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AN ANALYSIS OF UNSOLVABLE LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF ORDER ONE

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Abstract

It is difficult to underestimate the importance of differential equations in understanding the physical world. These equations, involving not just simple variables like temperature, speed or mass, but also the derivatives, i.e. the rate of change of these variables, are found in nearly every branch of science. Until the mid 20th century, all such equations were thought to be solvable. This was based on the discovery by Leonard Euler that certain differential equations, called ordinary differential equations (ODEs), are indeed always solvable. While ODEs deal with simple conditions, under which some quantity changes with some other quantity and its derivatives, there are more sophisticated differential equations known as Partial Differential Equations (PDEs), which describe how one quantity changes with respect to two or more other quantities and their derivatives. The hopes of an entire generation of mathematicians were dashed when it was discovered that there exist very simple linear PDEs that are unsolvable – and thus the worst objects that a mathematician could possibly face. It is the goal of this research to present one such example in a form accessible to anyone who has a basic knowledge of differential equations. Understanding of such equations is an extremely important step in developing numerical methods for estimating the extent to which PDEs may not be solvable, thus giving scientists valuable tools in unlocking the secrets of the physical world, many of which are hidden in Partial Differential Equations.

Introduction

Most undergraduate science and engineering majors are familiar with the topic of differential equations. These mathematical objects are found in every branch of science, modeling real-life events such as radioactive decay, population growth, heat flow, stock market fluctuations and much more. The ability to solve a differential equation means being able to predict the behavior of the phenomenon the equation describes, and the more accurately a differential equation models the phenomenon, the closer the prediction will be to the future outcome. Many times, the number of parameters involved in an accurate description of an event is so large that the functions involved depend on more than just one variable. In this case, the mathematical model involves *partial derivatives* of the unknown function, thus giving rise to *partial differential equations.*

Depending on how the unknown function and its derivatives occur in the mathematical model, one can have different kinds of differential equations. In this work, we are interested in the most elementary type, namely, the so-called *linear differential equations*. Undergraduate students are very familiar with the solution method of linear *ordinary differential equation* (ODE), i.e. models for events depending on just one variable, such as time or temperature. The solution method of linear *partial* differential equations (PDEs) resembles the method for linear ODEs, but only to a certain extent. In the late 1950's, it came as a great surprise to the mathematical world when Hans Lewy showed that there exist very simple but striking examples of linear PDEs of order one which are not solvable. This is in great contrast to the case of linear ODEs.

It is the goal of this research to give an easy to understand presentation of one such example. Although the proof that this equation is unsolvable relies heavily on subtle techniques of complex analysis, we have made every effort to convey the fundamental ideas in a manner accessible to any undergraduate student with a basic familiarity with differential equations.

Classical Methods of Solving Partial Differential Equations

A linear ODE of order one is an equation of the form

$$h'(x)+p(x)h(x) \qquad (0)$$

where $p(x)$ and $g(x)$ are given functions and $h(x)$ denotes the unknown function (and $h'(x)$ denotes the derivative of h). Under minimal assumptions on the coefficients $p(x)$ and $g(x)$, one can show that all solutions h can be explicitly represented as an integral involving p and g.

A linear PDE of order one takes the form

$$a\,(x,y)\;(fu\,/fx\,)+ b\,(x,y)\;(fu\,/fy) + c\,(x,y)\;u = f\,(x,y) \qquad (1)$$

where, once again, $a(x,y)$, $b(x,y)$, $c(x,y)$ and $f(x,y)$ denote given coefficients, whereas u denotes the unknown function. If a, b, c and f are reasonably smooth real-valued functions, it is possible to show that there is always a change of variables from (x,y) to, say, (î,ç), such that with respect to the new variables, (1) takes the equivalent form

$$(fu\,/f\S) + c\,(\S,n)\;u = f\,(\S,n) \qquad (2)$$

Note that, with respect to the variable î, equation (2) is now, formally, just the same as the ODE (0), so that one can solve (2) using methods for Ordinary Differential Equations. However, matters are completely different if one allows at least one of the coefficients of (1) to take complex values, as it may no longer be possible to perform an "ad hoc" change of variables. The next section will examine such a example.

An Example of an Unsolvable Partial Differential Equation

We now consider the following equation:

$$(fw\,/fx) + ix\,(fw\,/fy) = f\,(x,y) \qquad (3)$$

where i denotes the imaginary variable ($i^2 = -1$), and f denotes a function to be suitably constructed as follows. First consider an infinite sequence of points $\{x_n\} = x_1, x_2, x_3, x_4 \ldots$ which tend to zero as n goes to infinity. One could, for example think of x_n as $1/n$ or $(1/2)^n$. Now consider a sequence of closed, non-intersecting discs, D_n, each of which is centered at the corresponding points x_n. Since these discs must not intersect one another, their radii must get smaller and smaller as they approach the origin. By using a procedure known as the "convolution integral," we then construct a function $f(x,y)$ such that:

(a) $f(x,y)$ is an even function of x. That is, $f(x,y) = f(-x,y)$ for all x, regardless of y.

(b) $f(x,y) = 0$ whenever (x,y) is not in any of the little discs D_n

(c) The integral of f is non-zero over each disc D_n.

We now show that for such a choice of the datum f, the equation above is *never* solvable. To begin, we observe that any function, w, can be written as the difference of an even function u^e and an odd function u^o (by odd function, we mean $-u(x,y)=u(-x,y)$; for example, the function ix is odd, since $i(-x)=-(ix)$). Next, by applying the definition of derivative, it is easy to show that the derivative of an odd function is even and the derivative of an even function is odd. Finally, we mention that the product of two even or two odd functions is even, whereas the product of an even function and an odd function is odd. We now apply all of these facts to the solution $w(x,y)$ with respect to the variable x, i.e. we write w as the sum of an even function of x and an odd function of x (no requirements on y). We obtain

$$w(x,y) = u^e\,(x,y) + u^o\,(x,y) \qquad (4)$$

This leads to:

$$(\;fu/\;\;fx) + ix\,(fu\;\;/fy\;\;)^e = f^e = f \qquad (5)$$

where the second equality follows in equation (5) by property (a), and by the observations on even and odd functions mentioned above, we note that if w solves (3), then u^o solves

$$(fu^o\,/fx) + ix\,(fu^o\,/fy) = f\,(x,y) \qquad (6)$$

We now want to show that (5) has no solution u^o with continuous partial derivatives in any little disc centered at the origin. To this end, we begin by supposing, to the contrary, that there is such a solution. We will then show that this would violate property (c) of the definition of f.

Notice, for future use, that an important feature of odd functions is that they must vanish at the origin. Indeed

$$u^o\,(0,y) = -u^o\,(-0,y) = -u^o\,(o,y) \qquad (7)$$

And hence:

$$u(0,y) = 0 \qquad (8)$$

Now we introduce a substitution of variables in the right half space from (x,y) to (s,y), namely $s = \sqrt{x}$, such that in the new variables (s,y) equation (6) becomes

$$(fU\,/fs) + (fU\,/fy) = F\,(s,y) \qquad (9)$$

Where U and F is obtained from u^0 and f via the change of variable; F still satisfies the properties (a)-(c) with respect to a new series of sets, $\ddot{I}_n$. which are obtained from D_n via the change of variables from s to x. The main advantage in going from (6) to (9) is that this new form of the equation is "more symmetric," i.e. the coefficients of ?U/?s and ?U/?y only differ by the constant i rather than the function ix. This fact has a major bearing on the main features of U. Indeed, equation (9) implies that U is now a so-called *analytic function* with respect to the complex variable $s+iy$ in the region where F vanishes, in particular, outside of the $\ddot{I}_n$'s. The main drawback in going from (5) to (7) is that now U is only defined on the right half-plane, whereas u^0 was defined in the whole of the plane. Note that property (6) is preserved under the change of variables. This, together with the fact that U is analytic allows us to apply a famous theorem from Complex Analysis (the "*Reflection Principle for Analytic Functions*") to conclude that U is actually well defined and analytic on the left half plane as well. Property (8) now forces U to be zero in a little disc centered at the origin (this is another consequence of the properties of analytic functions, namely the fact that the so-called zero set of an analytic function must be *discrete*). At this point, we apply the famous *Green's Identity*, which, roughly speaking, states that the integral of the derivative of a function over a region is equal to the integral of the function over the boundary of the region. In our particular case, this means that the integral over some disc Γ_n of the function F is zero, contradicting part (c) in the definition of f. Thus, a continuously differentiable solution of (9) and thus of (3) does not exist in any neighborhood of the origin.

Faculty comments

Loredana Lanzani, Ms. Field's project mentor, made the following comments regarding this project:

> I have known Laura since January 2000, when she took in my advanced undergraduate Complex Analysis course for Mathematics and Physics majors. Although all the students in the class had been carefully selected and proved to be unusually talented and dedicated, Laura was, by all measures, the very best. I was impressed by the depth of her mathematical reasoning and by her ability to apply the new mathematics notions she had just learned to problems in Physics and Engineering, one of the main goals of this course.
>
> Laura showed so much promise that I thought it might be a good idea to get her involved in an undergraduate research project under my direction. In order to help Laura build the sophisticated mathematics skills she would need for the project, and wishing to protect her from stressful competition against the more mature graduate students, I suggested that she audit the graduate complex analysis course that I was going to teach the next fall. I thought that this might allow Laura to benefit from my lectures on a topic I believed to be too difficult to manage for an undergraduate student. It would also exempt her from having to deal with the difficult weekly homework assignments that I took mostly from the fundamental text, Complex *Analyst's* by L. Ahlfors–a strenuous but instrumental step in the preparation of our graduate students for the comprehensive exams.
>
> With her typical understated pragmatism, Laura decided, instead, that she might as well register to the course. This was indeed a good decision since Laura turned out to be, once again, among the very best in the class. Laura's excellent talent for mathematics was confirmed one more time. Not only was she not in the least intimidated by her graduate student classmates (and I should add that this was an unusually good and large class), but she often came up with very nice and original ideas for the solution of the homework assignments and in-class test problems. By the end of December Laura was ready to embark on her project, which involved a thorough study of ground-breaking research papers and a firm grasp of the interplay of two different areas of mathematics: Partial Differential Equations and Complex Analysis. Laura has now achieved a full understanding of all the main points of the proofs in the papers and by now I know her too well to be surprised by her originality, independence and enthusiasm. Yet when I read her manuscript I was amazed by her ability to integrate all the guiding principles of this project into a succinct yet simple and compelling reading. Laura has made this professional-research-level, highly technical and difficult material understandable, indeed enjoyable, by any undergraduate student with a basic knowledge of differential equations and a minimal interest in mathematics and its applications. I would not be surprised if, after reading this paper, more than one student gained refreshed interest and enthusiasm for mathematics.
>
> The challenge of communicating advanced mathematics in a clear yet unintimidating and appealing manner is only too familiar to the professionals in the field, and it is perhaps the main reason why mathematics is often an unpopular subject in our culture. In this respect, I feel that Laura J. Fields has successfully completed the most ambitious project I could hope to expect from an undergraduate student.

William Oliver, vice-chair of the physics department, knows Ms. Fields well and is very complimentary about her research abilities. He says:

> I would like to give my highest endorsement of the research project of Ms. Laura J. Fields, a Bachelor of Science Physics and Mathematics. Indeed it is an honor to do so! Ms. Fields holds the prestigious Sturgis Fellowship in our J. William Fulbright College of Arts and Sciences. Two years ago she was selected as the-recipient of the Fulbright College Presidential Scholarship. In, addition, she was awarded a Stevens Foundation award for academic excellence and most recently a Science Information Liaison Office Student Undergraduate Research Fellowship and the prestigious Barry M. Goldwater Fellowship. Laura is a brilliant young student and a wonderful person, and it is my great privilege to work with her as advisor and research mentor. I am completely confident that she will fulfill her desire to become a successful physicist, and whether she chooses academia or industry, she will be a great role model for women in science.
>
> Ms. Fields had not yet declared a major when she enrolled in the honors section of our department's *University Physics I* course. Midway through the semester she had captured the attention of the professor for the course, who told me she was one of the strongest students that she had seen. I contacted Laura and began a dialog with her about physics in general as well as our programs for undergraduates. To my delight she formally declared a major in physics during the spring of 1999 and I have served as her

departmental advisor since that time.

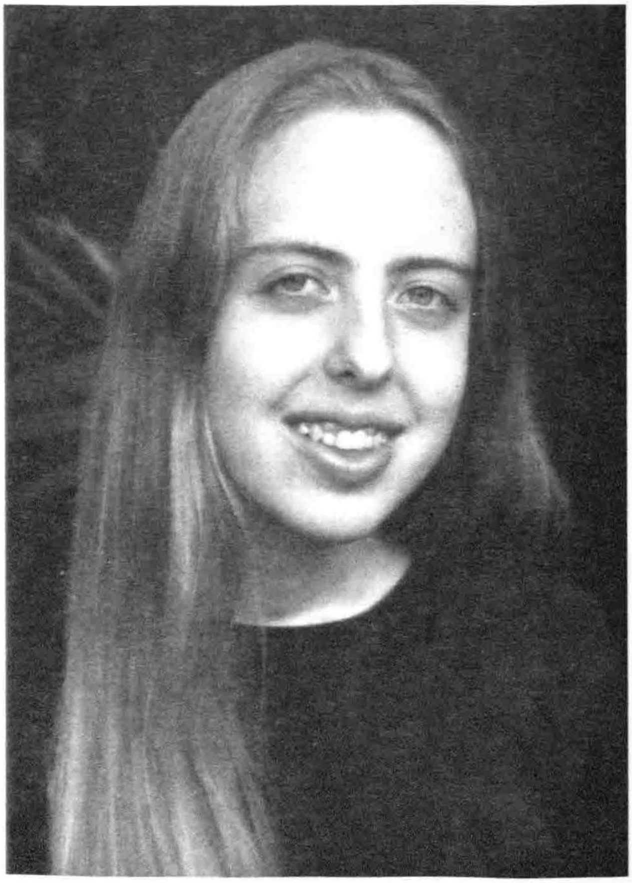
During the fall semester of 1999, Laura took *University Physics I* and an *Intro to Electronics* course from me. She finished each course at or near the top of the class. She was the most diligent student in the electronics course—a course that requires a large measure of self-motivation and discipline—and she seldom missed any points on the exams. Professor Stewart tells me that she was a delight to have in *University Physics II* and that her honors project was well executed and "refreshingly well written." I asked my colleague, Dr. Fihpkowski, how Laura did in her first 3000-level course (in *Modern Physics)* and was told that she had a perfect score. Laura has continued to set a standard of excellence in her studies. Academically, Ms. Fields is a superlative student

Laura is also a wonderful person to be around. She is always very polite and friendly; she interacts well socially and academically with her peers, and has taken an active role in all aspects of our department's programs. Currently, she is serving as president of our chapter of the Society of Physics Students. Laura has provided me with invaluable feedback and insightful comments about our department and our courses. This has often been of great help in carrying out my duties as Vice Chair of the department

Finally, I would like to comment on Laura's research capabilities and potential. Two summers ago, she participated in a summer NSF-REU program in Alaska on atmospheric physics and her research advisor there had stellar things to say about her. Last summer, she was accepted into all of the REU programs that she applied for, including that at the Smithsonian Astrophysical Observatory at Harvard, which she turned down to accept a summer research position at CERN in Geneva, Switzerland, a world-class research center for particle and nuclear physics. Before going to Geneva, Laura participated in a European Studies tour of Europe sponsored by the Fulbright College. During these two experiences, Laura gained both confidence and valuable experience in traveling and living abroad. She also began to narrow her research interests toward particle astrophysics, although I sense that she is still exploring different areas of physics to some degree.

During the spring of last year, she began an honor's research project under my direction on a biophysical topic involving light scattering studies of protein denaturation. She was in Little Rock with her family for the break when she contacted me to find out when we could begin. I told her that any time was fine with me and she immediately drove (3.5 hours) to Fayetteville and showed up in my office. To start, I asked her to do a literature search on dynamic fight scattering and proteins. Two hours later she returned with over a hundred references, each marked with information as to which library they were in. We looked them over to choose those most appropriate for our project and the next morning she was in my office with photocopies, asking questions about what she was reading. This is typical of Laura! She is highly motivated, self-disciplined, and has excellent time management skills. In contrast to her peers, her successful SURF/SILO proposal was finished almost two weeks before its deadline last fall. As she has continued on this project, Laura has demonstrated the self-motivation, tenacity, and creativity necessary for successful research. For the first time in my laboratory, we are now measuring reliable diffusivities for proteins in solution as well as protein sizes, all as a result of Laura's hard work.

From my interactions with Sturgis, Chancellor, and other fellowship awardees and from my experience sitting on many honor's thesis committees, Laura Fields is ranked with the absolute best of our University's students.



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