Journal of the Arkansas Academy of Science

2006

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Variational Symmetries and Conservation Laws in Linearized Gravity

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Abstract.—The methods of symmetry group analysis are applied to the action functional of linearized gravity to derive necessary conditions for the existence of variational symmetries. Two classes of variational symmetries of linearized gravity are discussed, and the local conservation laws associated with these variational symmetries are presented by applying Noether's theorem.

Key words.—Gravity, variational symmetries, linearized, conservation laws, Noether's theorem.

Introduction

Local conservation laws play a pivotal role in several branches of physics (Barrett and Grimes 1995, Goldberg 1958). The conserved quantities derived from conservation laws permit the characterization of a given physical system in terms of a relatively small number of physical quantities. For example, quantities such as energy, linear momentum, angular momentum, and charge, are often encountered in a wide range of classical and quantum systems because of the conservation laws associated with these quantities. In the quantum field theoretic description of the fundamental interactions, the existence of conservation laws is often the guiding principle that dictates the correct choice of the field theory that describes the fundamental interactions (Kaku 1993). In general relativity, it has often been argued that the existence of local conservation laws would lead to the construction of observables of the gravitational field, which could play a significant role in any quantum theory of gravity (Torre 1993). Conservation laws also play an important role in a variety of mathematical issues such as integrability, existence and uniqueness, and stability (Olver 1993).

In 1918, the German mathematician Emmy Noether proved two important theorems concerning the existence of conservation laws for physical systems that admit a Lagrangian formulation (Noether 1918). Her first theorem proved that if the Lagrangian admits a variational symmetry (a symmetry transformation that leaves the action functional invariant), then the system admits a local conservation law. The second theorem states that if a variational symmetry depends on arbitrary functions, then the differential equations governing the system must satisfy an identity.

In this paper, I discuss the variational symmetries and local conservation laws admitted by the linearized, vacuum Einstein equations of general relativity. I begin with a brief review of the Lagrangian formulation of linearized gravity and then proceed to apply methods of symmetry group analysis to derive necessary conditions for the existence of variational symmetries of linearized gravity. By applying Noether's theorems, the conservation laws associated with the variational symmetries are derived. I conclude with a discussion of two classes of variational symmetries admitted by linearized gravity — the Poincaré group of symmetries and the gauge symmetry of linearized gravity.

Methods

Linearized gravity—In Einstein's general theory of relativity, spacetime is assumed to be a 4-dimensional manifold M endowed with a Lorentzian metric g_{ab}. The spacetime metric g_{ab} satisfies the Einstein equations (in units where $c = 8\pi G = 1$),

$$G_{ab} = T_{ab}, \tag{1}$$

where G_{ab} is the Einstein tensor and T_{ab} is the energy-momentum tensor of the matter distributions (Wald 1984). In the absence of any matter distributions (for example, outside a star or planet), $T_{ab} = 0$ and the metric g_{ab} solves the vacuum Einstein equations of general relativity,

$$G_{ab} = 0. \tag{2}$$

Furthermore, if we restrict attention to regions of spacetime where the gravitational field is weak (for example, very far away from a star or planet), we can always choose local coordinates on spacetime such that the metric g_{ab} takes the form

$$g_{ab} = \eta_{ab} + h_{ab}, \tag{3}$$

where h_{ab} can be viewed as a symmetric $(0,2)$-type tensor field propagating on a flat spacetime endowed with the Minkowski metric η_{ab}. The tensor h_{ab} is assumed to be a small perturbation of the background Minkowski metric. This assumption allows us to restrict attention to terms linear in h_{ab}, which in turn implies

$$g^{ab} = \eta^{ab} - h^{ab}, \tag{4}$$

where $g^{ac}g_{cb} = \delta_b^{\ a}$ and $h^{ab} = \eta^{ac}\eta^{bd}h_{cd}$.

The linearized, vacuum Einstein equations are obtained by substituting equations (3) and (4) in equation (2) and expanding G_{ab} to linear order in h_{ab}. This yields

$$G_{ab} = \frac{1}{2}\left(\partial_c\partial_a h^c{}_b + \partial_c\partial_b h^c{}_a - \partial_a\partial_b h - \partial_c\partial^c h_{ab} - \eta_{ab}\partial_c\partial_d h^{cd} + \eta_{ab}\partial_c\partial^c h\right) = 0, \tag{5}$$

where $\partial_a = \frac{\partial}{\partial x^a}$ and $h := \eta_{cd}h^{cd} = h^c{}_c$ (Carroll 2004).

Variational Symmetries and Conservation Laws in Linearized Gravity

The linearized, vacuum Einstein equations do admit a Lagrangian formulation; they can be derived from a variational principle. The action functional $S[h^{ab}]$ of the variational problem is

$$S[h^{ab}] = \int_\Omega L(h, \partial h)\, d^4x, \qquad (6)$$

where the Lagrangian density,

$$L(h, \partial h) = \frac{1}{2}\left[(\partial_c h^{cd})(\partial_d h) - (\partial_c h^{db})(\partial_d h^c{}_e) + \frac{1}{2}\eta^{cd}(\partial_c h^{ef})(\partial_d h_{ef}) - \frac{1}{2}\eta^{cd}(\partial_c h)(\partial_d h) \right] \qquad (7)$$

is defined on a compact region Ω of the spacetime manifold M with a smooth boundary $\partial\Omega$ (Carroll 2004). Let

$$h_\varepsilon^{ab} = h^{ab} + \varepsilon \delta h^{ab} \qquad (8)$$

represent a one-parameter family of symmetric tensor fields on Ω. The tensor field δh^{ab} satisfies the boundary condition

$$\delta h^{ab}\big|_{\partial\Omega} = 0. \qquad (9)$$

In order to derive the linearized equations, we assumed the tensor field $h_0^{ab} = h^{ab}$ extremizes the action functional $S[h^{ab}]$. That is,

$$\delta S := \frac{d}{d\varepsilon}\big|_{\varepsilon=0} S[h_\varepsilon^{ab}] = 0. \qquad (10)$$

Equations (6) and (10) imply

$$\delta S = \int_\Omega \delta L\, d^4x = \int_\Omega \frac{d}{d\varepsilon}\big|_{\varepsilon=0} L\, d^4x = 0. \qquad (11)$$

By virtue of equations and , equation can be written as

$$\int_\Omega G_{ab}\, \delta h^{ab}\, d^4x + \int_\Omega \partial_a J^a\, d^4x = 0, \qquad (12)$$

where G_{ab} is the linearized Einstein tensor given in equation (5), the vector field

$$J^a = \frac{1}{2}\left[(\partial_b h)\delta h^{ab} + (\partial_b h^{ab})\delta h - 2(\partial_c h^a{}_b)\delta h^{cb} + (\partial^a h_{bc})\delta h^{bc} - (\partial^a h)\delta h \right] \qquad (13)$$

and the trace of the variation δh^{ab} is $\delta h = \eta_{ab}\delta h^{ab}$ (Wald 1984). The divergence theorem implies

$$\int_\Omega \partial_a J^a\, d^4x = \int_{\partial\Omega} n^a J_a\, d^3x, \qquad (14)$$

where n^a is the unit outward normal to the three-dimensional boundary $\partial\Omega$. Equations (9) and (13) imply

$$\int_{\partial\Omega} n^a J_a\, d^3x = 0,$$

which forces the second integral in equation (12) to vanish, leaving

$$\int_\Omega G_{ab}\, \delta h^{ab}\, d^4x = 0 \qquad (15)$$

to hold for all symmetric tensor fields δh^{ab} satisfying the boundary condition. The fundamental lemma in the calculus of variation (Courant and Hilbert 1989) implies that equation (15) holds in Ω only if the linearized Einstein tensor G_{ab} vanishes in Ω, proving the result that a necessary condition for the tensor field h^{ab} to extremize the action functional $S[h^{ab}]$ is that it satisfy the linearized, vacuum Einstein equations.

Variational symmetries and conservation laws—The linearized, vacuum Einstein equations are a system of ten differential equations $G_{ab} = 0$, involving 4 independent variables, or spacetime coordinates x^a, and ten dependent variables h_{ab} representing the components of a symmetric $(0,2)$-type tensor field. Let M represent the space of independent variables (or coordinates) x^a and U represent the space of dependent variables with coordinates h_{ab}. Consider a one-parameter family of infinitesimal transformations on the product space $M \times U$ given by

$$\widetilde{x}^a = x^a + \varepsilon \xi^a(x,h) + O(\varepsilon^2) \quad \text{and} \qquad (16)$$

$$\widetilde{h}_{ab} = h_{ab} + \varepsilon \gamma_{ab}(x,h) + O(\varepsilon^2), \qquad (17)$$

where $\varepsilon \ll 1$, and the functions $\xi^a(x,h)$ and $\gamma_{ab}(x,h)$ are the components of a smooth vector field and a smooth, symmetric, $(0,2)$-type tensor field, respectively, on $M \times U$. The infinitesimal transformations (16) and (17) transform the action functional given in equation (6) to

$$S[\widetilde{h}^{ab}] = \int_\Omega L(\widetilde{h}, \widetilde{\partial h})\, d^4\widetilde{x}, \qquad (18)$$

where

$$L(\widetilde{h}, \widetilde{\partial h}) = \frac{1}{2}(\widetilde{\partial}_c \widetilde{h}^{cd})(\widetilde{\partial}_d \widetilde{h}) - \frac{1}{2}(\widetilde{\partial}_c \widetilde{h}^{db})(\widetilde{\partial}_d \widetilde{h}^c{}_e) + \frac{1}{4}\eta^{cd}(\widetilde{\partial}_c \widetilde{h}^{ef})(\widetilde{\partial}_d \widetilde{h}_{ef}) - \frac{1}{4}\eta^{cd}(\widetilde{\partial}_c \widetilde{h})(\widetilde{\partial}_d \widetilde{h}), \qquad (19)$$

and $\widetilde{\partial}_a := \frac{\partial}{\partial \widetilde{x}^a}$ represents partial derivatives with respect to the transformed coordinates $\widetilde{x}^a$. Expanding equations (18) and (19) in a Taylor series about $\varepsilon = 0$ and reorganizing the resulting expression gives

$$S[\widetilde{h}^{ab}] = S[h^{ab}] + \varepsilon \int_\Omega \left[\delta L + \partial_c\left(\xi^c L\right) \right] d^4x + O(\varepsilon^2), \quad (20)$$

where

$$\delta L = \frac{1}{2}\left[(\partial_c h^{cd})(\partial_d Q) + (\partial_c Q^{cd})(\partial_d h) - (\partial_c Q^{db})(\partial_d h^c{}_e) - (\partial_c h^{db})(\partial_d Q^c{}_e) + \eta^{cd}(\partial_c Q^{ef})(\partial_d h_{ef}) - \eta^{cd}(\partial_c Q)(\partial_d h) \right] \qquad (21)$$

and the characteristic Q_{ab} is defined as

$$Q_{ab} = \gamma_{ab} - \xi^c \partial_c h_{ab} \qquad (22)$$

with $Q^{ab} = \eta^{ac}\eta^{bc}Q_{cd}$ and $Q = \eta^{ab}Q_{ab}$. In deriving equation (20) we have also used the fact that, to first order in ε, the volume element $d^4\tilde{x}$ transforms as

$$d^4\tilde{x} = \left(1 + \varepsilon\,\partial_c\xi^c\right)d^4x + O(\varepsilon^2).$$

The infinitesimal transformations (16) and (17) represent a variational symmetry of the action functional if they leave the action functional invariant up to an overall surface term for all symmetric tensor fields h_{ab} on M (Olver 1993). In other words,

$$S[\tilde{h}^{ab}] = S[h^{ab}] + \varepsilon\int_{\partial\Omega}n_a\Lambda^a d^3x + O(\varepsilon^2), \quad (23)$$

where Λ^a are the components of a vector field on M and n_a is a one-form field normal to the boundary $\partial\Omega$. The divergence theorem applied to the surface term in Equation (23) implies that if the fields ξ^a and γ_{ab} represent variational symmetries of linearized gravity then they must satisfy the condition

$$\delta L + \partial_c\left(\xi^c L\right) = \partial_c\Lambda^c. \quad (24)$$

In equation if the vector field $\Lambda^a = 0$, the variational symmetry is called a strict variational symmetry; otherwise the symmetry is referred to as a divergence symmetry (Olver 1993).

A local conservation law of the linearized, vacuum Einstein equations is a vector field P^a built from the coordinates x^b, the tensor field h_{ab}, and derivatives of the tensor field h_{ab} to any arbitrary but finite order that satisfies the condition

$$\partial_a P^a = 0$$

on solutions of the field equations (5) of linearized gravity. In order to elicit the relationship between variational symmetries and local conservation laws established in Noether's theorems, we rewrite equation (24) by integrating the term δL by parts. This yields, after some algebra,

$$\delta L = Q^{ab}G_{ab} + \partial_a S^a, \quad (25)$$

where G_{ab} is the linearized Einstein tensor defined in equation (5) and

$$S^a = \tfrac{1}{2}[(\partial_b h)Q^{ab} + (\partial_b h^{ab})Q - (\partial_b h^a{}_b)Q^{cb} - (\partial_b h^a{}_c)Q^{cb} + (\partial^a h_{bc})Q^{bc} - (\partial^a h)Q] \quad (26)$$

Substituting equation in equation and reorganizing the terms gives

$$\partial_a P^a = -Q^{ab}G_{ab}, \quad (27)$$

where

$$P^a = \Lambda^a - S^a - \xi^a L. \quad (28)$$

It is clear that on solutions of equation (5), if Q_{ab} represents the characteristic of a variational symmetry, then P^a represents a conserved vector field of linearized gravity.

We now explore two distinct types of variational symmetries that are significant in linearized gravity - the Poincaré symmetries and gauge symmetries. Assume the vector field ξ^a depends only on the coordinates on the manifold M (i.e., $\xi^a = \xi^a(x)$). Consequently, the infinitesimal transformations (16) represents a one-parameter family of coordinate transformations on M. Since h_{ab} is a tensor field on M, it must transform according to the tensor transformation law, namely,

$$\tilde{h}_{ab} = \tilde{\partial}_a x^c\tilde{\partial}_b x^d h_{cd} = h_{ab} - \varepsilon\left(\partial_a\xi^c h_{cb} + \partial_b\xi^c h_{ac}\right) + O(\varepsilon^2). \quad (29)$$

Equations (17) and (29) imply

$$\gamma_{ab} = -\left(\partial_a\xi^c h_{cb} + \partial_b\xi^c h_{ac}\right)$$

and the characteristic Q_{ab} takes the form

$$Q_{ab} = -L_\xi h_{ab}, \quad (30)$$

where $L_\xi h_{ab}$ is the Lie derivative of the tensor field h_{ab} with respect to the vector field ξ^a (Wald 1984). Furthermore, let us assume that the infinitesimal coordinate transformations generated by the vector field ξ^a leave the Minkowski metric η_{ab} invariant. In other words, ξ^a is a Killing vector field of the Minkowski metric and hence satisfies the Killing equation

$$L_\xi\eta_{ab} = \partial_a\xi_b + \partial_b\xi_a = 0. \quad (31)$$

Substituting equations (30) and (31) in equation (21) and simplifying the resulting equation using the properties of the Lie derivative gives

$$\delta L = -\xi^c\partial_c L + F^{ab}{}_c\partial_a\partial_b\xi^c, \quad (32)$$

where

$$F^{ab}{}_c = \tfrac{1}{2}\{\delta^a_c h^{bd}\partial_d h + \delta^d_c h^{ab}\partial_d h - h^{bd}\partial_c h^a{}_d - h^{bd}\partial_d h^a{}_c \\ - h^a{}_d\partial_c h^{bd} + h^d{}_c\partial_d h^{ab} + 2\eta^{ad}h^{be}\partial_d h_{ce}\}$$

Substituting equation (32) back into equation (24) yields

$$-\xi^c\partial_c L + F^{ab}{}_c(h,\partial h)\partial_a\partial_b\xi^c + \partial_c\left(\xi^c L\right) = \partial_c\Lambda^c.$$

Since ξ^a is a Killing vector field, it follows that $\partial_c\xi^c = 0$ and $\partial_a\partial_b\xi^c = 0$ (Crampin and Pirani 1994). Setting $\Lambda^a = 0$ proves that all coordinate transformations on the spacetime manifold M that leaves the Minkowski metric invariant are strict variational symmetries of the action functional of linearized gravity. These coordinate transformations are the ten-parameter

group of Poincaré symmetries.

To determine the conserved vector field associated with the Poincaré symmetries, we substitute equation (30) in equation (28) and set $\Lambda^a = 0$ to obtain

$$P^a = \frac{1}{2}(\partial_b h)L_\xi h^{ab} + \frac{1}{2}(\partial_b h^{ab})L_\xi h - (\partial_c h^a{}_b)L_\xi h^{cb} + \frac{1}{2}(\partial^a h_{bc})L_\xi h^{bc} - \frac{1}{2}(\partial^a h)L_\xi h$$
$$+ \frac{1}{2}\xi^a\left[(\partial_c h^{cd})(\partial_d h) - (\partial_c h^{de})(\partial_d h^c{}_e) + \frac{1}{2}\eta^{cd}(\partial_c h^{ef})(\partial_d h_{ef}) - \frac{1}{2}\eta^{cd}(\partial_c h)(\partial_d h)\right]$$

The other variational symmetry is the infinitesimal gauge transformation obtained by setting $\xi^a = 0$ and $\gamma_{ab} = \partial_a X_b + \partial_b X_a$ in **equations (16) and** (17), where X^a is an arbitrary vector field **on the manifold** M. Substituting the characteristic $Q_{ab} = \partial_a X_b + \partial_b X_a$ in equation (21) and recursively applying the integration by parts formula to the resulting equation gives

$$\delta L = \partial_a \Lambda^a - 2\partial_a G^{ab} X_b, \qquad (33)$$

where

$$\Lambda^a = S^a + 2G^a{}_b X^b, \qquad (34)$$

and S^a is the vector field defined in equation (26) with $Q_{ab} = \partial_a X_b + \partial_b X_a$. The linearized, vacuum Einstein equations satisfy the contracted Bianchi identity

$$\partial_a G^a{}_b = 0, \qquad (35)$$

which reduces equation (33) to a form that clearly indicates that the gauge transformation is not a strict variational symmetry of the linearized theory, but instead a divergence symmetry. Equations (28) and (34) imply that the conserved vector field is

$$P^a = 2G^a{}_b X^b.$$

However, note that the conserved vector field $P^a = 0$ on solutions of equation (5) and hence defines a trivial conservation law. This is because the gauge symmetry depends on an arbitrary vector field X^a and hence falls under the purview of Noether's second theorem, which states that the linearized Einstein equations must satisfy a constraint equation. This constraint equation is the contracted Bianchi identity given in equation (35).

Conclusions

I derived necessary conditions that must be satisfied by a variational symmetry of the linearized, vacuum Einstein equations and investigated two classes of variational symmetries: the ten-parameter group of Poincaré symmetries and the gauge symmetry. I showed that the Poincaré symmetries

admit a ten-parameter family of local conservation laws, while the gauge symmetry is a divergence symmetry admitting a trivial conservation law. Looking ahead, it would be interesting to investigate the various conservation laws associated with the Poincaré symmetries and explore their geometric and physical significance. Another interesting research direction is the classification of all local conservation laws of the linearized, vacuum Einstein equations. This is achieved by investigating solutions of equation , where the characteristic Q_{ab} depends on derivatives of the the tensor field h_{ab} to any arbitrary but finite order.

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