^{-1}(\mathbf{x}) = \mathbf{e}_k(\mathbf{v}(\mathbf{x})\beta^{-1}(\mathbf{x}))$. This determines β^{-1} up to an additive constant (say δ).

Taking second derivatives, which are symmetric in arbitrary indices ($\partial_j \partial_k \beta^{-1} = \partial_k \partial_j \beta^{-1}$ for all j and k), $\mathbf{e}_j \partial_k(\mathbf{v}(\mathbf{x})\beta^{-1}) = \mathbf{e}_k \partial_j(\mathbf{v}(\mathbf{x})\beta^{-1})$.

Distinct indices anti-commute in the chosen basis, so $\mathbf{e}_i\mathbf{e}_j \partial_k(\mathbf{v}(\mathbf{x})\beta^{-1}) = \mathbf{e}_i\mathbf{e}_k \partial_j(\mathbf{v}(\mathbf{x})\beta^{-1})$ is anti-symmetric in its first two indices ($j \neq i \neq k$) and symmetric in the last two indices . By the Braid Lemma, $\mathbf{e}_i\mathbf{e}_j \partial_k(\mathbf{v}(\mathbf{x})\beta^{-1})$ is identically zero. Multiplying $\mathbf{e}_i\mathbf{e}_j \partial_k(\mathbf{v}(\mathbf{x})\beta^{-1})$ on the left by $\mathbf{e}_j^{-1}\mathbf{e}_i^{-1}$ gives $\partial_k(\mathbf{v}(\mathbf{x})\beta^{-1}) = 0$. Then $\mathbf{v}(\mathbf{x})\beta^{-1}(\mathbf{x})$ is a constant; call it γ .

Now to state and prove Liouville's Theorem.

Theorem 4.4.5 (Liouville's Theorem) Under the conditions of the preceding theorem and corollary, ϕ is a Möbius transformation.

Proof Our goal is to find a Möbius transformation that is equivalent to ϕ and represented by an Ahlfors-Vahlen matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Note that a Möbius transformation

$g : \mathbf{x} \to (a\mathbf{x} + b)(c\mathbf{x} + d)^{-1}$ in the point $\mathbf{x} = 0$ is particularly simple: $g(0) = bd^{-1}$. Suppose then, to simplify matters, that $0 \in \Omega$ (if $0 \notin \Omega$, then a translation must be included first). Suppose further that $\beta(0) = 1$ (if $\beta(0) \neq 1$, then consider $\phi \circ \nu(\beta^{-1}(0))$ defined on $\nu(\beta(0))(\Omega)$, which is equivalent to ϕ on Ω. This reduces to $\beta(0) = 1$, except that one must include an orthogonal transformation first.)

Choose $b = g(0)$ and $d = 1$. The pseudo-determinant $ad^* - bc^*$ is only determined up to a multiplicative nonzero scalar (constant), so one may choose $ad^* - bc^* = 1$ and therefore $a = 1 + bc^*$. By Lemma 4.4.2, a is the product of vectors if c is a vector. The coefficient c is determined by the differential of ϕ, to which the differential of g must be equal. This differential condition is

$$dg_{\mathbf{x}} X = (ad^* - bc^*)(c\mathbf{x} + d)^{*-1} X (c\mathbf{x} + d)^{-1} = (c\mathbf{x} + 1)^{*-1} X (c\mathbf{x} + 1)^{-1}$$

$$= sig(\beta(\mathbf{x}))\beta(\mathbf{x}) X \overline{\beta(\mathbf{x})} = d\phi_{\mathbf{x}} X = \beta(\mathbf{x}) X (\beta(\mathbf{x}))^*,$$

where in the last line was used $(\beta(\mathbf{x}))^* = sig(\beta(\mathbf{x}))\overline{\beta(\mathbf{x})}$, which holds at $\mathbf{x} = 0$ (by the choice of $\beta(0)$, $(\beta(0))^* = 1$; the image of $\beta(0)$ under the main and reversion anti-automorphisms is also 1) and throughout Ω (by the continuity of β and its anti-automorphisms, together with the constancy of $sig(\beta)$ established in an earlier argument). By Corollary 4.4.4, $\beta^{-1}(\mathbf{x}) = \mathbf{x}\gamma + \delta$ for Lipschitz-valued constants γ and δ. If one lets $c = \gamma = \mathbf{v}(\mathbf{x})\beta^{-1}(\mathbf{x}) = \frac{1}{2}(D(\beta\overline{\beta})/(\beta\overline{\beta}))\beta^{-1}$ and $d = \delta = 1$, the equality above follows readily using $(a^*)^{-1} = (a^{-1})^*$ for $a \in C\ell_n$. Using $\beta(0) = 1$, it follows that c is a vector.

Thus has been found a matrix A whose entries satisfy conditions (a) and (c) for the above definition of an Ahlfors-Vahlen matrix. It remains to show that condition (b) is satisfied. Since b and c are vectors and $d = 1$, $bd^* = b$ and $c^*d = c^*$ are also vectors. Also since b and c are vectors, b^*b and c^*c^* are real. Then $ac^* = (1 + bc^*)c^* = c^* + b(c^*c^*)$ and $a^*b = (1 + cb^*)b = b + c^*(b^*b)$ are vectors. Hence condition (b) of the above for an Ahlfors-Vahlen matrix is satisfied. So ϕ is a Möbius transformation.

It is useful to recapitulate the main result with all hypotheses stated.

Theorem 4.4.5 (Liouville's Theorem Restated) Let Ω and Π be domains in $\mathbb{R}^n$ with $n > 2$. Then every conformal map $\phi : \Omega \to \Pi$ that is at least three times continuously differentiable is a Möbius transformation. Moreover, this Möbius transformation can be represented by an Ahlfors-Vahlen matrix.

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