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Thesis Digest: Mathematical Interpretation of Political Power and the Arkansas State Government

Andrew King
University of Arkansas, Fayetteville

THESIS DIGEST:

MATHEMATICAL INTERPRETATION OF POLITICAL POWER AND THE ARKANSAS STATE GOVERNMENT

By Andrew King
Department of Mathematics

Faculty Mentor: Professor Bernard L. Madison
Department of Mathematics

Abstract

On the whole, political power can he very difficult to quantify. A person may be powerful due to his or her personal charm, wealth, fame, credibility, or influential connections. Political bodies do not account for these qualities when creating voting procedures; they only assign voting rules to specific positions. For example, most would say that in the United States government that a Senator is more powerful than a Representative, but less powerful than the President, without knowing any way to quantify or verify those differences.

Since the 1950's, mathematicians and political scientists have attempted to create mathematical models that partially describe an individual's power as a voting member of a committee, board, or legislative body. These models have resulted in four major "power indexes" that describe the percentage of a body's total power held by each individual member. The four most prominent power indexes are the Shapley-Shubik, Banzhaf, Johnston, and Deegan-Packel, each of which uses a different theory to calculate the probability that an individual's vote will decide whether a proposal passes or fails.

The research in this paper develops formulas to calculate the four-power indexes for legislatures that are unicameral, bicameral, unicameral with committees, and bicameral with committees. These formulas have several variables (up to ten) and have many (up to several thousand) terms for typical sizes of state legislative chambers. Using Mathematica computer software the four power indices are computed for various legislative configurations and the indices' behavior are studied. Then these methods are applied to the Arkansas State Government by calculating the power indexes of the Governor, Senate, House, House Committee members, and Senate Committee members. By examining the theories behind the four power indexes and available historical evidence, the paper concludes by analyzing which indexes, if my, provide the best model for the political power structure of the Arkansas State Government.

Democracy frequently requires a way to translate the various preferences of several individuals into a single group decision. The most common way of doing this is the *yes-no voting system*, which allows individual voters to decide between a single proposal (such as a bill or resolution) and the status quo. A yes-no voting system is defined as a set of rules that specifies exactly which collections of "yes" votes will pass a proposal [4].

Yes-no voting systems are defined in terms of *coalitions*, or specific collections of voters. A coalition is called winning if a proposal passes when all of its members vote 'yes' and losing if it does not meet this condition. In almost all yes-no voting systems, the winning coalitions are those that meet or exceed a quota, or the minimal number of votes for a proposal to pass. Usually, the quota is a *majority*, or the smallest whole number of votes that is greater than one-half of all possible votes.

The Four Power Indices:

Power indices are methods of computing the influence of an individual voter on whether a proposal passes or fails in a yes-no voting system It is always a number between 0 and 1, and the sum of the power indices for all voters in a yes-no voting system will always be 1. The Shapley-Shubik Index was the first developed.

1. Shapley-Shubik:

In 1954, Shapley and Shubik applied Shapley's previous work regarding multi-person cooperative games to the measure of political power. Their definition of power was based on the probability that an individual player's vote will be *pivotal*, that is, be the qth voter when the quota is q. To understand this, suppose that all voters willing to vote for a proposal line up in some random order and vote in turn. Once q voters have voted 'yes,' the motion is declared passed and the last voter is deemed pivotal. The index assumes that all such orderings are equally probable [3]. It is computed for an individual voter A by taking the fraction of all voter orderings in which A is pivotal.

In the process of developing the formulas and applying them to various configurations, some very interesting properties of the power indices were discovered. For example, two odd-sized chambers in a pure bicameral system using simple majority votes, share Shapley-Shubik power equally. Contrary to published literature, the same is not necessarily true if one of the chambers has an even number of voters. This result has a side effect of producing some very complex combinatorial identities.

Another way of understanding the *Shapley-Shubik Index* is to assume that players align themselves in order of enthusiasm for a proposal, with the strongest supporter first and the strongest opponent last. There will be a number of voters "on the fence," which the others will have to persuade to join their respective coalitions. The player who brings the coalition to winning strength is "pivotal," and may determine, in the words of Shapley and Dubey, "how strong a law will be enacted, or how much money will actually be appropriated for some purpose, or how hard a candidate will have to campaign, etc [2]."

2. Banzhaf:

To lawyer John Banzhaf III, it was clear that [5], "voting power is not proportional to the number of votes a legislator may cast." In a 1960s lawsuit involving the Nassau County, New York, Board of Supervisors, he was able to demonstrate that three of the six supervisors were *dummies*, or representatives with no actual voting power, and that the voting system of the board was therefore unconstitutional. To do so, he invented a new way to measure political power, which became known as the *Banzhaf Index of Power* [5].

The Banzhaf Index is based on the idea of a *critical defector*, a voter in a winning coalition who will cause the coalition to lose if that voter is removed from it. To calculate the Banzhaf Index for a voter p, one must first count the number of coalitions from which p is a critical defector. This is called the *Total Banzhaf Power* of p. The Banzhaf Index of p is the Total Banzhaf Power of p divided by the sum of all voters' Total Banzhaf Power.

3. Johnston:

In 1977, R.J. Johnston, an English geographer, made some variations to the Shapley-Shubik and Banzhaf voting models and invented a new power index [6]. The *Johnston Index of Power* is similar to the Banzhaf Index in that it is based on critical defections. But this time, each time a voter is a critical defector from a winning coalition, it is divided by the number of critical defectors in that coalition. As before, one must calculate the *Total Johnston Power* of a voter p, denoted TJP (p), where $n_1, n_2, \ldots n_i$ are the number of critical defectors from each coalition:

$$TJP(p) = 1/n_1 + 1/n_2 + \ldots + 1/n_i$$

To calculate the Johnston Index, JI (p), one must divide a voter's Total Johnston Power by the sum of all voters' Total Johnston Power [4].

4. Deegan-Packel :

Among the basic assumptions common among the Shapley-Shubik, Banzhaf, and Johnston indices is that the power to effect change is the same as blocking power. In 1978, mathematicians John Deegan and Edward Packel suggested that these two kinds of power are actually different, and proposed an index that could measure the power to initiates changes [7]. It is based on *minimal winning coalitions*— those coalitions that will be losing if any one member is removed.

The *Deegan-Packel Index of Power* is based on three assumptions:

1) Only minimal winning coalitions are relevant when determining power.

2) Each minimal winning coalition forms with equal probability.

3) Any member of a minimal winning coalition gains the same mount of power from belonging to that coalition as all other members of the coalition [7].

To calculate the Deegan-Packel Index for a voter, one must first calculate the *Total Deegan-Packel Power* for that voter— the number of minimal winning coalitions of which he/she is a member. Then that number is divided by the sum of all voters' Total Deegan-Packel Power.

Research Results:

The goals of this research had two central thrusts:

1) To develop formulas for computing each of the four power indices for four legislative models: unicameral, bicameral, unicameral with committees, and bicameral with committees.

2) Apply the formulas to the Arkansas General Assembly and its current committee structure.

Both these goals were accomplished and several significant discoveries were made along the way.

The Formulas:

Formulas were developed in each of the sixteen situations described above, and these formulas were implemented in several situations using *Mathematica* computer software [9]. The formulas are very complicated, having hundreds of terms and requiring calculations involving very large numbers. As an example, the formula below is for the Shapley-Shubik index for one committee member in a bicameral legislature with committees. The assumption is that proposals before either chamber are first referred to a committee for a recommendation (pass or do not pass in the Arkansas legislature). If the committee recommends that the proposal be adopted, then the chamber can approve with some specified majority. If the committee does not recommend that the proposal be adopted then a larger majority is likely required for the chamber to pass the proposal. Consequently there are ten variables, N1, C1, c1, k1, n1, N2, C2, c2, k2, and n2, which represent the number of non-committee members, number of committee members, committee quota, chamber quota, and override quota for each house.

Power Indices for the Arkansas State Government:

Both the Arkansas House of Representatives (100 members) and the Arkansas Senate (35 members) have committees that function as described above. Most House committees have 20 members [11], and most Senate committees have 7 members [12]. Executive officers with veto power that require super majorities to override (such as the U. S. President) will share power with the legislative chambers. Although the Arkansas Governor has veto power, the House and Senate can override a Governor's veto with simple majority vote [10]. Consequently, all four of the indices give the Governor zero power. The four power indices were computed for the Arkansas House and

$$
\begin{aligned}
&\frac{1}{(C1+C2+N1+N2)!} \\
&\Bigg(\binom{C1-1}{c1-1} \\
&\quad \sum_{i=0}^{n1-k1}\binom{N1}{-c1+i+k1}\Bigg(\sum_{n=c2}^{C2}\binom{C2}{n}\sum_{m=k2}^{n+N2}\binom{N2}{m-n}(i+k1+m-1)!\,(C1+C2-i-k1-m+N1+N2)!+ \\
&\qquad \sum_{n=0}^{c2-1}\binom{C2}{n}\sum_{m=n2}^{n+N2}\binom{N2}{m-n}(i+k1+m-1)!\,(C1+C2-i-k1-m+N1+N2)!\Bigg)+ \\
&\sum_{i=c1}^{C1-1}\binom{N1}{-i+k1-1}\binom{C1-1}{i} \\
&\quad \Bigg(\sum_{n=c2}^{C2}\binom{C2}{n}\sum_{m=k2}^{n+N2}\binom{N2}{m-n}(k1+m-1)!\,(C1+C2-k1-m+N1+N2)!+ \\
&\qquad \sum_{n=0}^{c2-1}\binom{C2}{n}\sum_{m=n2}^{n+N2}\binom{N2}{m-n}(k1+m-1)!\,(C1+C2-k1-m+N1+N2)!\Bigg)+ \\
&\sum_{i=0}^{c1-2}\binom{N1}{-i+n1-1}\binom{C1-1}{i} \\
&\quad \Bigg(\sum_{n=c2}^{C2}\binom{C2}{n}\sum_{m=k2}^{n+N2}\binom{N2}{m-n}(m+n1-1)!\,(C1+C2-m-n1+N1+N2)!+ \\
&\qquad \sum_{n=0}^{c2-1}\binom{C2}{n}\sum_{m=n2}^{n+N2}\binom{N2}{m-n}(m+n1-1)!\,(C1+C2-m-n1+N1+N2)!\Bigg)\Bigg)
\end{aligned}
$$

Senate with committees of the current usual size. These are listed in the following tables and show that committee members have considerably more power than non-committee members in passing a particular bill. They also show that the power indices of the committees are affected by the existence of the other chamber. The last two columns of each table show how the relative power of each chamber is affected by the introduction of an internal committee structure.

	One Chamber and Its Committee			Combined Two Chambers and Their Committees				
Shapley-Shubik	Committee Member	Non Committee Member	Ratio	Committee Member	Non Committee Member	Ratio	Sum of Power without Committees	Sum of Power with Committees
Arkansas House, size 20 committee	0.03042	0.00489	6.21	0.01856	0.00227	8.18	0.5204	0.5528
Arkansas Senate, size 7 committee	0.05204	0.02270	2.29	0.02401	0.00997	2.41	0.4796	0.4472

	One Chamber and Its Committee			Combined Two Chambers and Their Committees				†
Banzhaf	Committee Member	Non Committee Member	Ratio	Committee Member	Non Committee Member	Ratio	Sum of Power without Committees	Sum of Power with Committees
Arkansas House, size 20 committee	0.02023	0.00744	2.72	0.01510	0.00416	3.63	0.6453	0.6350
Arkansas Senate, size 7 committee	0.05291	0.02249	2.35	0.01893	0.00830	2.28	0.3548	0.3650

	One Chamber and Its Committee			Combined Two Chambers and Their Committees				
Johnston	Committee Member	Non Committee Member	Ratio	Committee Member	Non Committee Member	Ratio	Sum of Power without Committees	Sum of Power with Committees
Arkansas House, size 20 committee	0.37751	0.00306	123.28	0.01687	0.00161	10.45	0.4060	0.4666
Arkansas Senate, size 7 committee	0.09458	0.01207	7.84	0.04762	0.00715	6.66	0.5941	0.5334

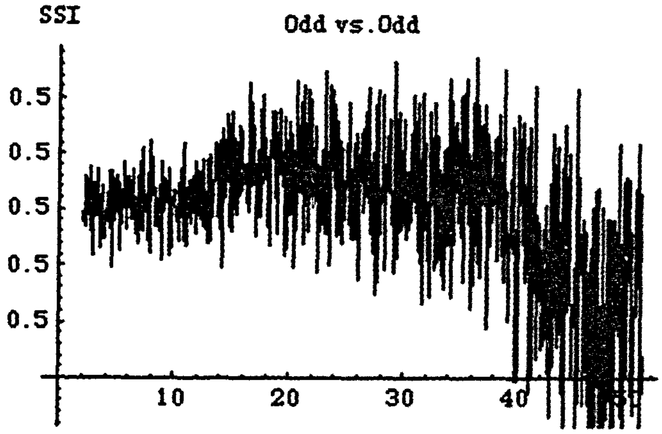
	One Chamber and Its Committee			Combined Two Chambers and Their Committees				†
Deegan-Packel	Committee Member	Non Committee Member	Ratio	Committee Member	Non Committee Member	Ratio	Sum of Power without Committees	Sum of Power with Committeees
Arkansas House, size 20 committee	0.01178	0.00955	1.23	0.00864	0.00700	1.23	0.73193	0.732822
Arkansas Senate, size 7 committee	0.03200	0.02771	1.15	0.00854	0.00741	1.15	0.26807	0.267178

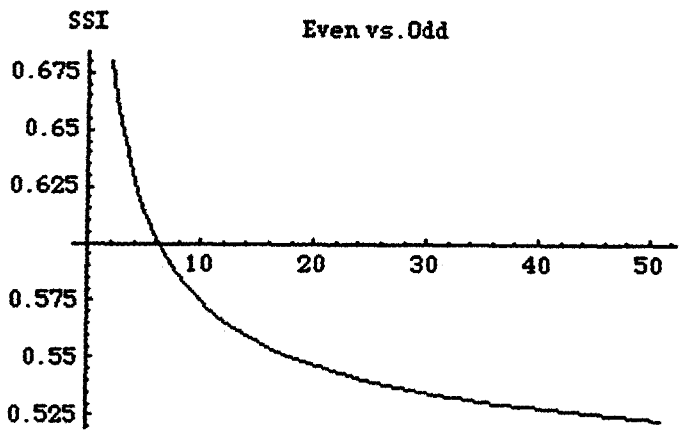
Properties of the Indices:

The work to establish techniques for analyzing the power structure of bicameral legislatures has yielded several important discoveries. First of all, several complex combinatorial identities have been discovered, including the complex expression for the Shapley-Shubik Index of a committee member shown above. By programming each index into *Mathematica*, one can create two- and three-dimensional graphs that illustrate the behavior of each power index as a function of committee size, chamber size, and quotas. Examples of such behaviors are that increasing the quota of both chambers almost always increases the power of the larger chamber, that a committee is always more powerful than a voting block of the same size, and that the four indices exhibit widely different behaviors at extreme values.

One of the most notable results of this analysis actually contradicts an assertion originally made by Shipley and Shubik in 1954. Regarding the properties of the Shapley-Shubik Index, they stated,°∞In pure bicameral systems using simple majority votes, each chamber gets 50% of the power (as it turns out), regardless of the relative sizes [3].∞± This is true if both chambers are the same size, or if both chambers have an odd number of members, which was proven by simplifying the formula for the Shapley-Shubik Index with *Mathematica*. The simplified expression was proven to equal 1/2 by Emeritus Professor John Duncan, using double induction on the residues of the tangent and gamma functions of a complex variable.

But examples show that two chambers do not always share power equally when one or both of them have an even number of members, such as the Arkansas State Legislature, which has a 100-member chamber with S2% of the power and a 35-member chamber with 48% of the power. This may be confirmed by visually inspecting the following two graphs. The graph on the left plots the Shapley-Shubik Index of a variable odd chamber with a majority quota against a fixed odd chamber with 49 members and quota of 25. The graph on the right plots the Shapley-Shubik Index of a variable even chamber with a majority quota against the fixed odd chamber with 49 members. Each point on the horizontal axis represents the quota of the variable chamber. Notice that the Shapley-Shubik Index of the odd chamber remains constant





at.5, while the Shapley-Shubik Index of the even chamber declines as it gets larger.

Conclusions:

This research has provided several revealing insights into the political power structure of bicameral legislatures, including the Arkansas General Assembly. The four power indices all confirm the strong influence of legislative committees, and all show that a committee is always more powerful than a voting block of the same size. The power indices also indicate some relationships that are somewhat surprising. For example, internal committees affect the relative power of two chambers. It also seems that the relative power of a committee is affected by the size and existence of another chamber.

The ability to program formulas for each power index and situation into *Mathematica* has lead to several meaningful discoveries about the properties of each power index. Among these properties are the effect of relative chamber size, the effect of quotas, and the overall influence of committees on a bicameral legislature.

Being able to understand the four power indices as mathematical functions lays the basis for evaluating which index provides the best measure of power for the Arkansas State Government. There are unique characteristics of each index that should be considered. For example, the Deegan-Packel Index is very consistent across the three systems—bicameral, unicameral with committees, and bicameral with committees—but does not place much power in legislative committees. The Banzhaf Index varies the most widely among the three voting situations, but gives a moderate amount of power to legislative committees. The Shapley-Shubik Index is based on fundamental concepts of game theory and distributes power in a fairly consistent manner. When complemented by historical and political analysis, these characteristics could be used to determine which, if any, of these indices are effective measures of voting power in bicameral legislatures such as the Arkansas General Assembly.

From the UN Security Council to the University of Arkansas student government, voting bodies make many important decisions that affect large numbers of people. Therefore, ascertaining the power of individual voters is a worthwhile undertaking. This research takes advantage of the latest computing technology, establishing several techniques that can be used to evaluate a wide variety of voting bodies. These techniques provide a way to quantify voting power and could be used to design more effective legislative systems.

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Faculty Comment:

Mr. King's faculty mentor, Professor Bernard Madison had this to say about Mr. King's research:

Andrew's research area is outside anything we offer in undergraduate or graduate studies. The only related courses in mathematics are in introductory combinatorics. Consequently, Andrew began by reading through Alan Taylor's *Mathematics and Politics* (Springer-Verlag, 1995) and several research journal papers to establish a knowledge base for this research.

From the beginning, Andrew wanted to investigate mathematical measures of political power. This led to a study of yes-no voting systems, which can be considered as part of cooperative/competitive game theory. The use of game theory to study distribution of power in voting systems can he traced back to John von Neumann and Oskar Morgenstern in their 1944 classic, *Theory of Games and Economic Behavior*. Results in this area moved into the public eye with the book and movie, *A Beautiful Mind*, about the life of Nobel Laureate John Forbes Nash.

Over the past half-century, four indices of political power have been developed, often as results of legal arguments over legislative apportionment and voting rights. The four indices carry the names of their creators: Shapley-Shubik (1954), Banzhaf (1965), Johnston (1978), and Deegan-Packel (1978). The Shapley-Shubik index is best known and emerged in Andrew's work as the one receiving most attention and often making the most sense. Andrew's research focused on the application of these indexes to legislatures: unicameral, bicameral, unicameral with committees, and bicameral with committees. His main results were obtained by achieving the following two rather ambitious goals:

1. Developing expressions for computing each of the four power indices for each of the four legislative models — for committee members, non-committee members, and the full chambers.

2. Applying the four indices to the Arkansas General Assembly and its current committee structures.

Until recently, computing these indices for a state legislative body was virtually impossible because of the complexity and the large numbers involved. For example, computing the Shapley-Shubik index for the Arkansas Senate and House of Representatives requires summing several thousand terms involving ten variables and then dividing by 135, a 231-digit number! Andrew developed expressions to do this and then used *Mathematica* for the computations. The formulas are extremely complex sums of binomial coefficients and factorial expressions and represent a major accomplishment—in my view an accomplishment that is most extraordinary for an undergraduate research project.

Along the way to these two main goals, Andrew discovered results that give glimpses of the interrelationships of the indices and reveal very interesting aspects of their behavior when applied to bicameral systems. For example, Shapley and Shubik, in their 1954 paper, state, "In pure bicameral systems using simple majority votes, each chamber gets 50% of the power (as it turns out), regardless of the relative sizes." They were referring to what is now known as the Shapley-Shubik index as a measure of power. Andrew was able to prove this (using *Mathernatica* and computing residues of the tangent and gamma functions of a complex variable) if the size of each house in the bicameral system is odd and if each quota for passing a proposal is the simple majority. However, if the sizes of both chambers are not both odd (as in the Arkansas General Assembly), then the chambers need not share power equally, even when using simple majority votes. Andrew showed this by constructing examples with small numbers as well as with the Arkansas General Assembly. A side result of knowing that the above result for odd-sized chambers is a wealth of very complex combinatorial identities.

Andrew's research is significant, complex, and highly relevant. As far as I am able to discern, many of his results are original; indeed, he developed all results independently. Although mathematical power indexes cannot account for many of the aspects of political power (for example, all the indexes give the Arkansas Governor zero power in the legislative process because only simple majorities are required to override a veto), they do give insight into structural issues in legislatures. Not surprisingly, they show that committee members considering a bill have considerably more power over that bill than do non-committee members. So, Andrew's work should have wider appeal than most research in mathematics. His accomplishments far exceed what I consider normal for a strong undergraduate honors thesis, and he has achieved these accomplishments with minimal guidance from me. He has showed uncanny ability to sort through and organize some enormous counting problems, to program the resulting expressions in *Mathematica* and to interpret the results with maturity far beyond his experiences. His intuition about both mathematics and politics helped immensely.