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Inventory Models for Intermittent Highly Variable Demand and Policy Parameter Adjustments to Meet Desired Service Level Requirements

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INVENTORY MODELS FOR INTERMITTENT HIGHLY VARIABLE DEMAND AND
POLICY PARAMETER ADJUSTMENTS TO MEET DESIRED
SERVICE LEVEL REQUIREMENTS

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of the requirements for the degree of
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ABSTRACT

This dissertation consists of three assays. The first assay examines the robustness of lead time demand models for the continuous review (r, Q) inventory policy. A number of classic distributions as well as distribution selection rules are examined under a wide variety of demand conditions. First, the models are compared to each other by assuming a known demand process and evaluating the errors associated with using a different model. Then, the models are examined using a large sample of simulated demand conditions. Approximation results of inventory performance measures − ready rate, expected number of backorders and on-hand inventory levels are reported. Results indicate that distribution selection rules have great potential for modeling the lead time demand.

Incorporating distributions that preserve higher moment information into an inventory control system to determine the desired performance measures is a challenging task. One difficulty in applying such distributions is estimating the parameters from the data. In most cases only the demand per period is available. Thus, the demand per period moment data must be combined with the knowledge of the lead-times to represent the moments of the lead-time demand. The other difficulty lies in deriving closed form expressions that utilize an appropriate parameter fitting procedure. The second assay addresses these challenging issues by utilizing new parameter fitting strategies. The experiment results, collected under across a large number of simulated demand conditions, indicate that the models that preserve more flexible distributional form yield more accurate inventory performance measure results.

The focus of the third assay is to develop generic simulation optimization techniques based on sample average approximation (SAA) in order to set policy parameters of classical inventory systems having constrained service levels. This work introduces a policy optimization procedure for the continuous review (r, Q) inventory system having a ready rate service level constraint. Two types of SAA optimization procedures are constructed based on sampling from two different simulation methods: discrete-event and Monte-Carlo simulation. The efficiency of each sampling method is evaluated through a set of experiments under a compound Poisson demand process. In addition, the applicability of the proposed optimization procedure to the other re-order type inventory systems is discussed.

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DEDICATION

This dissertation is dedicated to my mother, Fatma Ünlü.

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1 INTRODUCTION

Inventory control under intermittent demand is a challenging task, which is mostly due to the nature of the demand pattern. The transaction (demand incidence) variability (sporadicity) and also demand size variability (lumpiness) are two factors that together make it hard to model the lead time demand, and accordingly set an appropriate inventory control policy. Intermittent demand occurs when a demand series has a significant portion of the periods with no demand and that when a demand does occur the size can have significant variability. Because of this, the variance of the demand process for intermittent demand is often significantly high. Intermittent demand can be observed for the items of engineering spares and stock keeping units (SKUs) observed at the level of a warehouse or retailer or various products at different stages in a supply chain. For such organizations, the dedicated amount of intermittent items can be a significant amount of opportunity cost since even small improvements in their inventory management may result in a substantial amount of cost savings. The primary concern in this research is to determine whether inventory control models can achieve the target service levels during the planning period so that an inventory manager is able to run the underlying lead time demand model, reach the planned service level and determine the optimal policy levels. Therefore, the scope of this research is to identify and develop effective inventory control models in the face of intermittent and highly variable demand situations. The research plan includes three major objectives: 1) developing mathematical procedures for modeling lead-time demand and for inventory systems with intermittent and highly variable demand situations, 2) identifying and developing inventory control models that are well-suited for intermittent highly variable demand situations and 3) investigating simulation based procedures for adjusting selected parameters to hit targeted service levels.

The first objective examines the use of standard (r, Q) inventory control policies for situations involving intermittent and highly variable demand. The purpose of the research is to develop a better understanding of how various distributions and distribution selection rules perform in terms of the related error in predicting the operational performance measures. Historical data from industry are analyzed and used to develop realistic demand scenarios for the purpose of experimentation. A number of classic distributions, namely, normal, lognormal, gamma, Poisson and negative binomial are examined. In addition, a number of distribution selection rules are evaluated for their effectiveness across a wide variety of demand conditions. First the models are compared to each other to examine their comparative error statistics. Then, the models are examined versus a large sample of simulated conditions. The results will indicate the risks associated with various distributional families and distribution selection rules. These results can help inventory software vendors to better tune their software to handle intermittent and highly variable demand situations.

The second objective extends the first objective by incorporating lead time demand models that have more flexible distributional forms into the intermittent and highly variable demand environment. Typically, parameter fitting procedures are employed based on matching the first two moments. The parameter fitting procedure can be strengthened by exploiting higher moments information, which is essentially the focus of this research objective. In this research, moment matching procedures are developed on higher moments for a number of distributions. For other lead time demand models that also preserve more flexible distributional forms, special parameter fitting procedures are applied by taking into account the structure of the intermittent and highly variable demand. This, in principle, enhances the process of extracting and using more information about the demand pattern that exhibits special structural forms as in the case of intermittent and highly variable demand. The experiments are considered to be two fold. The first set of experiments will compare the results with the results of a known lead time demand process. The second set of experiments will be performed in a more sophisticated experimental environment in order to mimic more realistic situations. Throughout the experiments, it is intended to show that such lead time demand models preserving more general form will yield more quality results in terms of approximating the desired inventory performance measures. Clearly, this serves the objective of the overall research that aims to meet the target service level.

The third research area investigates and develops simulation optimization-based methods that can be used without relying on an explicit lead time demand model in order to allow target service levels (e.g. ready rate) to be met for planned inventory policies. The proposed methods allow for the joint optimization of policy parameters of a given classic stochastic inventory system with a service level. This will obviously cover the gap of any risk of not meeting the desired service levels in the case of using the offered models.

2 EVALUATING LEAD TIME DEMAND MODELS FOR (R, Q) INVENTORY MODELS UNDER INTERMITTENT AND HIGHLY VARIABLE DEMAND

2.1 Introduction

In determining inventory performance measures, the modeling of the lead time demand (LTD) plays a key role. In practice it is common to assume that the LTD follows a particular distributional family. The parameters of the assumed distribution are often estimated by employing forecast estimates and the forecast error (or other knowledge). Then, an inventory policy is used and inventory policy parameters are determined. During this process a number of assumptions are made, starting with the assumed form of the distribution. In addition, a number of other specification errors (e.g. demand assumptions, forecasting selection procedures, estimation methods, policy setting heuristics, etc.) may cause the planned inventory system performance to not be met in practice. This is a problem in the use of inventory models in general, but it is especially so for the case of hard to forecast demand situations where the characterization of the LTD is even more problematic. This paper, focuses on the error of arbitrarily picking a distributional model used as an approximation to the LTD. The key emphasis is to develop a better understanding of the robustness of the chosen model in characterizing the LTD under complex demand situations. In this respect, we study a number of classic distributions, namely, normal, lognormal, gamma, Poisson and negative binomial. In addition, this paper examines several distributional selection rules for modeling the LTD. Therefore, a LTD model in this paper refers to a distribution or a distribution selection rule that selects a most promising distribution by relying upon the given LTD parameters (e.g. mean and variance).

In this paper, we approach LTD modeling from the view point of a practitioner. Since, in most situations, the LTD model is not known exactly, performance measures can only be approximated. In this respect, the practitioner can employ a LTD model regardless of whether it is in continuous or discrete distributional form. The characteristics of the inventory control system in this paper are well described in section 2.1 in (Zhao et al., 2007) (without any storage constraints). Inventory is managed for a single item at a single location. The inventory is controlled by a continuous review (r, Q) inventory system with discrete policy parameters of r and Q. In simulating the inventory system, a special demand generator is used to generate demand scenarios. The demand generator creates demand incidences at arbitrary points in time followed by the generation of individual demand sizes in discrete values. The parameters of the demand generator are set by taking into account real data obtained from industry. The details of the demand generator can be found in section 2.5.1. Demands that cannot be immediately fulfilled are fully backordered. Lead times are assumed to be deterministic. All parameters associated with the demand, lead time, and inventory policy are directly supplied (as test cases) by an associated test case generation algorithm in order to cover a wide range of possible scenarios. Hence, no forecasting methodology is considered in this paper. The mean and variance estimate values of LTD are matched to the parameters of the given LTD model. Then, the LTD model can be employed to approximate the inventory performance measures.

In the experiments, the robustness of an arbitrarily picked LTD model is evaluated by the accuracy of approximations for the inventory performance measures. The analysis concentrates on errors associated with computing the ready rate, the expected number of backorders, and the on-hand inventory level. The ready rate is discussed here, instead of the fill rate, because of its analytical tractability. The definitions of these performance measures can be found in section 2.3.

The lead time demand models are evaluated under two demand groups: Group 1 and Group 2. Both of these groups have significant levels of variability, with Group 2 having more variability than Group 1. These groups were selected based on analysis of industrial datasets and to provide challenging test conditions that may provide insights into the robustness of the models. Thus, the term robustness is not meant to be interpreted in some statistical sense, but rather in the more colloquial sense of "able to withstand or overcome adverse conditions". The experiments provide a statistical analysis of the modeling error, which is the difference between the true and approximated performance measure values. The error statistics are tabulated via an *analytical evaluation*, and a *simulation evaluation*. For the analytical evaluation, the actual LTD is assumed to follow a known distribution. Therefore, the true performance measure value can be calculated using the analytical formula associated with the known distribution. Then, a different LTD model is used in order to investigate how well the performance measures are approximated by using a wrongly selected LTD model. During the simulation evaluation, the true performance measure values are estimated by employing a simulation model. The approximated performance measure values are calculated by using the analytical formula associated with a LTD model. The analytical formulas considered in this paper are used to approximate performance measures under a continuous review (r, Q) policy in the face of a unit demand process (Zipkin, 2000, pg 188). Therefore, in order to have comparable results, the simulation model processes the demand as individual units, even if it arrives in sizes greater than 1. The simulation model was verified and validated by comparing with analytical formulas in the case of a compound Poisson process with logarithmic demand sizes (i.e. negative binomial LTD process, (Axsäter, 2006, pg 80).

The paper is organized as follows. In the next section, background on LTD distributions is presented by reviewing relevant literature. In section 3, distributions and distribution selection rules to be evaluated within the paper are presented. Section 4 describes the error metrics and the results associated with the use of a specific LTD model assuming that another model is actually more appropriate. Section 5 discusses the effectiveness of the LTD models when compared to an extensive set of simulated test cases. Finally, section 6 summarizes the findings and discusses the possibilities for future research.

2.2 Literature Review

In the application of traditional inventory models, it is often assumed that the LTD distribution is normal (Silver et al., 1998; Axsäter, 2006). (Lau and Lau, 2003) present a review of the literature summarizing the appropriateness of using the normal distribution to model the LTD. In particular, they show cases where the use of the normal distribution can result in significant cost penalties, even if the coefficient of variation (CV) is low. They also found that the shape of the distribution matters, especially its skewness and kurtosis. (Janssen et al., 2007) argue against the normal distribution's popularity in inventory control management by asserting several reasons. First of all, the normal distribution does not fit the criteria discussed in (Burgin, 1975): Demand distributions can generally be represented with nonnegative values of demand and the shape of the density function changes from monotonic decreasing (low mean demand) to a normal type distribution that is truncated at zero (high mean demand). However, there exists a probability of a normally distributed random variable being negative and the normal distribution is symmetric. The gamma distribution, on the other hand, is nonnegative and the value of the shape parameter can be adjusted to get all three forms. Hence, the gamma distribution does fit the criteria described in (Burgin, 1975).

The authors study the applicability of the gamma distribution in inventory control. According to their findings, the gamma distribution is more applicable for representing the demand of different items than the normal distributions, since it is defined only for non-negative values. The Erlang distribution, known as a special case of the gamma distribution with an integer parametrization, is considered in (Leven and Segerstedt, 2004) for LTD modeling due to its facilitating inventory calculations. The authors compare the results of the inventory systems with Erlang and normal distributed LTD cases. In their study, the inventory system with the Erlang distribution outperforms, although the results are mostly predicated on the special structure of the inventory control system. The authors also point out that skewed distributions are important in obtaining more accurate inventory calculations as opposed to what is claimed in (Silver et al., 1998).

(Tadikamalla, 1984) examined the normal, logistic, lognormal, gamma, and Weibull distributions for adequacy in representing the LTD, arguing that these distributions represent "typical" symmetric and asymmetric distributions. The results indicated that if the coefficient of variation is small that no practical difference between the distributions exists with respect to quality of the optimal solutions. He concluded that the normal and logistic distributions are inadequate when the coefficient of variation is large and that the skewness becomes an important consideration. A number of other distributions have been used for modeling the LTD including the uniform and truncated exponential (Das, 1976), Weibull (Tadikamalla, 1978), Tukey's lambda (Silver, 1977), Pearson and Schmeiser-Deutsch (S-D) distributions (Kottas and Lau, 1980).

Most modeling is predicated on a two moment matching procedure; however, (Kottas and Lau, 1980) advocate for more flexible distributional forms, relying on other moments, to better capture the shape of the distribution. They describe the use of the 4-parameter Pearson and the 4-parameter Schmeiser-Deutsch (S-D) distributional families, and beta distributions. The challenge in applying these distributions is in estimating the parameters from the data. In most cases only the demand per period is available (not actual observations of the LTD). Thus, the demand per period moment data must be combined with knowledge of the lead times to represent the moments of the LTD. (Heuts et al., 1986) refute earlier results in (Naddor, 1978) and show that the shape of the distribution is a key factor in its ability to represent the LTD. In particular, they show that the cost of the inventory policy is significantly affected by the skewness of the distribution and that relying on only the first two moments of the distribution is problematic. They base their analysis on the S-D distribution.

(Shore, 1986) provides a number of useful approximations for computing inventory performance based on approximations for the normal, Poisson, gamma, and negative binomial distributions. He suggests that these approximations may also be useful for more complex inventory systems. (Kumaran and Achary, 1996) compare the generalized type distribution to the approximations in (Shore, 1986). The generalized type distribution is a flexible 4-parameter distribution specified by the pth quantile function. This distributional family includes both symmetric and skewed distributions. It has the advantage of a relatively straightforward method of fitting the parameters but does not have a closed form cumulative distribution function. (Lordahl and Bookbinder, 1994) describe distribution free calculations for the performance measures of (r, Q) inventory systems. Their method relies on having available LTD data and using an approach based on order statistics. That is, using the order statistics to estimate appropriate pth fractiles and directly setting policy parameters. The approach also can be based on bootstrapping methods applied to generating the LTD distribution. They carry out a simulation study to empirically show the value of their method. An important addition of their work is the consideration of bi-modal distributions having positive and negative skew. They conclude that the method gives reasonable results that could easily be implemented in practice.

(Bartezzaghi et al., 1999) examined the impact of skewness and multi-modality on inventory cost calculations. They also looked at the impact of shape with increasing levels of coefficient of variation and the interaction with target service levels via simulation. The focus was on comparing these factors and not necessarily on the ability of the distributions to represent actual LTD distributions. They conclude that practitioners should be aware of the impact of coefficient of variation and shape and how it can dramatically change the analysis of cost considerations. They indicate that better planning methods should be available for specific situations identified by the characteristics of the demand. (Zotteri, 2000) continues this line of research via a more comprehensive simulation study, especially for the case of lumpy demand. The simulation was performed within the context of a manufacturing planning and control system that uses a lot-for-lot rule with rolling horizons, without backlogs. Again, the key finding is that the coefficient of variation and the shape of the distribution is important. The combination is extremely relevant when considering high service levels.

(Shore, 1999) discusses the challenges in using 4-parameter distributional families. The key issue in this approach is adequately estimating the 3rd and 4th moments, which typically have significant sampling error. He introduces a new 4-parameter distribution that has the advantage of a closed form quantile function and a method to specify its 4 parameters by using only the first and second moments. He shows that the procedure adequately preserves the skewness and kurtosis. In addition, optimal solutions to the continuous review (r, Q) inventory model are presented in terms of the new distribution. The results are compared to the normal, gamma, lognormal, and Weibull distribution. He concludes that the method is highly accurate especially in situations where the LTD distribution is highly skewed. (Ramaekers and Janssens, 2008) also describe a procedure to use two-moment information when specifying a distribution from the 4-parameter

Pearson family. They provide a useful summary of past research. They conclude that if the demand process follows a Poisson distribution with some arbitrary demand size then the use of a unimodal or J-shaped beta distribution for the LTD distribution should be considered because the impact of shape is a very important factor. In more recent work, (Gallego et al., 2007) study the effect of non-negative distributions on the optimal base stock levels for inventory systems having highly uncertain demand (extremely large coefficients of variation e.g. greater than 3). A key finding for the gamma, negative binomial, and lognormal distributions was that very little should be ordered when the variance is excessively large. They conclude that when demand variability is very high, it may be enormously expensive and unnecessary to insist on based-stock levels as suggested by the normal distribution.

As for modeling the LTD, distribution selection rules are also advocated by the literature. (Silver et al., 1998) discuss the use of the coefficient of variation in determining the LTD distribution. They state that if the estimated coefficient of variation of the demand is less than 0.5, the normality assumption is often deemed appropriate. (Axsäter, 2006) proposed a rule of thumb, which we call "Axsäter's Rule," in selecting the distribution to fit for the LTD. The LTD with estimated variance-to-mean ratio (VMR) that lies within the range 0.9 and 1.1 can be fit with Poisson distribution. If the estimated VMR is less than 0.9, then a Poisson or a mixture of binomials (described in (Adan et al., 1995)) can be considered; and if the estimated VMR is greater than 1.1, the negative binomial can be selected.

2.3 Models For the Lead Time Demand

This section discusses the inventory performance measures, LTD distributions and distribution selection rules. Three inventory performance measures are considered: ready rate (*RR*), the expected number of backorders (*B*) and on-hand inventory level (*I*). The ready rate is the fraction of time with positive stock on-hand (Axsäter, 2006, pg 94 and Silver et al., 1998, pg 245). Let *IN* be the net inventory level. Then, ready rate is $Pr\{IN > 0\}$ (i.e. $1 - Pr\{IN \leq 0\}$) by definition. The analytical formula used for *RR* in this paper is based on (Zipkin, 2000, p 188, formula 6.2.11). Another commonly used performance measure in practice is called the fill rate (*FR*), which is the fraction of total demand that can be immediately satisfied from inventory without backordering (Silver et al., 1998, pg 245). It should be noted that the expression of $1 - Pr\{IN \leq 0\}$ represents both the ready rate and fill rate in the case of a pure Poisson process (i.e. demand size is 1 and inter-arrivals are exponentially distributed) (Zipkin, 2000, pg 183 and Axsäter, 2006, pg 95). However, if the demand follows a process different from pure Poisson, then it is observed that $FR \leq RR$. The reader is referred to (Larsen and Thorstenson, 2008) for further discussion. We examine ready rate since a tractable analytical formula exists for the underlying inventory environment considered in this paper. In addition, fill rate is clearly related to ready-rate and thus some of its behavior can be inferred by the results on ready-rate. The expected number of backorders can be defined as the long-run average number of backordered demands. On-hand inventory level refers to the long-run average number of inventory units on-hand. Under the assumption that *F* is the LTD model, $\overline{RR}_F$, $\overline{B}_F$ and $\overline{I}_F$ represent the ready rate, expected number of backorders and on-hand inventory levels, respectively. In case of picking a distributional model to be used as an approximation to the LTD, the desired performance measures are computed by using the following general formulations:

$$\overline{RR}_F = 1 - \frac{1}{Q}\left[G_F^1(r) - G_F^1(r+Q)\right] \tag{1}$$

$$\overline{B}_F = \frac{1}{Q}\left[G_F^2(r) - G_F^2(r+Q)\right] \tag{2}$$

$$\bar{I}_F = \frac{1}{2}(Q+1) + r - \mu + \overline{B}_F \tag{3}$$

where μ, r and Q are the mean of the LTD, the re-order point and the order quantity while $G_F^1(.)$ and $G_F^2(.)$ are the first and second order loss functions of F, respectively. We explain how we compute the parameters of F as follows. Suppose that the gamma distribution is the underlying lead time demand model. The gamma distribution has two parameters α (shape) and β (scale). We represent gamma distributed random lead time demand variables as $X \sim \Gamma(\alpha, \beta)$ where $(X \geq 0)$. The parameters of the gamma distribution should be determined in order to be used as an approximation to LTD. One practical way to determine the parameters of the gamma distribution is to use the demand per period data (Tyworth and Ganeshan, 2000). The estimate values of the mean and standard deviation of demand (μ_D, σ_D) and the mean of lead time (μ_L) can be used in order to compute the mean and standard deviation of LTD (μ, σ). The following standard equations: $\mu = \mu_D \mu_L$ and $\sigma = \sigma_D \sqrt{\mu_L}$ are used. The mean and variance estimate values of LTD are matched to the gamma distribution parameters $\mu = E[X] = \alpha\beta$, $V[X] = \sigma^2 = \alpha\beta^2$, and therefore, the parameters of the gamma distribution are determined with μ and σ as: $\alpha = \frac{\mu^2}{\sigma^2}$ and $\beta = \frac{\sigma^2}{\mu}$. Then, expressions (1), (2) and (3) along with the loss functions of F are used to calculate inventory performance measures. The term $\frac{1}{2}(Q+1)$ in expression (3) is replaced with $\frac{1}{2}Q$ in the case where F has a continuous distributional form. The formulae are exact in the case of the discrete demand processes. Otherwise, they are approximations (Zipkin, 2000, p 211). The reader is referred to (Zipkin, 2000) for details and further explanation of the formulations.

In this paper, the classic distributions of normal (*N*), lognormal (*LN*), gamma (*G*), Poisson (*P*) and negative binomial (*NB*) are considered for F. In addition, F can be some distribution recommended by a distribution selection rule. The distribution selection rules evaluated in this paper are as follows: Adan *et al.* Rule (*ADR*), Gamma-Adan Rule (*GADR*), Axsäter's Rule (*AXR*), the mixture of normal and negative binomial (*MNNB*) and the mixture of gamma and negative binomial (*MGNBA*).

In order to employ mixture distributions to calculate inventory performance measures by using (1), (2) and (3), their loss functions should be derived. Let $G^1_{MD}(.)$ and $G^2_{MD}(.)$ be the first and second order loss functions of a given mixture distribution, respectively. We derive the first and second order loss functions of a mixture distribution with the following.

$$G^1_{MD}(x) = (1-q)\,G^1_1(x) + qG^1_2(x) \tag{4}$$

$$G^2_{MD}(x) = (1-q)\,G^2_1(x) + qG^2_2(x) \tag{5}$$

where $G^1_1(.)$ and $G^1_2(.)$ are the first order loss functions and $G^2_1(.)$ and $G^2_2(.)$ are the second order loss functions of the two distributions being mixed, respectively. The proofs of expressions (4) and (5) are given in Appendix A. It should be noted that if F is modelled by a mixture distribution, the calculation of inventory performance measures is quite easy based on (4) and (5) as they facilitate the corresponding parameter fitting procedure. The mean and variance estimates of the LTD are matched to the parameters of the distributions that are mixed. In addition, the mixture fraction q is always provided by the associated mixture distribution in this paper.

Based on these preliminaries, the distribution selection rules considered in this paper are given as follows. Adan *et al.* Rule (Adan et al., 1995) selects a distribution from the set of distributions {the mixture of binomial (*MB*), the mixture of negative binomial (*MNB*), the mixture of geometric (*MG*) and (*P*)}. The rule decides which distribution to select with respect to the parameter a. In order to utilize the mean (μ) and variance (σ^2) of the LTD, the parameter a is defined as $\frac{\sigma^2-\mu}{\mu^2}$. The rule selects *MB* if $\mu > \sigma^2$ (i.e. $a < 0$); *MNB* if $\mu < \sigma^2$ (i.e. $a > 0$) and the parameter fitting is possible (i.e. $a < 1$) for *NB*; *P* if $\mu = \sigma^2$ (i.e. $a = 0$); *MG* for large coefficient of variation values (i.e. $(a \geq 1)$). The rule *ADR* is presented in Exhibit A.1.

The mixture distributions used in the rule are re-defined based on (Adan et al., 1995) as follows:

The Mixture of Binomial Distribution: This distribution consists of the mixture of two binomial distributions ($BIN_i\left(k,p\right)$ where k is the number of trials and p is the probability of success). If u is defined as the random variable generated from $U\left(0,1\right)$ (uniform distribution), then the random variable Y is determined by the following:

$$Y = \begin{cases} BIN_1\left(k,p\right), & if\, u \leq q \\ BIN_2\left(k+1,p\right), & if\, q < u \leq 1 \end{cases}$$

where $k = \left|\left\lfloor \frac{1}{a} \right\rfloor\right|$, $q = \frac{1+a(1+k)+\sqrt{-ak(1+k)-k}}{1+a}$ and $p = \frac{\mu}{k+1-q}$.

The Mixture of Negative Binomial Distribution: This distribution consists of the mixture of two negative binomial distributions ($NB_i\left(k,p\right)$) where k is the desired number of success and p is the probability of success). The random variable Y is determined by the following:

$$Y = \begin{cases} NB_1(k,p), & if\, u \le q \\ NB_2(k+1,p), & if\, q < u \le 1 \end{cases}$$

where $k = \left|\left\lfloor \frac{1}{a} \right\rfloor\right|$, $q = \frac{a(1+k)-\sqrt{(1+k)(1-ak)}}{1+a}$ and $p = \frac{k+1-q}{k+1-q+\mu}$.

<u>*The Mixture of Geometric Distribution*</u>: This distribution consists of the mixture of two geometric distributions ($GEO_i(p_i)$ where p_i is the probability of successes). The random variable Y is determined by the following:

$$Y = \begin{cases} GEO_1(p_1), & if\, u \le q \\ GEO_2(p_2), & if\, q < u \le 1 \end{cases}$$

where $p_1 = 1 - \frac{\mu\left(1+a+\sqrt{a^2-1}\right)}{2+\mu\left(1+a+\sqrt{a^2-1}\right)}$, $p_2 = 1 - \frac{\mu\left(1+a-\sqrt{a^2-1}\right)}{2+\mu\left(1+a-\sqrt{a^2-1}\right)}$ and $q = \frac{1}{1+a+\sqrt{a^2-1}}$.

Axsäter's Rule recommends a distribution from the distribution set {MB, P, NB} based on the variance-to-mean ratio (VMR) of the LTD. The rule AXR is given in Exhibit A.2.

In this paper, we also present three additional LTD models, which we developed based on preliminary results from the experiments. The rule called Gamma-Adan Rule ($GADR$) is given in Exhibit A.3. We directly give the first and second order loss functions of the proposed rules called the mixture of normal and negative binomial distribution ($MNNB$) and the mixture of gamma and negative binomial ($MGNBA$) in Exhibit A.4 and Exhibit A.5, respectively.

We explain how we developed these models with an example. The other models presented in this paper are developed by exploiting similar ideas. When the LTD is approximated by N and NB to calculate B, the overlay plots of error across each target RR are depicted in Figure A.6 and

Figure A.7, respectively. The definition of error can be obtained in Section 2.4.2. These two plots indicate an opposite error behavior over target *RR*. Therefore, in order to decrease the size of the error, a mixture distribution can be built by using *N* and *NB* and a mixture fraction $q = 0.5$ (arbitrarily selected for convenience). The model called *MNNB* basically uses the defined mixture distribution whenever parameter fitting is possible for *NB*. If the parameter fitting is not possible, then the model *MNNB* selects *G* which exhibits somewhat similar error behavior with *NB*. The associated error plot for *MNNB* is given in Figure A.8. As can be seen from the figure, the error for each target *RR* are scattered symmetrically around 0 in relatively smaller amounts. The error behavior indicates that *MNNB* is a more robust LTD model as compared to both *N* and *NB*. We also present the associated error plot for *ADR* in Figure A.9. The robustness of this rule can also be noticed by observing the error behavior.

In what follows, the effectiveness of each of these LTD models will be tested on a set of test cases. The next section presents the characteristics of the industrial data sets and describes the test case generation methods utilized in this paper.

2.4 Analytical Evaluation

The models and procedures given in this section allow for the generation of test cases that represent example data found in industrial practice and to cover a large portion of the possible parameter values for the (r, Q) inventory policy. The industrial data sets consist of many monthly recorded demand series. Let σ_{NZ} be the standard deviation of the nonzero demand within a demand series. Based on this parameter, the demand series were classified into two groups: 1) The demand series showing the statistical property of $0 < \sigma_{NZ} < 4$. This is the first data group and labeled Group 1 within the analysis. This group represents 1017 demand series or 92% of the data sets. The demand series showing the statistical property of $4 \leq \sigma_{NZ} < 1000$ represents the second data group and called Group 2 within the analysis. This group has relatively high variability and represented 103 demand series or 8% of the datasets. Thus, the industry data used to motivate the generation of test cases consisted of a total of 1120 demand series. In this analysis, a set of cases will be generated and then performance computed for each of the LTD models. The following subsection describes how the test cases are generated.

2.4.1 Test Case Generation

In the analytical evaluation, a test case refers to a vector of (r, Q, μ, σ, F). Re-order point (r), re-order quantity (Q), the mean (μ) and the standard deviation (σ) of the underlying LTD (F) are given as input parameters in order to calculate the inventory performance measures through the analytical formulas. After randomly generating a pair of (μ and σ), the policy parameters (r and Q) are obtained based on a specified service level of *RR*. Policy parameters are enumerated to cover a range of many possible service levels. It should be noted that parameters r and Q are not selected arbitrarily. They are determined based on (Algorithm A.1).

Since each test case requires values for the mean (μ) and the standard deviation (σ) of the LTD, we needed a method for generating instances of these parameters. While, in general, μ and σ, can take on any positive values, we thought it would be more realistic to generate their values based on the characteristics of real industrial datasets. For the datasets, we performed a bi-variate analysis of the estimated values of μ and σ. Our intuition is that these parameters should have some relationships (e.g. if μ is high σ might be high). Thus, we wanted to model possible relationships in the generated test cases. While we could have simply used the parameters implied by each dataset directly, a model for generating test cases allows greater control during the experiments. Based on an analysis of industrial data (not described here), the pair of μ and σ can be adequately represented by a bivariate lognormal distribution. Two bivariate lognormal distributions were defined for Group 1 as $BVML$ and for Group 2 as $BVMH$.

In what follows, the policy parameters r and Q are determined under a constraint on RR. For a range of values of Q, the value of r that approximately meets RR is determined. The approximation is performed by using the gamma distribution. The values of RR are enumerated over the range from 0 to 1 with an emphasis over the higher values. Let W = {0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.62, 0.64, 0.66, 0.68, 0.7, 0.72, 0.74, 0.76, 0.78, 0.8, 0.82, 0.84, 0.86, 0.88, 0.9, 0.91, 0.92, 0.93, 0.94, 0.95, 0.96, 0.97, 0.98, 0.99} be the set of target RR values. Notice that many service levels are enumerated to generate test cases so that a large range of scenarios are covered. The ideas of this section are brought together in the algorithm given in Appendix A (Algorithm A.1). The given algorithm generates the test cases which captures many possible scenarios for the analytical evaluation procedure. In using the algorithm, some target levels (w_L) of RR (especially very low ones) may not be hit for a randomly generated pair of (μ, σ). This causes the generation of more test cases that hit the higher RR values. Thus, the test cases have a higher sampling frequency of larger RR values. This is typically desirable, since in practice higher values of RR are more pertinent. This algorithm can be used by other researchers to generate additional test cases.

2.4.2 Evaluation Procedure and Results

In calculating inventory performance measures, one basic question arises under the assumption that the true LTD model is known with certainty (F_{LTD}): What would be the effect on inventory performance measures if the LTD model is approximated by some other arbitrarily selected model (F_a). Clearly, the aforementioned specification errors are of interest in order to identify the magnitude of the effect. In this respect, we define two types of errors.

Type-I error: error of not picking the LTD model given that it is actually the true model. This type error is concerned with the potential error of not picking the true LTD model. For example, consider the potential error of not approximating the LTD via the normal distribution in the case where the LTD is actually distributed normal.

Type-II error: error of approximating the LTD via a distribution or rule given that the LTD actually follows some other distribution. That is, type II error deals with the error of arbitrarily picking a LTD model. For example, consider the potential error of approximating the LTD via Adan et al. Rule although the true LTD actually follows a distribution (e.g. gamma).

The idea behind the foregoing error types can be observed in Table A.1. Let E^i_j and $\left|RE^i_j\right|$ be the error and absolute relative error for the known LTD model i and test case j. Let θ^i_j be the value of a performance measure for the true LTD model i as F_{LTD} and the test case j. Let θ^a_j be the value of a performance measure for the test case j under the assumption that the LTD model is arbitrarily selected as F_a. Type-I error statistics are obtained for each distribution by collecting the following error results across all the LTD models. Thus,

$$E^i_j = \theta^i_j - \theta^a_j \qquad i \in \{F_{LTD} : N, G, P, LN, NB\} \tag{6}$$

$$\left|RE^i_j\right| = \left|\frac{\theta^i_j - \theta^a_j}{\theta^i_j}\right| \qquad i \in \{F_{LTD} : N, G, P, LN, NB\} \tag{7}$$

The ideas for type-I error are implemented by Algorithm 1 to record the associated statistics for all generated test cases.

Algorithm 1 Type-I Error Calculation in Analytical Evaluation

Generate test case j *of* (r, Q, μ, σ)
For $i \in \{F_{LTD} : N, G, P, LN, NB\}$ *do*
    Match μ, σ *to the parameters of* F_{LTD}
    Evaluate θ_j^i *for* $\overline{RR}_F$, $\overline{B}_F$ *and* $\overline{I}_F$ *where* $F = F_{LTD}$ *for test case* j
    For $a \in \{F_a : N, G, P, LN, NB, AXR, ADR, GADR\}$ *do*
        Match μ, σ *to the parameters of* F_a
        Evaluate θ_j^a *for* $\overline{RR}_F$, $\overline{B}_F$ *and* $\overline{I}_F$ *where* $F = F_a$ *for test case* j
        Evaluate E_j^i *and* $\left|RE_j^i\right|$ *for test case* j *using*
        $E_j^i = \theta_j^i - \theta_j^a$ *and* $\left|RE_j^i\right| = \left|\frac{\theta_j^i - \theta_j^a}{\theta_j^i}\right|$
        Record the error statistics
    End-do
End-do

Type-II error statistics of each LTD model are collected across all the distributions by using the following expressions:

$$E_j^i = \theta_j^i - \theta_j^a \qquad a \in \{F_a : N, G, P, LN, NB, AXR, ADR, GADR\} \tag{8}$$

$$\left|RE_j^i\right| = \left|\frac{\theta_j^i - \theta_j^a}{\theta_j^i}\right| \qquad a \in \{F_a : N, G, P, LN, NB, AXR, ADR, GADR\} \tag{9}$$

Algorithm 2 implements the ideas for type-II error to record the associated statistics for all generated test cases.

Algorithm 2 Type-II Error Calculation in Analytical Evaluation

Generate test case j *of* (r, Q, μ, σ)
For $a \in \{F_a : N, G, P, LN, NB, AXR, ADR, GADR\}$ *do*
    Match μ, σ *to the parameters of* F_a
    Evaluate θ_j^a *for* $\overline{RR}_F$, $\overline{B}_F$ *and* $\overline{I}_F$ *where* $F = F_a$ *for test case* j
    For $i \in \{F_{LTD} : N, G, P, LN, NB\}$ *do*
        Match μ, σ *to the parameters of* F_{LTD}
        Evaluate θ_j^i *for* $\overline{RR}_F$, $\overline{B}_F$ *and* $\overline{I}_F$ *where* $F = F_{LTD}$ *for test case* j
        Evaluate E_j^i *and* $\left|RE_j^i\right|$ *for test case* j *using*
        $E_j^i = \theta_j^i - \theta_j^a$ *and* $\left|RE_j^i\right| = \left|\frac{\theta_j^i - \theta_j^a}{\theta_j^i}\right|$
        Record the error statistics
    End-do
End-do

In what follows, the two types of error statistics are collected for each data group. For both Group 1 and Group 2, the statistics of these error results are collected for 30,000 randomly generated test cases which give at least 2 digits of accuracy based on a classic half-width analysis. The descriptive statistics of the generated test cases with respect to target *RR* levels (w_L) are presented for Group 1 and Group 2 in Table A.2 and Table A.3, respectively. In the tables, it can be seen that the targeted *RR* (i.e. service level) covers the range over (0, 1) and yields a high frequency on the high service levels, as mostly desired in industrial practice. Low and high variability of the generated test cases can also be seen in the tables.

The error results are presented in Table A.4, Table A.5, Table A.6, Table A.7, Table A.8 and Table A.9. Along with the descriptive statistics, these tables also present the statistics related to the probability that the absolute relative error is less than or equal to 0.10 (PRE(.10)), 0.05 (PRE(.05)) and 0.01 (PRE(.01)). The statistics of *PRE* $(\%)$ provide a reliability measure for the use of the LTD model. The discussion of the results related to the first and second type errors is given as follows.

Type-I Error Statistics Results: The error results for Group 1 and Group 2 are tabulated in Table A.4 and Table A.5, respectively. In some sense, the results indicate how bad the error can be if the true LTD is some distribution and the modeler chooses to apply a different distribution or rule. In Table A.5 the value $PRE(.10)$ means that if NB is the true LTD distribution and some other distribution or the rule is used for approximating the LTD, then 70.3% of the time the approximated performance will be within 10% of the true performance (i.e. $PRE\left(\left|\frac{\theta_j^i - \theta_j^a}{\theta_j^i}\right| \leq 0.10\right) = 70.3\%$ where $i = F_{LTD} = NB$). In almost all the cases, the error results indicate that the performance measures of B, RR and I are mostly overestimated. In the case where the LTD actually follows the normal distribution, $PRE(\%)$ statistics show that the other LTD models produce the worst performance results. That is, if the true LTD model is actually normal and a different distribution or the rule is used, then there is a higher risk of error if the modeler uses a different LTD model. This also yields the highest range of variation on the error results of both performance metrics in case of the actually normal distributed LTD. In the case where the Poisson distribution is the true LTD model, $PRE(\%)$ statistics show that the performance measures of B, RR and I can be reasonably approximated by the other LTD models. Thus, if the underlying model is actually Poisson, then the use of a different LTD model appears to be reasonable. Other LTD models yield the smallest deviation for the estimated performance measures in case of the Poisson distributed LTD.

Type-II Error Statistics Results: The results are presented for Group 1 and Group 2 in Table A.6, Table A.7, Table A.8 and Table A.9. The tables represent how well a randomly picked LTD model performs in approximating the performance measures of RR, B and I. In Table A.6, the value $PRE(.01)$ means that if ADR is used and the true LTD is something else, then using ADR will result in 90.2% of the time being within 1% of the true performance (i.e. $PRE\left(\left|\frac{\theta_j^i-\theta_j^a}{\theta_j^i}\right| \leq 0.01\right) =$ 90.2% where $a = F_a = ADR$). In almost all the cases, the error values associated with *B* are higher than those of *RR*. For the test cases in Group 2, there is much more variability in error results. We may even observe excessively large error results as it can be seen by the minimum and maximum error values from Table A.8 and Table A.9. Although for test cases in Group 1 the models performs somewhat well in approximating the service levels, some of them produce significantly poor results for cases in Group 2. However, the LTD model *GADR* that we propose in this study produces consistently better error results for cases both in Group 1 and Group 2. For cases in Group 2, $PRE\,(\%)$ statistics show that the performance measures of *B* and *I* are best approximated if the *GADR* is picked for approximating the LTD. The model also yields the smallest deviation for the estimated performance measures. For the same statistical measures, *GADR* and *ADR* produce similar statistical results for *RR* for cases in Group 2. As far as the statistics of $PRE(.01)$ are concerned, the quality of *GADR* in approximating *B* is noticed easily. The model yields 55.7% for the cases in Group 1 and 54.1% for the cases in Group 2 although all the other models remain much below 50%. This means that for more than half of all the cases, *GADR* yields error sizes less than 1% of the true value of *B*.

The gamma and negative binomial distributions produce competitive error results especially for the cases in Group 2 where we observe much more variability. This points out that if a distribution has to be employed other than a rule, then, either the gamma or the negative binomial may be preferred to approximate the LTD in the case of high variability in demand. It should also be noted that we observe the worst error results with the Poisson distribution. Throughout our experiments, we observed that the Poisson distribution is inclined to yield better error results if the mean and variance of the LTD values are small and the ratio of variance to the mean is around 1. More specific results for the performance of the Poisson distribution are available in Table A.17. For test cases where $\mu \leq 1$, $\sigma^2 \leq 1$ and $0.9 \leq \sigma^2/\mu \leq 1.1$, Table A.17 shows the error results for Group 1. As can be seen from $PRE(\%)$ statistics, the Poisson distribution yields better results as compared to the results given in Table A.7. For example, as far as B results for P(10%) statistics are concerned the Poisson distribution gives only 27% as can be seen from Table A.7. This means that only 27% of the time absolute error results will be within 10% of the true performance. However, for test cases where $\mu \leq 1$, $\sigma^2 \leq 1$ and $0.9 \leq \sigma^2/\mu \leq 1.1$, the performance of the Poisson distribution increases up to 72% as can be seen from Table A.17. As can be noted from Table A.7, the Poisson distribution gives somewhat good results for the cases where we observe less variability in demand. However, its performance significantly decreases for the cases in Group 2 (Table A.8 and Table A.9) where we frequently observe large variance values. This also explains why AXR gives poor results for Group 2 as it relies heavily on the Poisson distribution.

There are two issues related to using the Poisson distribution as an approximation to the LTD. First, the true LTD model may not be the Poisson distribution. In this case, we observe large error values since the Poisson is able to model the LTD by using only the information of the mean. Second, suppose that the true LTD is the Poisson distribution. In this case, the other LTD models are able to approximate LTD fairly well since they utilize not only the information of the mean but also the variance, which results in smaller error sizes. These two issues suggest that the Poisson distribution should not be used to approximate the LTD unless the ratio of variance to the mean is exactly 1.

2.5 Simulation Evaluation

In this section, the specification errors are determined using the results of a simulation study. In addition to the LTD models in the analytical evaluation, two additional LTD models are also investigated; namely, *MNNB* and *MGNBA*. The simulation experiments were done in a rigorous environment in terms of empirically generated test cases based on a special demand generator. The demand generator (batch on/off model) is used to characterize the demand process. The parameters generated by the batch on/off model also feed the (r, Q) algorithms for test case generation and the simulation model. The created test case is utilized by both the simulation model and the analytical models to evaluate the errors. The details of the simulation model are given in the following section.

2.5.1 Simulation Model

The JSL is an open source simulation library developed for discrete event simulation modeling by supporting random number generation, statistical collection and basic reporting. The JSL includes some packages for easily modeling complex inventory and supply chain systems.

The simulation model consists of 4 modules. The first module utilizes the demand generator to draw the demand observations from the specified demand distributions. The most significant part of the simulation model is that it generates demands by employing a flexible arrival process. The second module receives the demand parameters and randomly generates lead times to approximate the mean and the variance of the LTD. These two parameters are utilized for the test case generation procedure. In the third module, by employing the demand generator, the standard (r, Q) model is simulated while performance measures of *RR*, *B* and *I* are monitored during the simulation. This module is also responsible for capturing the LTD parameters (i.e. μ and σ) by implementing the methodology described in (Brown, 1982). The last module calculates the performance measures using the LTD models (N, G, P, LN, NB, AXR, ADR, $GADR$, $MNNB$, $MGNBA$) and tabulates the error statistics.

We model the demand process with a continuous time batch-on/off model based on (Galmes and Puigjaner, 2003 and also described in Rossetti et al., 2010). The batch-on/off model consists of an arrival process which has two states: *ON* in which demand may be permitted to arrive and *OFF* in which no demands are permitted to arrive. Let $S(t)$ be the stochastic process that determines whether demand can arrive at time t. Therefore,

$$S(t) = \begin{cases} 1, & \text{state: } ON \\ 0, & \text{state: } OFF \end{cases}$$

Let X_I be the length of time spent in the *OFF* state. Let X_B be the length of time spent in the *ON* state. We assume that X_I and X_B are independent. Thus, $S(t)$ is an alternating renewal process with period (length) $X_I + X_B$. The steady-state probability of state *ON* is

$$P_b = P(S(t \to \infty) = 1) = \frac{E[X_B]}{E[X_I] + E[X_B]}$$

(Tijms (2003), pg 43). The potential arrival process is a renewal process with inter-occurrence times, Y_i, having time points $T_0 = 0$ and $T_{i-1} < T_i < T_{i+1}$. Let D_i be the amount of demand at event i. Then,

$$D_i = 0, \quad if\, S(T_i) = 0$$

$$D_i \sim F_{NZ}, \quad if\, S(T_i) = 1$$

where F_{NZ} is the cumulative probability distribution of non-zero demand. In a simulation model in which demands are generated by the continuous time batch-on/off model, the demand sizes are determined by the variable of D_i and the events of demand arrivals are scheduled by Y_i. The alternating renewal process is scheduled by sampling, X_I and X_B. This gives an extensive amount of flexibility in modeling a wide variety of demand processes. For example, if $S(t)$ is always ON and the demand occurrence process is Poisson (i.e. Y_i is exponentially distributed), then the resulting demand has a compound Poisson process. An algorithm of the demand generator and more discussion can be obtained in (Rossetti et al., 2010).

In this study, the demand generator is set up as follows. The events of demand arrivals are scheduled by an exponential distribution (i.e. Y_i is exponentially distributed). A gamma distribution is assumed to generate the input random variables of X_I: whose mean and standard deviation are μ_I and σ_I, X_B: whose mean and standard deviation are μ_B and σ_B, D_i (integer-rounded): whose mean and standard deviation are μ_{NZ} and σ_{NZ}. Therefore, the set (μ_{NZ}, σ_{NZ}, μ_B, σ_B, μ_I, σ_I) represents the demand generator parameters. The demand generator parameters set is generated via a multivariate distribution which will be given in the next section. Based on the described set up, the algorithm of the demand generator is given in Appendix A (Algorithm A.2).

The generated demands are processed in order to be consistent with the underlying theory of the (r, Q) inventory model. Since the given demand generator drives demands in discrete units, it is possible to apply demand-splitting (Teunter and Dekker, 2008). In addition, the simulation model processes demands based on a first-come-first-served ordering. By doing so, the received (and backordered) demands can be partially or fully satisfied from the available stock on-hand. Therefore, based on these assumptions, a customer with a demand for multiple units can be regarded as multiple customers with demand of unit size. This allows the simulation model to face the compound demand process as if it faces a unit demand process, which allows comparable results of inventory performance measures with the analytical formulas (1), (2) and (3).

2.5.2 Test Case Generation

For the simulation evaluation, the test cases are generated as follows. The LTD parameters (i.e. μ and σ) are captured during the simulation by the method described in (Brown, 1982, pg 260). The policy parameters of r and Q are obtained with the same procedure given in section 2.4.1. However, in order to meet the target service levels of RR, we first approximate the LTD parameters (defined as μ_A and σ_A). A compound Poisson approximation method is applied in order to obtain μ_A and σ_A. Suppose that $T_i \sim exponential(\frac{1}{\lambda})$ where λ is the demand arrival rate. Thus, λP_b is the rate of non-zero demand. Then, $D(t) = \sum_{i=1}^{N(t)} D_i$ where $N(t)$ is the number of non-zero demand incidences that have occurred up to time t and, therefore, $N(t) \sim Poisson(\lambda P_b)$.

It follows that

$$E[D(t)] = \lambda P_b t E[D_i]$$

and

$$Var(D(t)) = \lambda P_b t E\left[D_i^2\right]$$

$$Var(D(t)) = \lambda P_b t \left[Var(D_i) + (E[D_i])^2\right]$$

The approximated mean and standard deviation of the LTD are expressed as follows:

$$\mu_A = E[D(L)] = \lambda P_b L \mu_{NZ}$$

$$\sigma_A = \sqrt{Var(D(L))} = \sqrt{\lambda P_b L \left[\sigma_{NZ}^2 + (\mu_{NZ})^2\right]}$$

In this study, observations of lead times were not directly available from industry sources;

therefore, we considered the data for lead times (monthly) given in (Strijbosch et al., 2000). In (Strijbosch et al., 2000), although the standard deviation of the lead time is not provided, the average of monthly lead times is given as 5.17. The largest lead time is given as 20 months and indicates the range of the lead time. Let μ_L, σ_L, $\overline{R}$ and d_2 be the mean and standard deviation of the lead time, the range of the lead times and a constant for σ_L, respectively. d_2 is assumed to be 3.9. The approach in (Eng., 2010) is used to calculate the standard deviation of the lead time which is $\sigma_L = \frac{\overline{R}}{d_2} = \frac{20}{3.9} \simeq 5$. Because of the parameter fitting and distribution shape issues, the gamma distribution was selected to generate "constant" lead times for each test case using the fitted parameters of μ_L and σ_L.

The algorithm given in Appendix A (Algorithm A.3) implements the ideas in this section. In the given algorithm, the enumerated set of ready rates is defined as W = {0.2, 0.4, 0.6, 0.62, 0.64, 0.66, 0.68, 0.7, 0.72, 0.74, 0.76, 0.78, 0.8, 0.82, 0.84, 0.86, 0.88, 0.9, 0.91, 0.92, 0.93, 0.94, 0.95, 0.96, 0.97, 0.98, 0.99}. In order to generate demand generator parameters set, two multivariate lognormal distributions are used based on an analysis of industrial datasets: one for the cases in Group 1 ($MVML$) and one for the cases in Group 2 ($MVMH$). The multivariate distributions are used to reflect the correlation among demand generator parameters. The algorithm generates both demand generator parameters sets of (μ_{NZ}, σ_{NZ}, μ_B, σ_B, μ_I, σ_I) and the test case vectors of (r, Q, μ, σ, F) which cover many possible scenarios for the simulation evaluation procedure.

2.5.3 Experiment Settings and Results

The simulation model for each generated case associated with Group 1 was run for 30 replications with 2,000,000 time units of run-time and 200,000 time units of warm-up period. Based on these simulation run parameters, at least 2 digits of precision is achieved for the cases in Group

1 for the performance measures of *RR*, *B* and *I*. The computational time for only one test case in Group 2 took approximately 8 minutes. For some cases in Group 2, computational times may even be much higher. Since there is a considerable constraint on the computational time of the simulation runs, at least 1 digit of precision is achieved for the cases in Group 2 for the performance measures through the experiments. Therefore, for cases in Group 2, the parameters setting for simulation runs was 30 replications with 850,000 time units of run-time and 85,000 time units of warm-up period. We define the variables E_j^S and $\left|RE_j^S\right|$ as the error and absolute relative error for the test case j. Let θ_j^S be the value of a performance measure estimated by the simulation model for test case j and let θ_j^a be the value of a performance measure approximated for test case j under the assumption that the LTD model is F_a. The same types of statistics with the analytical evaluation are recorded for each assumed LTD model by using the following formulas:

$$E_j^S = \theta_j^S - \theta_j^a \qquad a \in \{F_a : N, G, P, LN, NB, AXR, ADR, GADR, MNNB, MGNBA\} \tag{10}$$

$$\left|RE_j^S\right| = \left|\frac{\theta_j^S - \theta_j^a}{\theta_j^S}\right| \qquad a \in \{F_a : N, G, P, LN, NB, AXR, ADR, GADR, MNNB, MGNBA\} \tag{11}$$

Algorithm 3 summarizes the recording of the error statistics for all generated test cases.

Algorithm 3 Error Calculation in Simulation Evaluation

Generate $(\mu_{NZ}, \sigma_{NZ}, \mu_B, \sigma_B, \mu_I, \sigma_I)$ *and test case* j *of* (r, Q, μ, σ) *and*
Estimate θ_j^S *for RR, B and I through simulation model*
For $a \in \{F_a : N, G, P, LN, NB, AXR, ADR, GADR, MNNB, MGNBA\}$ *do*
    Match μ, σ *to the parameters of* F_a
    Evaluate θ_j^a *for* $\overline{RR}_F$, $\overline{B}_F$ *and* $\overline{I}_F$ *where* $F = F_a$ *for test case* j
    Evaluate E_j^S *and* $\left|RE_j^S\right|$ *for test case* j *using*
    $E_j^S = \theta_j^S - \theta_j^a$ *and* $\left|RE_j^S\right| = \left|\frac{\theta_j^S - \theta_j^a}{\theta_j^S}\right|$
    Record the error statistics
End-do

In the simulation experiments, a total of 2,400 test cases are used to record the statistics. The descriptive statistics of the generated parameters of (μ_{NZ}, σ_{NZ}, μ_B, σ_B, μ_I, σ_I, μ, σ, r, Q, L) and the associated w_L are given in Table A.10 and Table A.11 for 2,208 cases in Group 1 and 192 cases in Group 2, respectively. In these two tables, the target ready rate covers the range over (0, 1) by yielding high frequency coverage of the higher service levels. The tables also indicate a statistical distinction for low and high variability of the generated test cases. Test cases for which a low variability is observed (i.e. Group 1) have demand sizes ranging from 1.11 to 9.13. For the same test cases, policy parameters r and Q range from -28 to 80 and from 1 to 40, respectively. The cases in Group 2 have demand sizes ranging from 28.32 to 38112.17. The demand size variability is also very high. For the same test cases, policy parameter r ranges from -46 to 146031 while policy parameter Q ranges from 1 to 518.

For test cases in Group 1, the error results for distributions and distribution selection rules are tabulated in Table A.12 and Table A.13, respectively. The error results for test cases in Group 2 for distributions and distribution selection rules are tabulated in Table A.14 and Table A.15, respectively. The results indicate that in almost all the cases, the error values associated with B are higher than those of RR. In fact, all the LTD models produce poor results in terms of approximating B under these extremely variable demand scenarios. In addition, in most of the cases, the performance measure of B is overestimated by all the models except AXR and P. As far as test cases where $\mu \leq 1$, $\sigma^2 \leq 1$ and $0.9 \leq \sigma^2/\mu \leq 1.1$ are concerned, the performance of the Poisson distribution can be observed by comparing $PRE\,(\%)$ results in Table A.12 and Table A.18. For cases in Group 2, $PRE\,(\%)$ statistics show that the performance measures of RR and I are approximated much better as compared to B. As far as RR and I are concerned, ADR gives relatively good error results as compared to the other models, although its B estimation results are off as compared to the simulation model. As can be seen from the $PRE(\%)$ statistics in tables, the performance measure of B is estimated relatively well by ADR and $MNNB$ as compared to the other models. A key result is that the performance measure of B is overestimated. This means that a planning model that attempts to set policies by penalizing back orders either in the objective function or within constraints will set target policy parameter values higher than necessary because it will (on average) plan for more backorders than what will actually occur. Given that the results for B are so poor, it is recommended that additional LTD models or specialized inventory policy models be developed to try to improve the performance in this area.

We also present box plots of RR error results for Group 1 and Group 2 in Figure 5. These plots indicate that the more robust LTD models yield error results which condense around 0 in smaller sizes. It is clear that $MGNBA$, $MNNB$, ADR, $GADR$ and NB perform robustly for each group. In addition, error values associated with each model are inclined to be higher for Group 2 where we observe more variable demand cases.

The overall performance of the LTD models was analyzed by using the multiple comparison procedure referred to as "Tukey-Kramer HSD" (Tukey, 1953 and Kramer, 1956) found in statistical package MINITAB. The method compares the least square means for each pair of the LTD models and presents results in a categorized manner. Each category is represented by a letter in a column. Table A.16 tabulates the results of the procedure for Group 1 under 95% confidence level. The LTD models that share a letter are not significantly different. In this respect, we can sort the performance of the models in descending order as follows: $\{MGNBA\} > \{NB, ADR, GADR\} > \{AXR, LN, G, P\} > \{N\}$ and $\{MNNB\} > \{GADR\}$. As can be seen, $MGNBA, MNNB$ are the two models whose performance is significantly higher than many other LTD models. We applied the same comparison method to the models for Group 2. However, the results indicated that the performance of the models were not significantly different.

We further analyzed the overall performance of the LTD models by using another multiple comparison procedure referred to as "Hsu's multiple comparisons with the best (Hsu's MCB) (Hsu, 1981)." The difference of Hsu's MCB from Tukey-Kramer HSD is that Hsu's MCB reveals the best mean by comparing the best level and other levels while Tukey-Kramer HSD compares all possible pairwise comparisons. In case of determining minimum, the procedure tests whether means are greater than the unknown minimum. For the difference between each level mean, Hsu's MCB computes a confidence interval. A statistically significant difference can only be observed between corresponding means if an interval contains zero as an end point. The results, computed by setting the default options of statistical package MINITAB, are depicted in Exhibit A.11 and Exhibit A.12 for Group1 under 95% confidence level. Exhibit A.11 shows that *MGNBA* has the minimum mean among other LTD models meaning that it is selected as the best level to compare with other LTD models. Exhibit A.12 that shows the pairwise comparisons with respect to the performance of *MGNBA* determines whether other LTD models are significantly different. Since zero is contained in all other confidence intervals as an end point, it is safe to state that *MGNBA* is the LTD model whose performance is significantly higher than other LTD models. Exhibit A.13 and Exhibit A.14 provide the corresponding results when the same procedure is applied to Group 2. Exhibit A.14 indicates that *AXR* and *P* are not the best, and that there is no significant difference between the others for this group.

2.6 Summary and Future Work

In this paper, two types of evaluations were performed within an experimental framework. First, an analytical evaluation is carried out to present the error metrics and the results related to the use of a LTD under the assumption that the actual model is different from the one selected for use. The experimental results reveal that there is a high potential for using distribution selection rules. *MGNBA*, *MNNB*, *ADR* and *GADR* are the rules that give promising results in terms of producing small range of variability in the error metrics. It is observed that there is variability in the error metric results for the type of performance measure and generated test cases. In the case of approximating *B*, the sizes of the error metrics are often higher as compared to *RR*. Further, there is more error when approximating *B* as compared to *RR*. For cases in Group 2, excessively larger error values are observed as compared to cases in Group 1. It should be noted that the gamma and the negative binomial distributions yield fairly good results especially in Group 2 where there is much more variability. The experiments reveal not only promising models (distributions and selection rules) but also the models that produce poorest results. Two types of error results in analytical evaluation reveal that the LTD should not be approximated by the Poisson distribution unless the variance-to-mean ratio is exactly 1.

The second type of evaluation follows the simulation evaluation to reveal the effectiveness of the LTD models. The simulation evaluation results indicate that all the models except *AXR* and *P* overestimate the performance measure of *B*. One challenging issue that we experienced during the simulation evaluation is the computational time of the simulation runs for the cases in Group 2. Multiple computers (10 computers at a time) were dedicated for the runs in order to have the statistical results in a reasonable time period. Some key conclusions can be drawn from the experiments. Distribution selection rules are promising for LTD modeling. In the simulation evaluation part of the experiments, the potential use of the distribution selection rules is justified as *ADR* yields relatively good results while the gamma and negative binomial distributions have excellent potential in the case of high demand variability. The conclusion based on the (r, Q) analysis and the simulation provide insights on the use of the LTD models, which may also provide insight into the case of these distributions for other policies (e.g. (r, NQ), (s, S)). In this study, only the first two moments of LTD model are considered for parameter fitting. There have been studies within the literature that consider more than two moments in order to utilize more information associated with the LTD although no comprehensive work has been done to draw full conclusions of which model is more appropriate in LTD modeling. Therefore, one direction for further research is to consider a similar study under more moments for a better parameter fitting among distributions including other inventory policies (e.g. (r, NQ)), which will be the topic of forthcoming research efforts. Developing a different more robust distribution selection rule can be regarded as another research direction because of its potential use in LTD modeling.

3 EVALUATING THE USE OF HIGHER-ORDER MOMENTS FOR LEAD TIME DEMAND MODELS

3.1 Introduction

This study extends the ideas in (Rossetti and Ünlü, 2011) by incorporating distributions that include additional moments into LTD modeling. In the previous paper, the parameter fitting procedure utilized the first two moments. In this paper, moment matching procedures are developed on higher moments for a number of distributions.

This paper examines the use of classical continuous review (r, Q) inventory system under a number of lead-time demand models that have more general distributional forms within the context of four demand classes. These classes are named "Group1", "Group2", "Erratic" and "Smooth." The demand classes are determined based on the demand classification scheme proposed by (Boylan et al., 2008) and the demand variability observed in an industrial data set. The demand classes will be discussed in Section 3.4.2. The lead time demand models considered in this paper require moment matching procedures on higher moments. These are, namely, the distributions of phase-type (PT), four-parameter Pearson (PD), four-parameter Schmeiser-Deutsh (SD), generalized lambda-type (GL), and two Poisson model (TPM). All these LTD models were defined in the related literature. Other LTD models considered in this paper preserve special parameter structures such as zero-modified distributions (ZM) . The term "zero-modified" embraces many distributions. In this paper, the following distributions will be considered: zero-modified Poisson (ZMP), zero-modified negative binomial (ZMNB), zero-modified geometric (ZMG) and zero-modified binomial (ZMB) distributions. These distributions are also studied in the literature although they have not been traditionally used within inventory modeling. The motivation behind considering such distributions is because these distributions are capable of explicitly modeling zero and nonzero demands. Parameter derivations are necessary in order for zero-modified distributions to be employed by classical inventory policies as a lead time demand model. This paper also introduces distribution selection rules that utilize zero-modified distributions. The following distribution selection rules are considered in this study: zero-modified Adan's rules: ZMADR, ZMADR2, ZMADR2ADR and ZMADR2PT. These rules are developed based on the initial empirical results. The common name "LTD model" will be used to refer both a distribution and distribution selection rule.

No comprehensive study has been presented in the literature to show the effectiveness of the LTD models considered in this paper for modeling the demand during lead-time for the classical inventory control models in the face of different demand classes. Thus, the objective of this paper is to cover this gap by evaluating the LTD models that have more flexible distributional forms. The LTD models considered in this paper will be discussed in Section 3.3 in detail. The next section will give some background about the models that have more general distributional forms.

3.2 Literature Review

There is a sparse body of literature in regard to distribution based LTD modeling. Rossetti and Ünlü (2011) classify the literature into two groups: 1) LTD modeling predicated on one or two moments 2) LTD modeling predicated on higher moments and/or more flexible forms. This paper presents a review of the models that fall into the latter group.

A definition of phase-type distributions is provided by (Neuts, 1981). He defines a nonnegative random variable X that is distributed as phase-type if and only if X is characterized as the time until absorbed by some finite state continuous time Markov chain. The underlying Markov process consists of n transient states and one absorbing state. Thus, the total states defined in the Markov process is $n+1$. A phase-type distribution is characterized by a triple parameter set of (n, T, α). A phase-type distribution can be represented in many forms with respect to different forms of the matrix representations of T and α. However, in practice, the majority of the values in these matrices are taken as zero. For example the following shows the matrix representation of hypoexponential distribution (i.e. generalized Erlang):

$$\alpha = [1, 0, 0, 0] \text{ and } T = \begin{bmatrix} -\mu_1 & \mu_1 & 0 & 0 \\ 0 & -\mu_2 & \mu_2 & 0 \\ 0 & 0 & -\mu_3 & \mu_3 \\ 0 & 0 & 0 & -\mu_4 \end{bmatrix}$$

Since closed form expressions are available in terms of the Markov chain parameters, the phase-type distributions are applicable for various well-known problem areas in the literature. The phase-type distributions were investigated to a great extent by (Neuts, 1981) in the context of queuing and the associated stochastic problems. However, inventory modeling applications of these distributions are limited. (Ravichandran, 1984) uses the phases-type distribution in order to approximate the lead time which is defined as a random variable. In his work, the calculation of the inventory level distribution is shown for the case where the demand follows a Poisson process. He uses a continuous review (s, S) inventory policy. Later, (Federgruen and Zipkin, 1985) present an iterative algorithm to compute an optimal (s, S) policy for an inventory model with continuous demand. Since their approach is regarded as Markov decision process, the calculations can be carried out with closed form phase-type distributions. A remarkable work was done by (Zipkin, 1988) in terms of investigating the potential use of phase-type distributions for inventory models.

His work will be discussed in detail. His main contribution is the expressions in closed form that enable calculations of the performance measures. He applies the phase-type distribution in order to model demand and lead-time processes in the context of the (r, Q) inventory control policy. His approach is relatively simple as compared to previous phase-type distribution applications in inventory control. The demand is assumed to be processed in a discretized manner (i.e. demand occurs one at a time). In his paper, the phase-type distribution is utilized to characterize the time between two consecutive demands. Lead-times are also assumed to follow a phase-type distribution (i.e. stochastic lead times). Further assumptions regarding inventory system include that the unmatched demand is backordered while inventory position is uniformly distribution over the range [r+1, q]. These assumptions are still often applied today. Based on these assumptions, the lead-time demand distribution is derived. The resulting marginal distribution is a discrete phase-type distribution with the same number of states as the lead time distribution. This, actually, makes sense under the assumption that lead-times and demand processes are independent. For example, think about lead times as exponential random variables (i.e. phase-type distribution with one state). In this case, the resulting marginal distribution of lead-time demand will also have one state. The resulting lead time demand distribution can be used with inventory policy parameters (i.e. r and Q) for the calculations of performance measures (e.g. probability of no stockout, the expected number of backorders). Unfortunately, a parameter fitting methodology was not provided in his work. However, he states that the modeler has to specify the structure and parameters in order to make use of the distribution.

The assumption of lead-times and demand processes being independent was relaxed in (Boute et al., 2007), however, for a different inventory problem. They tackle the problem of periodic review base-stock policy where the replenishment orders are placed in a capacitated production facility. Therefore, a delay occurs due to this capacity because the production facility is regarded as a single queuing system in which the replenishment orders have to be completed sequentially. Thus, the inventory behavior is affected by the correlation between demand and lead times. They, similar to (Zipkin, 1988), make use of the phase-type distribution to model both demand and lead time processes. However, they choose to directly compute (derive) the probability distribution of the inventory levels. Clearly, the derived distribution allows an inventory manager to calculate performance measures such as fill rate, base-stock levels and optimal safety stock levels. Their further contribution is an algorithm to fit the parameters of the phase-type distribution by employing a method of moment matching procedure. By means of this approach, the number of states in the phase-type distribution can be decreased and accordingly the computational complexity can be reduced to some extent.

Notice that the state of the art related to parameter fitting procedure was still under development when (Zipkin, 1988) proposed his work. Therefore, the parameter fitting approach proposed later can also be utilized in Zipkin's initial work. On the other hand, many parameter fitting schemes have been proposed in the literature for the phase-type distributions. Basically, two types of fitting methods attributed to the phase-type distributions: (1) Maximum likelihood estimators (MLE) and (2) Method of moment matching (MOM) techniques. A special algorithm called "Expectation Maximization" (EM) algorithm is applied to compute MLE. A discussion can be found in (Asmussen et al., 1996) for incorporating the EM algorithm into fitting parameters of the phase-type algorithm. The MLE is often the most appreciated parameter estimation method in the literature in terms of accuracy that an estimator can supply. However, it could be too taxing for a phase-type distribution preserving large structural features. On the other hand, MOM is considered to be more computationally efficient; however, its applicability is often limited with respect to the underlying phase-type distribution (Johnson and Taaffee, 1989). A procedure is described in their study to match the first three moments of any distribution in order to represent it by a phase type distribution. The literature also provides other procedures for MOM in the context of the phase-type distributions. For example, (Bobbio et al., 2011) present an algorithm to match a set of sample estimations of the first three moments with acyclic phase-type distributions (APH). They show the possible sets that can be represented by an acyclic distribution of order n. Then they show how to match the first three moments in a minimal way, i.e. using the minimal number of phases needed to do it. (Pérez and Riaño, 2006) implement the algorithm in their object-oriented tool. One gets the transition matrix (T) and the vector (α) by supplying only the first three moments of the LTD.

A zero-modified distribution is known as a special type of mixture distribution. The distribution can either be continuous or discrete. Various mixtures of distributions have been studied in the literature using different original distributions. The most well known ones are zero-modified Poisson and zero-modified binomial distributions. Both distributions are discussed in (Johnson et al., 2005) with their parameter estimation methods. Zero-inflated distributions are proposed in the case where there are many zeros in the count data which is over than expected (e.g. Cameron and Trivedi, 1998) and if the count data include zeros whose number is less than the expected, then zero-deflated distributions are used. An example regarding two situations for the Poisson case can be seen in (Dietz and Böhning, 1995).

Zero-modified distributions were proposed in the literature due to the need to better characterize count data with excess number of zeros collected from economic series, agricultural surveys, manufacturing processes etc. Zero-modified distributions were proposed in the literature due to the need to better characterize this type of data. Another name used for such distributions is known as zero-inflated distribution. (Heilbron, 1994) provides methods constructing these distributions in the context of generalized linear models. (Böhning, 1998) presents a literature review for these distributions by providing a variety of examples from different disciplines. A number of examples for different application areas can be obtained in (Terza and Wilson, 1990; Lambert, 1992; Zorn, 1996; Yau et al., 2003; Lord et al., 2005). Such distributions may be especially suitable for intermittent demand due to their capability for explicitly modeling non-zero and zero demand cases. (Ünlü and Rossetti, 2011) examine the use of standard (r, Q) inventory control policies under a number of zero-modified distributions, namely, zero-modified lognormal, zero-modified Poisson, zero-modified negative binomial. However, their results are predicated on the probability of period demand parameter which is corrected in this paper as the probability of LTD being zero.

The generalized lambda distribution is represented with four parameters which allows flexibility in terms of taking wide variety of curve shapes. The generalized lambda distribution is extensively discussed in (Ramberg and Schmeiser, 1974). The authors provide a method of moment matching which fits the distribution's four parameters to the given sample estimates of the first four central moments. The method moment matching technique relies on optimizing the associated constrained non-linear model. In addition, as an alternative method for parameter estimation, the authors provide tables which include the estimated parameter values when kurtosis and skewness estimations are given. (Lakhany and Mausser, 2000) provide a review of the available parameter fitting procedures for the generalized lambda distribution. Based on two different parametrizations, the authors evaluate the following methods: the method of moment matching, least squares, starship and the method proposed by (Ozturk and Dale, 1985). Their evaluation is predicated on an adjusted Kolmogorov-Smirnov test. The method of moment matching technique is one of the competitive approaches in their study.

p^{th} quantile and density functions of the generalized lambda distribution are given in Kumaran and Achary (Kumaran and Achary, 1996). For the generalized lambda-type distribution a closed-form of cdf is not available. (Achary and Geetha, 2007) derive a formula for the k^{th} partial expectation for the generalized lambda-type distribution which can be used to derive the loss functions in order to approximate desired inventory performance measures.

The four-parameter Schmeiser-Deutsh distribution is proposed by (Ramberg et al., 1979). The method of moment matching is very similar to the procedure for the generalized lambda distribution. The distribution's four parameters can be fitted to the given sample estimations of the first four central moments by using the method of moment matching. The parameter fitting procedure is dependent on the solution of a constrained nonlinear optimization problem. The authors also provide tables that facilitate the parameter fitting procedure based on the given kurtosis and skewness estimations. The four-parameter Schmeiser-Deutsh distribution is used for inventory modeling in (Kottas and Lau, 1979) and (Kottas and Lau, 1980). The latter study delivers the derived first order loss function of the distribution as well as the some inventory calculations such as stock-out probability.

The two-Poisson model, utilized in (Bookstein and Swanson, 1974) and (Harter, 1975), is a mixture distribution of two Poisson distributions. The method of moment matching is used in (Harter, 1975) to fit the distribution's three parameters to the sample estimations of the first three moments around zero. The two-Poisson models are also used in (Church and Gale, 1995) where the authors show that Poisson mixtures fit the data better than standard Poissons.

3.3 Lead Time Demand Models

For some of the LTD models, moment matching procedure can be carried out based on higher moments. Incorporating such distributions that preserve higher moment information into an inventory control policy (e.g. (r, Q)) to determine the desired performance measures is a challenging task. One difficulty in applying these distributions is estimating the parameters from the data. As previously noted, in most cases only the demand per period is available (not actual observations of the lead-time demand). Thus, the demand per period moment data must be combined with the knowledge of the lead-times to represent the moments of the lead-time demand.

We are interested in the moments of the LTD which is a random variable X. Let $\left[\mu_x^k\right] = E\left[X^k\right]$ be the kth raw moment of random variable X. In addition, let μ_D, μ_D^2 and μ_D^3 be the first, second and third raw moments of the demand. (Grubbström and Tang, 2006) provide the moments of X (μ_X, μ_X^2, μ_X^3 and μ_X^4) as follows.

$$E\left[X\right] = \mu_X = \mu_L \mu_D \tag{12}$$

$$E\left[X^2\right] = \mu_X^2 = \mu_L \mu_D^2 + \left(\mu_L^2 - \mu_L\right)\left(\mu_D\right)^2 \tag{13}$$

$$E\left[X^3\right] = \mu_X^3 = \mu_L \mu_D^3 + 3\left(\mu_L^2 - \mu_L\right)\mu_Y \mu_D^2 + \left(\mu_L^3 - 3\mu_L^2 + 2\mu_L\right)\left(\mu_D\right)^3 \tag{14}$$

where μ_L, μ_L^2 and μ_L^3 are the first, second and third raw moments of L, respectively. We assume that μ_L is fixed and equals to L, which is a constant. Then,

First Raw Moment:

$$E\left[X\right] = \mu_X = L\mu_D \tag{15}$$

Second Raw Moment:

$$E\left[X^2\right] = \mu_X^2 = L\mu_D^2 + \left(L^2 - L\right)\left(\mu_D\right)^2 \tag{16}$$

Third Raw Moment:

$$E\left[X^3\right] = \mu_X^3 = L\mu_D^3 + 3\left(L^2 - L\right)\mu_D\mu_D^2 + \left(L^3 - 3L^2 + 2L\right)\left(\mu_D\right)^3 \tag{17}$$

Let $\mu_X^{'2}$, $\mu_X^{'3}$ and $\mu_X^{'4}$ be the second, third and fourth central moments of X, respectively. Let $\mu_D^{'2}$, $\mu_D^{'3}$ and $\mu_D^{'4}$ be the second, third and fourth central moments of demand, respectively. In addition, let $\mu_L^{'2}$, $\mu_L^{'3}$ and $\mu_L^{'4}$ be the second, third and fourth central moments of L, respectively. (Grubbström and Tang, 2006) derive the central moments of X as follows.

$$\mu_X^{'2} = \mu_L\mu'^{2}_{D} + \mu_L^{'2}\left(\mu_D\right)^2 \tag{18}$$

$$\mu_X^{'3} = \mu_L\mu_D^{'3} + 3\mu_L^{'2}\mu_D\mu_D^{'2} + \mu_L^{'3}\left(\mu_D\right)^3 \tag{19}$$

$$\begin{aligned} \mu_X^{'4} \quad = \quad & \mu_L\mu_D^{'4} + 4\mu_L^{'2}\mu_D\mu_L^{'3} + 3\left(\mu_L^{'2} + \mu_L\left(\mu_L - 1\right)\right)\left(\mu_D^{'2}\right)^2 \\ & +6\left(\mu_L^{'3} + \mu_L\mu_L^{'2}\right)\left(\mu_D\right)^2\mu_D^{'2} + \mu_L^{'4}\left(\mu_L\right)^4 \end{aligned} \tag{20}$$

In our analysis, we assume that L is fixed known quantity. Thus, $\mu_L^{'2} = 0$, $\mu_L^{'3} = 0$ and $\mu_L^{'4} = 0$. In this case, the central moments of X are expressed as follows:

Second Central Moment:

$$\mu_X^{'2} = L\mu'^{2}_{D} \tag{21}$$

Third Central Moment:

$$\mu_X^{'3} = L\mu_D^{'3} \tag{22}$$

Fourth Central Moment:

$$\mu_X^{'4} \quad = \quad L\mu_D^{'4} + 3\left(L\left(L - 1\right)\right)\left(\mu_D^{'2}\right)^2 \tag{23}$$

Denote α_3^* and α_4^* the estimated kurtosis and skewness statistics based on the following expressions:

$$\alpha_3^* = \frac{\mu_X^{\prime 3}}{\left(\mu_X^{\prime 2}\right)^{1.5}} \tag{24}$$

$$\alpha_4^* = \frac{\mu_X^{\prime 4}}{\left(\mu_X^{\prime 2}\right)^{2}} \tag{25}$$

The other difficulty lies in deriving closed form of expressions similar to the case of the first and the second loss functions utilized in the case of first two-moment matching procedure. A numerical-type-algorithm might be the procedure to determine the required performance measures. In this section, these challenging issues are addressed on the LTD models while taking into account the approaches already presented in the literature.

3.3.1 Phase-Type Distributions (*PT*)

(Zipkin, 1988) introduces the use of the phase-type distributions in modeling the lead time demand. The loss functions of the phase-type distribution derived by (Zipkin, 1988) will be used to approximate the inventory performance measures.

Loss Functions:

The following notation will be used to derive the first and second order loss functions of the phase-type distributions.

PT: used to indicate that the expression is derived for the phase-type distributions

LTD: lead time demand

G_{PT}^1: first order loss function

G_{PT}^2: second order loss function

(n, T_{LTD}, γ): phase-type parameter set for the random variable of lead-time demand

Q: replenishment order quantity

e: column vector of 1's

μ_{LTD}: mean of lead-time demand

σ^2_{LTD}: variance of lead-time demand

I: nxn identity matrix

(Zipkin, 1988) successfully computed the first and second order loss functions of the lead-time demand distribution as follows:

$$G^1_{PT}(x) = \gamma(I - T_{LTD})^{-1} T^x_{LTD} e \quad \text{for } x \geq 0 \tag{26}$$

$$G^2_{PT}(x) = \gamma(I - T_{LTD})^{-2} T_{LTD} T^x_{LTD} e \quad \text{for } x \geq 0 \tag{27}$$

(Zipkin, 1988) extended his work by further apply the phase-type distribution for the inventory policies of (s, S) and (r, NQ). Parameter fitting to a phase-type distribution is not provided in (Zipkin, 1988) work. In what follows, the parameter fitting procedure will be introduced for the phase-type distribution.

Parameter Fitting Procedure:

The parameter fitting procedure for the phase-type distribution follows two different moment matching procedures. The first procedure relies on the first three factorial moments of the lead time demand.

1) Using Higher Moments:

The first three factorial moments (f_1, f_2 and f_3) of X are expressed as follows:

$$f_1 = E[X] = \mu_X = L\mu_D \tag{28}$$

$$
\begin{aligned}
f_2 = E\left[X\left(X-1\right)\right] &= E\left[X^2\right] - E\left[X\right] \\
&= \mu_X^2 - \mu_X \\
&= L\mu_D^2 + \left(L^2 - L\right)\left(\mu_D\right)^2 - L\mu_D
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
f_3 = E\left[X\left(X-1\right)\left(X-2\right)\right] &= E\left[X^3\right] - 3E\left[X^2\right] + 2E\left[X\right] \\
&= \mu_X^3 - 3\mu_X^2 + 2\mu_X \\
&= L\mu_D^3 + 3\left(L^2 - L\right)\mu_D\mu_D^2 + \left(L^3 - 3L^2 + 2L\right)\left(\mu_D\right)^3 \\
&\quad -3\left[L\mu_D^2 + \left(L^2 - L\right)\left(\mu_D\right)^2\right] + 2L\mu_D
\end{aligned} \tag{30}
$$

(Telek and Heindl, 2002) use the above derived first three factorial moments in order to match the moments for acyclic discrete phase-type distributions of second order. They define the transition matrix (T) and the vector (α) for the phase type distribution of second order as follows:

$$
T = \begin{bmatrix} 1-\beta_1 & \beta_2 \\ 0 & 1-\beta_1 \end{bmatrix} \tag{31}
$$

$$
\alpha = \left[p, 1-p\right] \tag{32}
$$

First they define auxiliary variables (a, b, c and d) based on the factorial moments:

$$
d = 2f_1^2 - 2f_1 - f_2 \tag{33}
$$

$$
c = 3f_2^2 - 2f_1f_3 \tag{34}
$$

$$
b = 3f_1f_2 - 6\left(f_1 + f_2 - f_1^2\right) - f_3 \tag{35}
$$

$$
a = b^2 - 6cd \tag{36}
$$

Then show that parameters can be fitted as follows.

If $c > 0$ then

$$p = \frac{-b + 6f_1 d + \sqrt{a}}{b + \sqrt{a}} \tag{37}$$

$$\beta_1 = \frac{b - \sqrt{a}}{c} \tag{38}$$

$$\beta_2 = \frac{b + \sqrt{a}}{c} \tag{39}$$

if $c < 0$ then

$$p = \frac{b - 6f_1 d + \sqrt{a}}{-b + \sqrt{a}} \tag{40}$$

$$\beta_1 = \frac{b + \sqrt{a}}{c} \tag{41}$$

$$\beta_2 = \frac{b - \sqrt{a}}{c} \tag{42}$$

and if $c = 0$ then

$$p = 0, \beta_2 = \frac{1}{f_1}$$

Notice that if $c = 0$, then it is redundant to fit β_1.

(Telek and Heindl, 2002)also list a number of conditions where the moment matching is feasible. If any of the indicated conditions are not met then the parameter fitting is infeasible. In case of infeasibility, we follow the parameter fitting procedure provided by (Boute et al., 2007) whose approach relies on the mean and variance of X (i.e. two-moment matching).

2) Using Two Moments:

The calculation of the number of phases (n) needed to match μ_X and σ_X^2 is given by (Telek and

Heindl, 2002) as follows:

$$n = \max\left(2, \left\lceil \frac{\mu_X}{\mu_X CV_X^2 + 1} \right\rceil\right)$$

where $CV_X^2 = \frac{\sigma_X^2}{\mu_X^2}$. The authors choose α and T as

$$\alpha = \left(\beta^1, \left(1-\beta^1\right), 0, 0, .., 0\right)$$

$$T = \begin{bmatrix} 1-p_1 & p_1 & 0 & 0 & ... & 0 \\ 0 & 1-p_2 & p_2 & 0 & ... & 0 \\ 0 & 0 & 1-p_2 & p_2 & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & p_2 \\ 0 & 0 & 0 & 0 & ... & 1-p_2 \end{bmatrix}$$

This leaves 3 additional parameters: β^1, p^1 and p^2 along with two equations: $\mu_X = E[X]$ and $\sigma_X^2 = Var[X]$. The parameter of p_1 and p_2 are computed by $p_1 = \frac{\beta^1}{\mu_D}$ and $p_2 = \frac{n}{\mu_D}$ where $0 \leq p_1 \leq \beta^1$ and $0 \leq p_2 \leq 1$ while $\mu_X \geq n$. Also,

$$\beta^1 = \frac{2\mu_X}{2\mu_X + n\left[(n-\mu_X) + nCV_X^2\mu_X\right]}$$

where $0 \leq \beta^1 \leq 1$.

3.3.2 Four-Parameter Pearson Distribution (*PD*)

The four-Parameter Pearson distribution does not possess an explicit expression. As discussed in (Johnson et al., 2005) a 4 parameter Pearson distribution can be approximated by a standard normal cdf provided by (Barndorff-Nielsen, 1990) (Assuming distribution forms are placed in Type IV region). However, the applicability of this approximation has to be investigated in terms of representing lead-time demand modeling. According to (Kottas and Lau, 1980), 4 parameters of a distributional form are given as a, b, p and q.

(Kottas and Lau, 1980) provides the first order loss function of the 4-parameter Pearson Type-I distribution (i.e. beta) as follows:

$$r = \frac{x-a}{b-a}$$

$$G_{PD}^{1}(x) = \frac{(b-a)\,p}{p+q}\left[1 - I_r(p+1,q)\right] - (x-a)\left[1 - I_r(p,q)\right] \tag{43}$$

The second order loss function can be derived based on a similar method as follows:

$$G_{PD}^{2}(R) = \frac{1}{2}\left[\begin{array}{c} \frac{(b-a)^2 p(p+1)}{(p+q)(p+q+1)} I_r(p+2,q) + 2(b-a)\,a\left(\frac{p}{p+q}\right) I_r(p+1,q) + a^2\left[1 - I_r(p,q)\right] \\ -2R\left[(b-a)\left[\left(\frac{p}{p+q}\right) I_r(p+1,q)\right] + a\left[1 - I_r(p,q)\right]\right] \\ +R^2\left[1 - I_r(p,q)\right] \end{array}\right] \tag{44}$$

The steps to derive (44) are given in Appendix B.

Parameter Fitting Procedure:

Parameter fitting process for 4-parameter Pearson Type-I distribution is described in (Gudum and de Kok, 2002) as follows. The parameters of a and b are defined as the minimum and maximum observed lead time demand values, respectively. Then the parameters of p and q are given by

$$p = \frac{\mu_s^2\,(1-\mu_s)}{\sigma_s^2} - \mu_s$$

$$q = \frac{p\,(1-\mu_s)}{\mu_s}$$

where $\mu_s = \dfrac{\mu - a}{b-a}$ and $\sigma_s = \dfrac{\sigma}{b-a}$.

3.3.3 Four-Parameter Schmeiser-Deutsh Distribution ($S-D$)

(Kottas and Lau, 1980) states that 4-parameter Schmeiser-Deutsh (S-D) distributions have the following closed-form expressions:

$$F(x) = \left\{ \begin{array}{ll} d - [(a-x)/b]^m & if \quad B_1 \le x \le a \\ d + [(x-a)/b]^m & if \quad a \le x \le B_2 \end{array} \right\}$$

where the parameters of the distribution are given as a, b, c, and d. Also, $m = \frac{1}{c}$.

The distribution's lower limit is given by:

$$B_1 = a - bd^c$$

The distribution's upper limit is given by:

$$B_2 = a + b(1-d)^c$$

Also, the inverse of cdf function is represented by

$$x = F^{-1}(F(x)) = \left\{ \begin{array}{ll} a - b[d - F(x)]^c & if \quad F(x) \le d \\ a + b[F(x) - d]^c & if \quad F(x) > d \end{array} \right\}$$

Loss Functions:

Let G^1_{SD} and G^2_{SD} be the first and second order loss functions of the S-D distribution, respectively. (Kottas and Lau, 1980) give the first order loss function of S-D distribution as follows:

For $x < a$,

$$G^1_{SD}(x) = (a-x)\left[1 - \left(d - \left[\frac{a-x}{b}\right]^{1/c}\right)\right] + \frac{b}{c+1}\left[(1-d)^{c+1} - \left(\frac{a-x}{b}\right)^{\frac{c+1}{c}}\right] \qquad (45)$$

and for $x \geq a$,

$$G_{SD}^{1}(x) = (a-x)\left[1-\left(d-\left[\frac{x-a}{b}\right]^{1/c}\right)\right]+\frac{b}{c+1}\left[(1-d)^{c+1}-\left(\frac{x-a}{b}\right)^{\frac{c+1}{c}}\right] \qquad (46)$$

One can derive the second order loss function of S-D distribution as follows:

For $x < a$,

$$\begin{aligned} G_{SD}^{2}(x) &= 0.5\left\{a(a-2x)\left[1-\left(d-\left[\frac{a-x}{b}\right]^{1/c}\right)\right]+\frac{2ab}{c+1}\left[\left(\frac{a-x}{b}\right)^{\frac{c+1}{c}}\right]\right. \\ &\quad +\frac{b^2}{2c+1}\left(\frac{a-x}{b}\right)^{\frac{2c+1}{c}}+\frac{b}{c+1}\left[(1-d)^{c+1}-\left(\frac{a-x}{b}\right)^{\frac{c+1}{c}}\right] \\ &\quad \left. +x\left[1-\left(d-\left[\frac{x-a}{b}\right]^{1/c}\right)\right]\right\} \end{aligned} \qquad (47)$$

and for $x \geq a$,

$$\begin{aligned} G_{SD}^{2}(x) &= 0.5\left\{a(a-2x)\left[1-\left(d+\left[\frac{x-a}{b}\right]^{1/c}\right)\right]+\frac{2ab}{c+1}\left[\left(\frac{x-a}{b}\right)^{\frac{c+1}{c}}\right]\right. \\ &\quad +\frac{b^2}{2c+1}\left(\frac{x-a}{b}\right)^{\frac{2c+1}{c}}+\frac{b}{c+1}\left[(1-d)^{c+1}-\left(\frac{x-a}{b}\right)^{\frac{c+1}{c}}\right] \\ &\quad \left. +x\left[1-\left(d+\left[\frac{a-x}{b}\right]^{1/c}\right)\right]\right\} \end{aligned} \qquad (48)$$

Appendix B provides the derivation of (47) and (48).

Parameter Fitting Procedure:

(Schmeiser and Deutsch, 1977) describe the following procedure to determine the parameters a, b, c and d. First determine the parameters c and d by solving the following non-linear optimization model (P1).

$$\min\,(\alpha_3-\alpha_3^*)^2+(\alpha_4-\alpha_4^*)^2$$

$$s.t$$

$$0\leq d\leq 1$$

where α_3^* and α_4^* are the estimated kurtosis and skewness statistics which are collected during simulation using based on the following expressions:

$$\alpha_3^*=\frac{\mu_X^{\prime 3}}{\left(\mu_X^{\prime 2}\right)^{1.5}}$$

$$\alpha_4^*=\frac{\mu_X^{\prime 4}}{\left(\mu_X^{\prime 2}\right)^{2}}$$

where $\mu_X^{\prime 2}$, $\mu_X^{\prime 3}$ and $\mu_X^{\prime 4}$ are determined by (21), (22) and (23), respectively. In addition, α_3 and α_4 are determined by

$$\alpha_3=\frac{\mu_3}{\mu_2^{1.5}}$$

$$\alpha_4=\frac{\mu_4}{\mu_2^{2}}$$

where $\mu_3=E\left[X^3\right]-3E\left[X^2\right]E\left[X\right]+2\left(E\left[X\right]\right)^3$ and $\mu_4=E\left[X^4\right]-4E\left[X^3\right]E\left[X\right]+6E\left[X^2\right]\left(E\left[X\right]\right)^2-3\left(E\left[X\right]\right)^4$. Also, kth raw moment is determined as follows:

$$E\left[X^k\right]=\frac{\left[(-1)^k\right]d^{kc+1}+(1-d)^{kc+1}}{b^{-k}\left(kc+1\right)}$$

Notice that b^{-k} is canceled out during the calculation of α_3 and α_4. Next, a and b are determined as follows:

$$p = c + 1 \text{ and } q = 2c + 1$$

$$b = \sqrt{\frac{\sigma^2 p^2 q}{p^2 \left(d^p + (1-d)^q\right) - q\left((1-d)^p - d^p\right)^2}}$$

$$a = \mu - \frac{b\left((1-d)^p - d^p\right)}{p}$$

Notice that the constrained non-linear equation is based on the parameters c and d.

3.3.4 Generalized Lambda-Type Distribution (GL)

p^{th}quantile and density functions are given in (Kumaran and Achary, 1996). For the generalized lambda-type distribution a closed-form cdf is not available. Achary and Geetha (Achary and Geetha, 2007) derive the following formula for the k^{th} partial expectation for the generalized lambda-type distribution.

$$E\left[\left([X-x]^+\right)^k\right] = \int_r^\infty (x-r)^k f(x)\,dx$$

$$E\left[\left([X-x]^+\right)^k\right] = \sum_{j=0}^{k} (k,j)\,(\lambda_1 - r)^{k-j}\,\frac{1}{\lambda_2^j}\sum_{i=0}^{k} (j,i)\,(-1)^i\, I_R\left(1+\lambda_3\,(j-1)\,,1+\lambda_4 i\right) \tag{49}$$

where

$$I_R(m,n) = \int_{t \le R} t^{m-1}\,(1-t)^{n-1}\,dt \quad \text{for } 0 < R < 1 \tag{50}$$

Numerical procedures are available for (50). One can use (49) to derive the first and second order loss functions. Let G^1_{GL} and G^2_{GL} be the first and second order loss functions of the generalized lambda-type distribution. Then the first and second order loss functions of the generalized lambda-type distribution are derived by using (49) and given as follows:

$$G_{GL}^{1}(r) = (\lambda_1 - r)(1 - p_r) + \frac{1}{\lambda_2}\left(\frac{p_r^{\lambda_3+1}}{\lambda_3+1} - \frac{(1-p_r)^{\lambda_4+1}}{\lambda_4+1}\right) \tag{51}$$

$$\begin{aligned} G_{GL}^{2}(r) &= 0.5\left\{(\lambda_1 - r)^2(1-p_r) + 2\frac{\lambda_1 - r}{\lambda_2}\left[\frac{1-p_r^{\lambda_3+1}}{\lambda_3+1} - \frac{(1-p_r)^{\lambda_4+1}}{\lambda_4+1}\right]\right. \\ &\left. -\frac{1}{\lambda_2^2}\left[\frac{1-p_r^{2\lambda_3+1}}{2\lambda_3+1} + \frac{(1-p_r)^{2\lambda_4+1}}{2\lambda_4+1} - 2I_{(1-p_r)}(1+\lambda_3, 1+\lambda_4)\right]\right\} \end{aligned} \tag{52}$$

where $p_r = P(X \leq r)$ and $r = \lambda_1 + \dfrac{p_r^{\lambda_3} - (1-p_r)^{\lambda_4}}{\lambda_2}$ $0 \leq p_r \leq 1$.

Parameter Fitting Procedure:

Three algorithms (the moment matching, least squares and starship etc.) are described in (Lakhany and Mausser, 2000). In order to be consistent with the outline of the paper, the moment matching method is used for the parameter fitting procedure. The parameters of λ_3 and λ_4 are determined by solving the following non-linear optimization model (P1).

$$\min\left(G_3(\lambda_3,\lambda_4) - \alpha_3^*\right)^2 + \left(G_4(\lambda_3,\lambda_4) - \alpha_4^*\right)^2$$

$$s.t$$

$$\lambda_3, \lambda_4 \geq -0.25$$

where α_3^* and α_4^* are the estimated on the expressions (24) and (25), respectively. In addition, $G_4(\lambda_3,\lambda_4)$ and $G_3(\lambda_3,\lambda_4)$ are determined by the expressions given in (Ramberg and Schmeiser, 1974). The authors also derive expressions for λ_1 and λ_2 based on λ_3 and λ_4.

3.3.5 Two Poisson Model (TPM)

The two Poisson model was presented (Church and Gale, 1995) as follows.

$$P(x) = \alpha P_1(x, \theta_1) + (1-\alpha) P_2(x, \theta_2)$$

where

$$\alpha = \frac{\mu_X - \theta_1}{\theta_1 - \theta_2}$$

<u>*Loss Functions*</u>:

The two Poisson model is a mixture distribution whose loss functions of the mixing distributions (i.e. Poisson) are available. Thus, (58) and (59) can be used to compute the loss functions.

<u>*Parameter Fitting Procedure*</u>:

θ_1 and θ_2 are the roots of the quadratic equation: $a\theta^2 + b\theta + c = 0$, where:

$$a = (\mu_X)^2 + \mu_X - \mu_X^2$$

$$b = (\mu_X)^2 - \mu_X \mu_X^2 + 2\mu_X - 3\mu_X^2 + \mu_X^3$$

$$c = \left(\mu_X^2\right)^2 - (\mu_X)^2 + \mu_X \mu_X^2 - \mu_X \mu_X^3$$

where μ_X^2 and μ_X^3 are obtained by using (16) and (17), respectively.

3.3.6 Zero-Modified Distributions (*ZM*)

Zero-modified distributions were originally derived due to the need to better characterize data sets that have a larger than expected number of zeroes, when considering a standard distribution (e.g. Poisson). The analysis utilizes the degenerate distribution with all probability concentrated at the origin (zero point in the axis). Let $P_j = P(X = j)$ be the probability of j for the unmodified distribution where $j = 0, 1, 2, \ldots$ and, the random variable Y is defined by the following finite-mixture distribution.

$$\Pr(Y = j) = \begin{cases} w + (1 - w) P_0 & if \quad j = 0 \\ (1 - w) P_j & if \quad j \geq 1 \end{cases} \tag{53}$$

The mixture distribution (53) is referred to as a zero-modified distribution or as a distribution with added zeros. Notice that parameter w is easily computed by the expression $w = \frac{f_0 - P_0}{1 - P_0}$ where $P_0 = \Pr(X = 0)$ and f_0 is the probability of observing zero LTD in an experimental investigation. Let N be total number of data points in a data series. We estimate it by $\hat{f}_0 = \hat{p}_0^L$ where $\hat{p}_0 = \dfrac{zero\,count}{N}$, the probability of zero demand under the assumption that consecutive demand values are not correlated with each other.

Loss Functions:

Proposition: Let G^1_{ZM} and G^2_{ZM} be the first and second order loss functions of a zero-modified distribution, respectively. In addition, let G^1_U and G^2_U be the first and second order loss functions of the unmodified probability distribution, respectively. Then, the following expressions hold for G^1_{ZM} and G^2_{ZM}:

$$G^1_{ZM} = w + (1 - w) G^1_U \tag{54}$$

$$G^2_{ZM} = w + (1 - w) G^2_U \tag{55}$$

where w is the previously defined parameter. The proofs of expressions (54) and (55) are given in Appendix B.

Parameter Fitting Procedure:

The scope of the parameter fitting procedure is to estimate the parameters of the original distribution (i.e. $\theta_1, \theta_2, ..., \theta_m$) so that the first and second order loss functions of the resulting zero-modified distribution can be calculated. The procedure basically takes two steps:

(1) *Step 1*: This steps is carried out to estimate $\theta_1, \theta_2, ..., \theta_m$ by ignoring the observed frequency in the zero class. A regular procedure (e.g. method of moment matching) may be applied to estimate the parameters of $\theta_1, \theta_2, ..., \theta_m$. This leads to the estimation of the parameter w after estimating the probability of expecting zero value (P_0) based on $\theta_1, \theta_2, ..., \theta_m$ and the probability of observing zero LTD value (f_0) based on the given data (or other sources of information).

(2) *Step 2*: The parameters of $\theta_1, \theta_2, ..., \theta_m$ are updated by taking into account the observed frequency in the zero class.

An algorithm of the parameter fitting procedure is given as follows:

Algorithm 4 Parameter fitting procedure of the zero-modified distribution

1) Estimate $\theta_1, \theta_2, ..., \theta_m$ by matching them to the estimated mean and variance of the data by using method of moment matching. Denote the estimated parameters $\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_m$.
2) Compute P_0 with $\hat{P}_0 = f\left(0|\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_m\right) \leftarrow$ unmodified distribution,
3) Estimate f_0 from the count data by $\hat{f}_0 = \hat{p}_0^L$ where $\hat{p}_0 = \dfrac{zero\,count}{N}$,
4) Finally, estimate w with $\hat{w} = \dfrac{\hat{f}_0 - \hat{P}_0}{1 - \hat{P}_0}$,
5) Update $\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_m$ by taking into account the observed frequency in the zero class.

The parameters of the original distribution can be updated by exploiting the fact that the probability generating function (pgf) of a zero-modified distribution is derived from that of the original distribution. Let $H_O(z)$ and $H_{ZM}(z)$ be the pgf of the original distribution and pgf of the zero-modified distribution, respectively. Then, the following holds (Johnson et al., 2005).

$$H_{ZM}(z) = w + (1-w)\,H_O(z) \tag{56}$$

The expression (56) points out an important property in regard to the parameter estimation of the zero-modified distribution. Since the pgf of the original distribution is often known, then one can have pgf of the zero-modified distribution after estimating the parameter w as described previously. The probability generating function of the zero-modified distribution facilitates the derivation of expressions for the moments. After these expressions are derived, the parameters of the original distribution can then be updated based on the method of moment matching. This procedure will be detailed in the next section where the zero-modified Poisson distribution is introduced.

<u>*Zero-Modified Poisson Distribution (ZMP)*</u>: Let λ be the parameter of the original distribution (i.e. Poisson). The random variable Y characterizing the zero-modified Poisson distribution is defined by the following finite-mixture distribution.

$$Y = \begin{cases} \Pr[X=0] & = & w + (1-w)\,e^{-\lambda} \\ \Pr[X=j] & = & \dfrac{(1-w)\,e^{-\lambda}\lambda^j}{j!}, \quad j = 1,2,..., \end{cases}$$

In addition, let $H_{ZMP}(z)$ be the pgf of the zero-modified Poisson distribution. From (56), clearly,

$$H_{ZMP}(z) = w + (1-w)\,e^{\lambda(z-1)} \tag{57}$$

By using (57), the expression to update λ can be derived as follows. The first raw moment of the zero-modified distribution is obtained by

$$E[X] = H_{ZMP}^{(1)}(z=1).$$

It follows that

$$H_{ZMP}^{(1)}(z=1) = (1-w)\lambda e^{\lambda(1-1)}$$

$$E[X] = (1-w)\lambda.$$

Thus the updated parameter of λ is obtained by,

$$\lambda = \frac{E[X]}{1-w}.$$

In most cases, $E[X]$ is estimated as the average of the data. The loss functions of zero-modified Poisson distribution given in expressions (54) and (55) are then calculated using the loss functions of the Poisson distribution based on the updated parameter of λ.

The parameters of the zero-modified distributions studied in this paper can be derived based on the above discussed procedure. The related literature already presents the required expressions for the parameter fitting procedure, which will be given as follows:

Zero-Modified Poisson Distribution (ZMP): Let λ be the parameter of the original distribution (i.e. Poisson). The random variable Y characterizing the zero-modified Poisson distribution is defined by the following finite-mixture distribution.

$$Y = \begin{cases} \Pr[X=0] & = & w + (1-w)\,e^{-\lambda} \\ \Pr[X=j] & = & \dfrac{(1-w)\,e^{-\lambda}\lambda^{j}}{j!}, \quad j = 1,2,..., \end{cases}$$

$$E[X] = (1-w)\,\lambda.$$

<u>*Zero-Modified Negative Binomial Distribution (ZMNB)*</u>: Zero-modified negative binomial distribution can be written as

$$Y = \begin{cases} \Pr[X=0] = w + (1-w)\,t^{k}, \\ \Pr[X=j] = (1-w)\begin{pmatrix} j+k-1 \\ j \end{pmatrix} t^{k}\,(1-t)^{j}, \quad j = 1,2,3,..., \end{cases}$$

where $t = \frac{k}{k+\mu}$. The mean and the variance of the ZMNB are derived by (Yau et al., 2003) as follows:

$$E[X] = (1-w)\,\mu$$

$$Var[X] = (1-w)\,(1+\mu/k+w\mu)\,\mu$$

μ is the un-modified mean while k is the parameter of the original distribution.

<u>*Zero-Modified Binomial Distribution (ZMB)*</u>: (Johnson et al., 2005) express the zero-modified binomial distribution as follows:

$$Y = \begin{cases} \Pr[X=0] & = & w + (1-w)\, q^n \\ \Pr[X=j] & = & (1-w) \begin{pmatrix} n \\ j \end{pmatrix} p^j (1-p)^{n-j}, \quad j = 1, 2, ..., \end{cases}$$

$$E[X] = (1-w)\, np.$$

$$Var[X] = (1-w)\, np\, (1 - p + wnp)$$

Zero-Modified Geometric Distribution (ZMG): The zero-modified geometric distribution is expressed as follows as follows:

$$Y = \begin{cases} \Pr[X=0] & = & w + (1-w)\, p \\ \Pr[X=j] & = & (1-w)\, p\, (1-p)^j, \quad j = 1, 2, ..., \end{cases}$$

$$E[X] = (1-w) \left(\frac{1-p}{p} \right)$$

$$Var[X] = (1-w) \left(\sigma^2 + \mu^2 \right) - (1-w)^2 \mu^2$$

μ and σ^2 are the un-modified mean and variance while p is the parameter of the original distribution.

In what follows, a zero-modified distribution selection rule is proposed based on the above defined distributions.

3.3.7 A Zero-Modified Distribution Selection Rule (ZMADR1)

A selection rule determines which distribution to recommend based on (Adan et al., 1995). The rule selects a distribution from the set of distributions {the zero-modified binomial (*ZMB*), the zero-modified negative binomial (*ZMNB*), the zero-modified geometric (*ZMG*) and the zero-modified Poisson (*ZMP*)}. The rule decides which distribution to select with respect to the parameter a. In order to utilize the mean (μ) and variance (σ^2) of the LTD, the parameter a is defined as $\frac{\sigma^2-\mu}{\mu^2}$. The rule selects *ZMB* if $\mu > \sigma^2$ (i.e. $a < 0$); *ZMNB* if $\mu < \sigma^2$ (i.e. $a > 0$) and the parameter fitting is possible (i.e. $a < 1$) for *ZMNB*; P if $\mu = \sigma^2$ (i.e. $a = 0$); *ZMG* for large coefficient of variation values (i.e. $(a \geq 1)$). Let F be the lead time demand model to be selected by the rule. The rule is presented in Exhibit 1.

Exhibit 1: Zero-Modified Distribution Selection Rule

$$a = \frac{\sigma^2-\mu}{\mu^2}$$

if $a < 0$ *then* $F = ZMB$ *(zero-modified binomial)*
else if $a > 0$ *and* $a < 1$ *then* $F = ZMNB$ *(zero-modified negative binomial)*
else if $a = 0$ *then* $F = ZMP$ *(zero-modified Poisson)*
else (i.e. $a \geq 1$*) then* $F = ZMG$ *(zero-modified geometric)*

A similar distribution selection rule can be created based on mixtures of zero-modified distributions. The next section will provide the modeling steps of the foregoing distribution selection rule.

3.3.8 Zero-Modified Adan Rule (ZMADR2)

(Adan et al., 1995) selects a distribution from the set of distributions {the mixture of binomial (*MB*), the mixture of negative binomial (*MNB*), the mixture of geometric (*MG*) and (P)}. The rule decides which distribution to select with respect to the parameter a. In order to utilize the mean (μ) and variance (σ^2) of the LTD, the parameter a is defined as $\frac{\sigma^2-\mu}{\mu^2}$. The rule selects MB if $\mu > \sigma^2$ (i.e. $a < 0$); MNB if $\mu < \sigma^2$ (i.e. $a > 0$) and the parameter fitting is possible (i.e. $a < 1$) for NB; P if $\mu = \sigma^2$ (i.e. $a = 0$); MG for large coefficient of variation values (i.e. $(a \geq 1)$). Let F be the lead time demand model to be selected by the rule. The rule is presented in previous paper. The idea behind the Zero-Modified Adan Rule is to replace the mixture distributions with their zero-modified versions. The rule still selects a distribution based on parameters a. The Zero-Modified Adan Rule can be expressed in Exhibit 2.

Exhibit 2: Zero-Modified Adan Rule

$a = \frac{\sigma^2-\mu}{\mu^2}$

if $a < 0$ *then* $F = ZMMB$ *(mixture of zero-modified binomial)*

else if $a > 0$ *and* $a < 1$ *then* $F = ZMMNB$ *(mixture of zero-modified negative binomial)*

else if $a = 0$ *then* $F = ZMP$ *(zero-modified Poisson)*

else (i.e. $a \geq 1$*) then* $F = ZMMG$ *(mixture of zero-modified geometric)*

The mixture distributions used in the rule are re-defined based on (Adan et al., 1995) as follows:

The Mixture of Zero-Modified Binomial Distribution: This distribution consists of the mixture of two binomial distributions ($BIN_i(k,p)$ where k is the number of trials and p is the probability of success). If u is defined as the random variable generated from $U(0,1)$ (uniform distribution), then the random variable Y is determined by the following:

$$Y = \begin{cases} ZMBIN_1(k,p), & if\, u \leq q \\ ZMBIN_2(k+1,p), & if\, q < u \leq 1 \end{cases}$$

where $k = \left|\left\lfloor \frac{1}{a} \right\rfloor\right|$, $q = \frac{1+a(1+k)+\sqrt{-ak(1+k)-k}}{1+a}$ and $p = \frac{\mu}{k+1-q}$.

The Mixture of Zero-Modified Negative Binomial Distribution: This distribution consists of the mixture of two negative binomial distributions ($NB_i\left(k,p\right)$) where k is the desired number of success and p is the probability of success). The random variable Y is determined by the following:

$$Y = \begin{cases} ZMNB_1\left(k,p\right), & if\, u \leq q \\ ZMNB_2\left(k+1,p\right), & if\, q < u \leq 1 \end{cases}$$

where $k = \left|\left\lfloor \frac{1}{a} \right\rfloor\right|$, $q = \frac{a(1+k)-\sqrt{(1+k)(1-ak)}}{1+a}$ and $p = \frac{k+1-q}{k+1-q+\mu}$.

The Mixture of Zero-Modified Geometric Distribution: This distribution consists of the mixture of two geometric distributions ($GEO_i\left(p_i\right)$ where p_i is the probability of successes). The random variable Y is determined by the following:

$$Y = \begin{cases} ZMGEO_1\left(p_1\right), & if\, u \leq q \\ ZMGEO_2\left(p_2\right), & if\, q < u \leq 1 \end{cases}$$

where $p_1 = 1 - \frac{\mu\left(1+a+\sqrt{a^2-1}\right)}{2+\mu\left(1+a+\sqrt{a^2-1}\right)}$, $p_2 = 1 - \frac{\mu\left(1+a-\sqrt{a^2-1}\right)}{2+\mu\left(1+a-\sqrt{a^2-1}\right)}$ and $q = \frac{1}{1+a+\sqrt{a^2-1}}$.

It should be noted that the mixture distributions in the rule use the zero-modified versions of the classic distributions. We use the algorithm in (Ünlü and Rossetti, 2011) in order to estimate the parameter w while the parameters of the mixed distributions remain unmodified. That is, the parameters of each distribution in a mixed distribution are not updated. Those parameters are determined by the Adan et al Rule.

Loss Functions: Let $G^1_{MD}(.)$ and $G^2_{MD}(.)$ be the first and second order loss functions of a given mixture distribution, respectively. We derive the first and second order loss functions of a mixture distribution with the following.

$$G^1_{MD}(x) = (1-q)\,G^1_1(x) + qG^1_2(x) \tag{58}$$

$$G^2_{MD}(x) = (1-q)\,G^2_1(x) + qG^2_2(x) \tag{59}$$

where $G^1_1(.)$ and $G^1_2(.)$ are the first order loss functions and $G^2_1(.)$ and $G^2_2(.)$ are the second order loss functions of the two distributions being mixed, respectively. Thus, the loss functions of a mixture distribution that mixes two zero-modified distribution can be expressed as follows:

$$G^1_{MD}(x) = (1-q)\left(w1 + (1-w1)\,G^1_{U1}(x)\right) + q\left(w2 + (1-w2)\,G^1_{U2}(x)\right) \tag{60}$$

$$G^2_{MD}(x) = (1-q)\left(w1 + (1-w1)\,G^2_{U1}(x)\right) + q\left(w2 + (1-w2)\,G^2_{U2}(x)\right) \tag{61}$$

Let

$$G^1_{ZM1}(x) = \left(w1 + (1-w1)\,G^1_{U1}(x)\right)$$

$$G^1_{ZM2}(x) = \left(w2 + (1-w2)\,G^1_{U2}(x)\right)$$

$$G^2_{ZM1}(x) = \left(w1 + (1-w1)\,G^2_{U1}(x)\right)$$

$$G^2_{ZM2}(x) = \left(w2 + (1-w2)\,G^2_{U2}(x)\right)$$

It follows that

$$G^1_{MD}(x) = (1-q)\,G^1_{ZM1}(x) + qG^1_{ZM2}(x) \tag{62}$$

$$G^2_{MD}(x) = (1-q)\,G^2_{ZM1}(x) + qG^2_{ZM2}(x) \tag{63}$$

In this paper, we also present two additional LTD models, which we developed based on preliminary results from the experiments. The new rules ZMADR2ADR and ZMADR2PT are given in Exhibit 3 and Exhibit 4, respectively. The probability of zero LTD (i.e. $\hat{f}_0 = \hat{p}^L_0$) has the major role in determining the selection of the recommended model. As can be seen from the exhibits, ZMADR2 is selected if the probability of zero is greater than 0.50. Otherwise, one of the two other competitive LTD models are selected, which also specifies the rule.

Exhibit 3: Distribution Selection Rule 1: ZMADR2ADR

$$\textit{if } \hat{f}_0 > 0.50 \textit{ then determine } F \textit{ with } ZMADR2$$
$$\textit{else } F = ADR$$

Exhibit 4: Distribution Selection Rule 2: ZMADR2PT

$$\textit{if } \hat{f}_0 > 0.50 \textit{ then determine } F \textit{ with } ZMADR2$$
$$\textit{else } F = PT$$

The effectiveness of each of these LTD models will be tested within an extensive set of test cases. The next section describes the test case generation methods utilized in the simulation evaluation.

3.4 Simulation Evaluation

(Rossetti and Ünlü, 2011) tested a number of LTD models within a large sample of simulated demand conditions. They apply moment matching technique to fit the LTD parameters to the parameters estimated based on the captured LTD observations during simulation. In order to capture the LTD parameters, the authors use Brown's method (Brown, 1982). This study includes a similar simulation experimental environment while it differs in a number of aspects. First of all, most of the LTD models tested in this study includes more than 2 parameters such as the third and fourth moments. Next, by using the moment matching technique, the parameters are fitted to ones whose estimation is based on the computed LTD parameters. As introduced in Section 3.3, one can compute the LTD parameters by using the expressions derived by (Grubbström and Tang, 2006). The computation relies on the moments of the demand and the lead time, which is what is available in practice. The moments of the demand are estimated from simulation. The two parameter estimation procedures are given in Algorithm 5 and Algorithm 6.

Algorithm 5 (Rossetti and Ünlü, 2011) Parameter Fitting Procedure

1. Estimate the moments of LTD based on the captured LTD observations from simulation.
2. By using the method of moment matching technique, fit the required parameters of the distributional model to the ones estimated at Step 1.

Algorithm 6 Modified Parameter Fitting Procedure

1. Estimate the moments of demand based on the demand observations from simulation.
2. By using the lead time and the moments of demand estimated at Step 1, compute the moments of LTD using the expressions derived by (Grubbström and Tang, 2006).
3. By using the method of moment matching, fit the required parameters of the distributional model to the ones estimated at Step 2.

In order to apply the modified parameter fitting procedure, the period demand is captured during simulation. Algorithm 7 provides the steps of capturing the period demand during the simulation. Note that the observed total demand during 1 time unit is captured by CollectStatistics(.) which also provides the required statistics (i.e. estimated period demand moments). These statistics along with the formulas proposed by (Grubbström and Tang, 2006) allow for the estimation of the moments of the LTD. Given a LTD model, these moments are used to approximate the required inventory performance measures.

Algorithm 7 The Method of Demand Capturing During Simulation

***Initialization*:**
t ←current time just after warm-up period,
E_P ←period demand event for the end of a period,
$Sum = 0$ ←cumulative demand between two period demand events,
Schedule E_P at time t
***Event* E_P:**
CollectStatistics(Sum)
$Sum = 0$
Schedule E_P at time $t = t + 1$

Ideally, the modified parameter fitting procedure promises a better moment estimation than the one used in (Rossetti and Ünlü, 2011). This is because the technique utilized in (Rossetti and Ünlü, 2011) solely depends on the LTD observations from simulation. In a complex demand environment such as intermittent demand process, the LTD observations may never be adequate for an accurate moment estimation. On the other hand, the modified parameter fitting procedure proposed in this paper depends on the demand observations from simulation and the theoretically correct expressions that hold for the LTD moment estimations. The derived expressions utilize the estimated demand moments which are based on a much larger set of observations as compared the LTD observations case. Therefore, the modified parameter fitting procedure yields a more accurate moment estimation.

This paper compares the LTD models based on the quality in approximating the inventory performance measures of ready-rate (RR), the expected number of backorders (B) and the expected number of inventory on-hand levels (I). Along with the models discussed in Section 3.3, the models studied in (Rossetti and Ünlü, 2011) are also tested within the experimental analysis. The experiments collect the similar error statistics introduced in (Rossetti and Ünlü, 2011). Therefore, the reader is referred to (Rossetti and Ünlü, 2011) for the algorithms for computing the performance measure errors. In this paper, the LTD models are tested under two different types of investigations.

3.4.1 Case I: A Known LTD Process

The demand process follows the compound Poisson process with the logarithmic jump sizes. The foregoing demand process yields a known LTD process whose distribution is the negative binomial distribution. Thus, the analytical model (i.e. NB) is available for computing the true performance measure values.

The test cases are generated based on the combination of the low and high values of a number of experimental factors. These factors are given in Table 1. The given factors create 64 different test cases. A test case refers to the collection of parameters of $(\mu, \sigma, r, Q, \gamma)$. The policy parameters r and Q are obtained in a similar fashion through the test case generation algorithm given in (Rossetti and Ünlü, 2011). The same notation is used verbatim. The reader is referred to (Rossetti and Ünlü, 2011) for the definitions of the notation used in this paper.

Table 1: Experimental Factors

Level	Target Service Level	Lead Time	Mean LTD	Variance LTD
Low	0.90	1	1.8	4
High	0.95	4	3.6	8

The simulation model of the continuous review (r, Q) inventory system for each generated case was run for 30 replications with 2,000,000 time units of run-time and 200,000 time units of warm-up period. Based on these simulation run parameters, at least 3 digits of precision is achieved. The error results for distributions and distribution selection rules are tabulated in Table B.1, Table B.2, Table B.3, and Table B.4. The tables are very informative due to a number of aspects. First of all, NB results given in Table-B.1 indicate that the simulation model with the set-up parameters yields the most accurate results. In this respect, the results validate the simulation model. Secondly, the performance of the other LTD models can be determined based on the true performance results gained by NB. From the tables, the quality of ZMADR2ADR in approximating performance measures is noticed easily. The model's performance is very close to the true LTD model NB. As far as the statistics of PRE are concerned, for almost all the cases and performance measures, the performance results of ZMADR2ADR fall into 1% within the true performance results. The second best performance results are observed by ADR which yields 0.92 for B based on PRE(.01) statistic.

3.4.2 Case-II: Unknown LTD Process

As an initial step, it is of interest to reveal the test cases for which the performance of the zero-modified distributions and distribution selection rules is competitive. Therefore, the zero-modified distributions, the zero-modified distribution selection rule (ZMADR1) and the zero-modified Adan Rule (ZMADR2) are tested in 3 different phases. For each step, we apply a different fixed lead time generation strategy in order to control the zero lead-time demand probability. For Phase-i, the fixed lead times are randomly generated from the gamma distribution with parameters 1.5 and 5.01; for Phase-ii, the fixed lead times are randomly generated from the uniform distribution with parameters 2 and 5; for Phase-iii, the fixed lead times are randomly generated from the uniform distribution with parameters 0.1 and 2. The experimental observations reveal that

for Phase-i: the probability of zero lead time demand falls into a region that is mostly restricted by 0.01 and 0.35;

for Phase-ii: the probability of zero lead time demand falls into a region that is mostly restricted by 0.23 and 0.39; and for Phase-iii: the probability of zero lead time demand falls into a region that is mostly restricted by 0.58 and 0.80.

The experimental results are collected for 100 test cases. In the experiments, the simulation model of the continuous review (r, Q) inventory model for each test case was run for 10 replications with 100,000 time units of warm-up and 1,000,000 time units of run-time. The probability of zero lead time demand and error results for each model along with the models presented in (Rossetti and Ünlü, 2011) are tabulated for Table-B.9, Table-B.10 and Table-B.11 for Phase-i, Phase-ii and Phase-iii, respectively.

As can be seen from each table, the performance of models improves in the probability of zero-lead time demand. It should be noted that the distribution selection rules ZMADR and ZMADR2 produce much better results in the case where the probability of zero-lead time demand is very high. For example, for Phase-ii, ZMADR gives 0.84 for PRE(0.10) statistic for BO results and ZMADR2 gives 0.93 for the same statistic. However, the performance of ZMADR and ZMADR2 are only better than ADR for Phase-iii where the probability of zero-lead time demand is very high. It should be noted that zero-modified distributions (e.g. ZMP) are not competitive even for the case of high lead-time demand probability. On the other hand, the distribution selection rules including these models provide fairly good results especially for high lead time demand probability cases.

As a next step, all the models in (Rossetti and Ünlü, 2011) and the models considered in this study are evaluated. The demand process is created by using a special demand generator proposed in (Rossetti et al., 2010) and (Rossetti and Ünlü, 2011). The latter study proposes a special test case generator algorithm which sets the required parameters of the demand generator and generates the test cases. In this paper, the test case generator algorithm is similar except that the generated test cases cover many more possible outcomes. Therefore, the number of test cases are larger than the study in (Rossetti and Ünlü, 2011).

The demand generator used in (Rossetti and Ünlü, 2011) is set-up by assigning only one type distribution. For example, events are generated based on assigning an exponential distribution as shown by$Y_i \sim$ exponential(1). In this paper, a number of different distributions are randomly assigned for each parameter.

The events are randomly generated by randomly selecting one the distributions given in Exhibit 5:

Exhibit 5: Generation of Events in the Demand Generator

$$Y_i = 1$$
$$Y_i \sim \text{uniform(0,2)}$$
$$Y_i \sim \text{exponential(1)}$$
$$Y_i \sim \text{lognormal(fitted parameters of mean=1, variance=1)}$$
$$Y_i \sim \text{gamma(fitted parameters of mean=1, variance=1)}$$

The length of times spent in the OFF state are randomly generated by randomly selecting one the distributions given in Exhibit 6:

Exhibit 6: Generation of X_I in the Demand Generator

$$X_I = 1$$
$$X_I \sim \text{gamma}(fitted\ parameters\ of\, \mu_I\ and\, \sigma_I)$$
$$X_I \sim \text{lognormal}(fitted\ parameters\ of\, \mu_I\ and\, \sigma_I)$$
$$X_I \sim \text{exponential}(fitted\ parameters\ of\, \mu_I)$$
$$X_I \sim \text{uniform}(fitted\ parameters\ of\, \mu_I\ and\, \sigma_I)$$

The length of times spent in the ON state are randomly generated by randomly selecting one the distributions given in Exhibit 7:

Exhibit 7: Generation of X_B in the Demand Generator

$$X_B = 1$$
$$X_B \sim \text{gamma}(fitted\ parameters\ of\ \mu_B\ and\ \sigma_B)$$
$$X_B \sim \text{lognormal}(fitted\ parameters\ of\ \mu_B\ and\ \sigma_B)$$
$$X_B \sim \text{exponential}(fitted\ parameters\ of\ \mu_B)$$
$$X_B \sim \text{uniform}(fitted\ parameters\ of\ \mu_B\ and\ \sigma_B)$$

Demand size values are randomly generated by randomly selecting one the distributions given in Exhibit 8:

Exhibit 8: Generation of D_i in the Demand Generator

$$D_i = 1$$
$$D_i \sim \text{gamma}(fitted\ parameters\ of\ \mu_{NZ}\ and\ \sigma_{NZ})\text{(rounded up to positive integer)}$$
$$X_B \sim \text{lognormal}(fitted\ parameters\ of\ \mu_{NZ}\ and\ \sigma_{NZ})\text{(rounded up to positive integer)}$$
$$D_i \sim \text{geometric}(fitted\ parameters\ of\ \mu_{NZ}\ and\ \sigma_{NZ})$$

(Rossetti and Ünlü, 2011)specify only two demand classes for which the LTD are evaluated within experiments. In this study, the LTD models discussed in Section 3.3 are evaluated under 4 demand classes: Group 1, Group 2, Erratic and Smooth. The demand classes of Group 1 and Group 2 are determined based on the same method described in (Rossetti and Ünlü, 2011). Erratic and Smooth demand classes are determined based on the method described in (Boylan et al., 2008). The authors propose a classification method determining 4 demand classes namely; Intermittent, Lumpy, Erratic and Smooth. The percentages of these demand classes observed in an industrial data set are given in Table 2. As can be noticed from the table, Group 1 and Group 2 include only Intermittent and Lumpy demand classes.

Table 2: Percentages of Demand Classes

Demand Class	% of Total
Group 1 (Intermittent and Lumpy)	75%
Group 2 (Intermittent and Lumpy)	7%
Erratic	9%
Smooth	9%

In what follows, the error statistics are collected for each demand class. For Group 1, the statistics of these error results are collected for 20,000 randomly generated test cases which give at least 2 digits of accuracy based on a classic half-width analysis. Based on the percentages of demand classes as given in Table 2, 1740 test cases for Group 2; 2500 test cases for Erratic; and 2,500 test cases for Smooth are generated by using the the demand generator. The descriptive statistics of the generated test cases with respect to target RR levels are presented for Group 1, Group 2, Erratic and Smooth demand classes in Table B.5, Table B.6, Table B.7 and Table B.8, respectively. These statistics are captured during simulation. In the tables, one can notice that the targeted RR (i.e. service level) is covered between 0 and 1 by frequently having high values, which is desired in most industrial practices.

3.4.3 Experiment Settings and Results

The same simulation model parameters as in (Rossetti and Ünlü, 2011) are used for the experiments. For the demand classes of Group 1, Erratic and Smooth, the simulation model of the continuous review (r, Q) inventory model for each test case was run for 30 replications with 200,000 time units of warm-up and 2,000,000 time units of run-time. As far as decimal points are concerned for the estimated performance measures during simulation, the foregoing set-up provides at least 2 digits of precision for the test cases in Group 1; at least 3 digits of precision for the test cases in Erratic and 4 digits of precision for the test cases in Smooth demand class. For the demand classes of Group 2, the same simulation model was run for 30 replications with 85,000 time units of warm-up and 850,000 time units of run-time, which provides 1 digit of precision for the cases in Group 2. The experiments were carried out by using (HPC) (http://hpc.uark.edu/hpc/) which allows many simulation experiments to be done simultaneously.

The error results of 21 LTD models for each demand class and each performance measure are tabulated in 16 tables (Table B.12 - Table B.27) in the Appendix B. Four tables are given for each demand class. For each of four tables, the first two tables tabulate the error results if the LTD is approximated by a distribution while the next two tables tabulate the error results when the LTD is approximated by a distribution selection rule.

The error results of Group 1 are given in Table B.12 Table B.13 for distributions and Table B.14 and Table B.15 for distribution selection rules. For the cases where the LTD is approximated by a distribution, PRE statistics reveal that PT, TPM, GL and SD are the models whose performance is superior to other distributions. Table B.12 shows that as far as B results are concerned PT gives 0.35 for PRE(.10) statistic which is the highest value among other distributions. The performance of other distributions is poor for B approximation results. For example, NB gives only 0.09 for PRE(.10) statistic as can be seen from Table B.12. The distributions often yield fairly good results for RR while they give better I results than B. TPM is the distribution that provides the best performance approximation results for RR yielding 1.00 for PRE(.10) statistic shown by Table B.13. The overall performance of distribution selection rules is better performance approximation results as compared to the distributions alone. For example, it is possible to increase B approximation quality up to 0.39 via ZMADR2ADR rule under PRE(.10) statistic as shown by Table B.15. The other performance measure results are fairly well approximated via distribution selection rules.

The error results of Group 2 are given in Table B.16, Table B.17 for distributions and Table B.18 and Table B.19 for distribution selection rules. The performance of all models decreases for the test cases in Group 2 where the demand is highly variable. The models' performance for B is poor as compared to other performance measures. As Table B.16 shows, the best performance is observed by PT among distributions in terms of B approximation results by yielding 0.19 for PRE(.10) statistic. PT and TPM provide fairly good results for both B and I. Table B.18 and Table B.19 indicate that the distribution selection rules provide better approximation results. ZMADR2ADR and ZMADR2PT are the two models whose overall performance are better better than other models. According to PRE(.10) statistics, the best results are gained through ZMADR2ADR for all performance measures. As far as B results are concerned ZMADR2ADR gives 0.29 for PRE(.10) statistic. For the same statistic the model yields 0.99 for RR and 0.96 for I approximation results.

The error results of Erratic demand class are given in Table B.20, Table B.21 for distributions and Table B.22 and Table B.23 for distribution selection rules. The error results of Smooth class are given in Table B.24, Table B.25 for distributions and Table B.26 and Table B.27 for distribution selection rules. As far as Erratic and Smooth demand classes are concerned, the overall performance of all models much better than for Group1 and Group 2. As can be seen from Table B.20 and Table B.24, the best performance approximation results are gained through NB among distributions. As far as B results are concerned NB gives 0.96 and 0.95 for PRE(.10) statistic for Erratic and Smooth demand classes, respectively. The approximation results are much better when the LTD is approximated by distribution selection rules. ADR, GADR, ZMADR2, ZMADR2ADR provide excellent results as can be seen from Table B.22, Table B.23, Table B.26 and Table B.27. The performance results are gained through ZMADR2ADR which provide 0.96 for B as far as

PRE(.01) statistic is concerned.

For Group 1 and Group 2, PT yields fairly good approximation results while its results degrade for Erratic and Smooth demand classes. This can be explained by Table 3 which presents the usage percentages of LTD models GL, SD and PT. It should be noted that the performance of PT is fairly good as long as higher order moments are utilized. PT is used only about 30% for erratic and smooth demand classes.

Table 3: Usage Percentages of Models

	Smooth	Erratic	Group 1	Group 2
GL	82%	55%	37%	68%
SD	71%	54%	54%	83%
PT(higher moments)	33%	37%	69%	72%

3.4.4 Multiple Comparison Methods

In this section, we apply multiple comparison methods on a number of different error measures. We take into account only absolute error results. As discussed before, the LTD models are tested within an extensive set of simulation experiments in order to capture the error results of each performance measures of RR, B and I. For each of these results, we apply multiple comparison method to see if there exists a statistically significant difference among LTD models. In addition, we obtain another error measure based on RR, B and I error measure results. The new error measure is called standardized error measure. For a given test case, the standardized error measure is obtained by dividing each error value with the largest observed error value. This way each observed error value is standardized on the range between 0 and 1. For given performance measure and test case, the value of 1 represents the highest observed error value while the value of 0 represents the minimum observed error value. In order to get a single error measure across performance measures, each standardized value is summed. This allows for making a comparison among LTD models based on a single error measure. The following section will present the results of the foregoing error measure results.

For the standardized error measure, the overall performance of the LTD models was analyzed by using the multiple comparison procedure referred to as "Tukey-Kramer HSD" found in statistical package MINITAB. The method compares the least square means for each pair of the LTD models and presents results in a categorized manner. Each category is represented by a letter in a column. Table B.28 tabulates the results of the procedure across all demand classes within 95% confidence level. The LTD models that share a letter are not significantly different. In this respect, we can sort the performance of the models in descending order as follows: {ZMADR2ADR} > {ZMADR2} > {GADR} > {ADR} > {TPM} > {MGNBA, ZMADR2PT, PT, NB, MNNB} > {P, AXR, G, N} > {ZIP} > {LN} > {ZMNB} > {ZMG} > {ZMADR}. As can be seen, ZMADR2ADR is the model whose performance is significantly higher than many other LTD models. Hsu's MCB: We further analyze the overall performance of the LTD models by using another multiple comparison procedure referred to as "Hsu's multiple comparisons with the best (Hsu's MCB)" The difference between Hsu's MCB and Tukey-Kramer HSD is that Hsu's MCB reveals the best mean by comparing the best level and other levels while Tukey-Kramer HSD compares all possible pairwise comparisons. In case of determining minimum, the procedure tests whether means are greater than the unknown minimum. For the difference between each level mean, Hsu's MCB computes a confidence interval. A statistically significant difference can only be observed between corresponding means if an interval contains zero as an end point. The results, computed by setting the default options of statistical package MINITAB, are depicted in Exhibit B.1 and Exhibit B.2 for across all demand classes under 95% confidence level. As can be seen from the exhibits, Hsu's

MCB reveals that the performance of ZMADR2ADR is significantly better than others.

We apply the multiple comparison method for each performance measure of B, RR and I. We use their absolute error values. 1740 test cases are selected from each demand class to collect the multiple comparison results.

i) *Absolute B Error Measure*: The HSD results are depicted in Table B.29. We can sort the performance of the models in descending order as follows: {ZMADR2ADR, ZMADR2} > {GADR, ADR} > {MGNBA, NB, LN, MNNB, PT, ZMADR2PT, TPM, AXR, P} > {ZMG, ZMNB, ZMADR, N, G} > {ZIP}. As can be seen, ZMADR2ADR, ZMADR2 is the model whose performance is significantly higher than many other LTD models. Hsu's MCB results are depicted in Exhibit B.3 and Exhibit B.4. As can be seen from the exhibits, Hsu's MCB reveals that no single LTD model's performance is significantly different than others.

ii) *Absolute RR Error Measure*: The HSD results are depicted in Table B.30. We can sort the performance of the models in descending order as follows: {ZMADR2ADR, ZMADR2, GADR} > {ADR} > {PT, ZMADR2PT} > {AXR, TPM, N, LN, MNNB, G, MGNBA, NB} > {P} > {ZMG, ZIP} > {ZMADR} > {ZMNB}. As can be seen, ZMADR2ADR, ZMADR2 and GADR are the models whose performance are significantly higher than many other LTD models. Hsu's MCB results are depicted in Exhibit B.5 and Exhibit B.6. As can be seen from the exhibits, Hsu's MCB reveals that no single LTD model's performance is significantly different than others.

iii) *Absolute I Error Measure*: The HSD results are depicted in Table B.31. We can sort the performance of the models in descending order as follows: {ZMADR2, ZMADR2PT, ZMADR2ADR} > {TPM, ADR, PT, GADR} > {ZMG, ZMNB, P, AXR, N, MNNB, LN, G, MGNBA, NB, ZMADR} > {ZIP}. As can be seen, ZMADR2ADR, ZMADR2 and ZMADR2PT are the models whose performance is significantly higher than many other LTD models. Hsu's MCB results are depicted in

Exhibit B.7 and Exhibit B.8. As can be seen from the exhibits, Hsu's MCB reveals that no single LTD model's performance is significantly different than others.

3.5 Conclusion and Future Research

This paper evaluates a large set of LTD models under different demand classes within a rigorous experimental environment. A similar study is carried out in (Rossetti and Ünlü, 2011) that evaluate the LTD models whose parameter fitting procedure is predicated on matching the first one or two moments. This paper, on the other hand, focuses on the LTD models whose parameter fitting procedure is dependent on higher order moments. The LTD sample moments are estimated based on the information of the captured period demand during simulation and lead time. This strategy is different from the previous paper in which the LTD sample moments are directly estimated by capturing the LTD during simulation.

It is of interest to see whether the use of LTD models that have flexible distributional forms reveal better inventory performance approximation results. In this respect, this paper evaluates the LTD models whose parameter fitting procedure is predicated on first two moments (e.g. Normal), first three moments (e.g. Phase-Type), first four moments (e.g. Generalized Lambda) and a particular updating strategy (e.g. Zero-modified Negative Binomial). In addition, a number of distribution selection rules recommending the most appropriate lead time demand distribution are evaluated within the same experimental environment. The experiments are carried out to examine the underlying LTD models within four different demand classes which are determined based on the variability in the observed demand size and the variability in the frequency of the demand incident.

The following are the key results:

1. *The distribution selection rules are of great potential in modeling lead time demand.*

   As can be seen from Table B.28, ZMADR2ADR is the distribution selection rule whose performance is superior to other LTD models studied in this paper. In addition, ZMADR2, GADR and ADR produce significantly better inventory performance approximation results as compared to other LTD models.

2. *The new strategy on the parameter fitting procedure improves the performance measure approximation results.*

   The formulas developed for the moments of the LTD give a better parameter estimation, which also improves the approximation quality of the performance measures.

3. *The approximation of the expected number of backorders is very sensitive to the parameter estimation.*

   Clearly, the new parameter fitting strategy reveals improved parameter estimations. The new parameter fitting strategy improves the expected number of backorders results much better than other performance measures. This implies that the approximation of the expected number of backorders is very sensitive to the parameter estimation procedure.

4. *The LTD models preserving flexible distributional forms provides better approximation quality.*

   The experiment results reveal that the LTD models whose parameters are fitted to the first three moments yield better approximation results than the LTD models (TPM and PT) whose parameters are fitted to the first two moments. Clearly, supplying more information regarding the LTD process to the parameter fitting procedure results in better performance approximations. The experiment results also reveal that the performance measure results degrade when the LTD is approximated by the models (GL and SD) whose parameters are fitted to the first four moments. Although in most cases these models yield better results than classical distributional models, their performance is not competitive as compared to the models TPM and PT. This is because the higher moment estimation is very sensitive to the given sample. In principle, the more accurate higher moment estimations leads to the better parameter fitting and, accordingly, better performance approximation results.

5. *Distributions capable of explicitly modeling zero and nonzero demands are of great potential in modeling LTD.*

   Clearly, the outcomes of the previous papers results in insights to develop better models using zero-modified distributions. The developed model is able to determine particular service levels for a fully specified policy. The experiments reveal that zero-modified distributions yield better results for test cases where the zero LTD probability is high. The distribution selection rules that utilize zero-modified distributions are very promising in terms of yielding the most accurate performance approximations. The distribution selection rule ZMADR2ADR provides fairly good results across all demand classes.

6. *The performance of LTD models varies in different demand classes.*

   As can be seen from the experiment results the LTD models provide better results in Erratic and Smooth demand classes as compared to Group 1 and Group 2 which contains intermittent and lumpy demand classes. The best performance approximation results are observed in Erratic and Smooth demand classes while the performance LTD models is poor for Group 2 which contains highly variable demand cases.

In the context of modeling LTD, this paper evaluates the use of higher moments in approximating inventory performance measures. Significant improvement is observed along with the use of new models and parameter fitting procedures. The use of the zero-modified distributions within distribution selection rules is shown to be promising in terms of providing more accurate performance measure results. However, the approximation results of the expected number of backorders could be poor depending on the demand class. The future research is needed to develop different modeling/parameter-fitting strategies in order to improve the foregoing performance measure approximation results.

4 SIMULATION METHODS FOR ENSURING TARGET SERVICE LEVELS IN INVENTORY SYSTEMS

4.1 Introduction

In this paper, simulation optimization procedures will be considered as a potential approach to set optimal policy parameters that ensure a particular target service level. The main objective is to develop simulation-based methods for determining policy parameters of an inventory system so that the desired service level can be attained for the situations involving complex demand structures.

Policy parameter setting procedures can often be performed through the optimization of inventory control policies. Optimizing inventory control policies can be interpreted by two perspectives: optimality of inventory policies and optimality of inventory policy parameters. The following two sections will discuss these two interpretations to reveal the research area of interest in this paper.

4.1.1 Optimality of Inventory Policies

The optimality of inventory policies is measured by total long-run costs that an inventory policy provides. A particular inventory policy is said to be optimal if it provides required service level with a minimum cost. (r, Q) and (s, S) type policies are often applied for inventory management. One can naturally ask if there exists better policies. In general, this is not the case as pointed out in (Axsäter, 2006). As far as a single-stage inventory system is concerned, one of these policies are actually optimal (Axsäter, 2006). A total cost in an inventory system is often represented as the sum of ordering, holding and shortage costs. Since shortages result in a variety of undesired effects, it is common to impose a constraint on one of the service levels (i.e. performance measures).

This is often done by the performance measure of the stockout frequency or the fill rate. It may happen that a particular inventory policy may not give the desired service level. For such cases, the inventory policy is said to be not optimal. (Zipkin, 2000) points out that the main issue when using a particular inventory policy is whether it minimizes the cost. The optimal inventory control policy is selected to ensure that it actually minimizes the inventory costs. In this respect, some studies are presented in the literature to show whether a particular inventory control policy is optimal. (Axsäter, 2006) states that an (r, Q) and (s, S) policies are equivalent in the case of continuous or Poisson demand. Since (r, Q) policy is optimal in these assumptions, (s, S) policy is also considered as optimal policy. He also points out that for inventory problems with service constraints, (s, S) policies are not necessarily optimal since the policy may give a higher service level than the desired. On the other hand, (Chen, 2000) proves that an (r, NQ) policy is optimal in the case of no ordering costs.

4.1.2 Optimality of Inventory Policy Parameters

Parameter setting procedures are based on the solution of an inventory optimization problem that arises in the case of minimizing long-run average inventory costs. Modeling the total cost may differ with respect to the underlying inventory control policy. The total relevant cost function of the corresponding optimization problem is often selected as the combination of ordering-set up, holding and shortage-backordering costs. The optimization problem in this form is called "unconstrained" version for which no service level is imposed. The performance measures of the expected number of backorders (or shortages) and the number of orders (or set-ups) are controlled by penalty costs in the objective function. In practice, the unconstrained version of the problem is less frequently observed since the associated penalizing costs are hard to quantify. The other way of controlling performance measures of interest is to impose a service level constraint. From the managerial perspective, it is relatively easier to set a specific service level that determines the limits of performance measures allowed in the inventory system. This gives rise to a constrained inventory optimization problem.

4.1.3 Potential Solution Approaches for the Inventory Optimization Models

There are a number of analytical procedures that focus on determining optimal policy parameters for classical inventory control policies (Schneider and Ringuest, 1990; Tijms and Groenevelt, 1984). These analytical methods are computationally efficient procedures. However, (Bashyam and Fu, 1998) point out that analytical procedures to determine policy parameters are limited in terms of their range of validity. For example, for a periodic (s, S) inventory control system $S - s$ should be larger than the demand during lead-time (Izzet, 1990). The performance of analytical procedures reflects poor results in the case where there is any violation to this rule. In addition, the policy parameters are often kept constant during the analytical search in such procedures. For example, in (Tijms and Groenevelt, 1984), $S - s$ is previously determined by the economic order quantity formula and kept constant while only the parameter s is determined through an analytical search using a Lagrangian multiplier.

Simulation optimization is regarded as a promising tool to optimize the parameters of an inventory control policy. The main motivation behind considering a simulation optimization approach is because it allows much more flexibility that may relax certain restrictions that an analytical procedure imposes. There are a number of simulation optimization procedures applicable to both constrained and unconstrained version of the inventory optimization problem. The following approaches are available in the literature:

1. *Gradient-based approaches*: A simulation optimization algorithm can be built based on the gradient of the performance measure of interest for an inventory system. The policy parameters are gradually adjusted by exploiting the information of direction gained by the gradient estimations. The gradient estimation is performed by running a single independent simulation of the underlying inventory system. Each simulation run is initialized with updated policy parameters which are determined by an approximation algorithm. The simulation optimization algorithm is terminated when the estimations of gradient are close enough to zero. As far as optimization of inventory systems is concerned, two main techniques are used for gradient estimation: perturbation analysis and likelihood ratio method. These methods make use of explicit formulas associated with the underlying inventory control system. The methods give a considerable amount of flexibility for the target demand pattern. The method so-called "finite difference" is also considered as a gradient based approach. However, the gradient estimation is performed by two independent simulation runs. Therefore, the computational aspects of the corresponding simulation optimization algorithm is much more expensive as compared to other two approaches.

2. *Retrospective simulation approaches*: Suppose we have a known sequence of demand (possibly from historical data, possibly a forecast, or possibly a sequence of future requirements as determined by an MRP lot-sizing procedure). Then, given an initial starting policy parameters, inventory level and a lead-time, the performance of the policy can be recreated or simulated over time. From the sample path, the operational service level (e.g. fill rate, ready rate, etc.) can be computed. (Wagner, 2002) argues that a retrospective simulation using real data can be used to calibrate the inventory control policy or safety stock levels to increase the

likelihood that the planned for performance will actually be met in practice. This approach can be thought of as sample path optimization (or sample average approximation). The main idea is to take the large enough set of samples so that the underlying stochastic problem turns out to be a deterministic optimization problem.

3. *Metaheuristics*: A metaheuristic algorithm can be involved with a search routine which is applied to approximate the optimal policy parameters. The objective of the procedure is to determine the minimum cost at a desired service level. There are often a number of probabilistic or statistical procedures incorporated into the search algorithm.

4. *Response surface methods*: Statistical methods are incorporate in the iterative algorithms to build a regression model. In many cases, the model and its parameter values are unknown. A sequence of designed experiments are performed for the required parameter estimation.

Methodology (3) and (4) do not use the information embedded inside the simulation model and, therefore, treats the simulation as a black box. The other approaches, on the other hand, treat the simulation as a white box. For example, (2) uses the information gained during the course of the simulation. Specifically, (Fu and Healy, 1992) optimize the policy parameters of a periodic review (s, S) inventory model by using a retrospective simulation algorithm which exploits the piecewise linearity and convexity of the objective function of the corresponding inventory optimization problem. The methodology in (Gudum and de Kok, 2002) keeps track of the net stock during the simulation and builds an empirical probability distribution to adjust the safety stock level for the desired service level. (1) utilizes explicit formulas that are built in the simulation model. These approaches, among others, are discussed in the next section in detail while the approach in this paper will be introduced in section 4.4.

4.2 Literature Review

(Wagner, 2002) points out that a retrospective simulation using real data can be used to calibrate (1) safety stock levels or (2) the inventory control policy to increase the likelihood that the planned for performance will actually be met in practice. From the standpoint of determining safety stocks to achieve a particular service level via simulation, the following key papers appear in the inventory literature:

(Callarman and Mabert, 1978) investigates the potential use of so-called “Service Level Decision Rule (SLDR)” which is developed through a linear regression analysis in order to estimate the service level. The rule is developed using a response surface mapping procedure that captures the changes in the service level against the change in safety stock buffer levels. By changing the safety stock levels systematically, the rule is built with the simulation of experimental factors of coefficient of variation of demand, forecast error (expressed as a percentage of average demand), the amount of safety stock (expressed as a percentage of average demand) and the time between orders. A search routine was applied with SLDR in order to achieve the desired service level.

SLDR is also used in (Callarman and Mabert, 1978) to determine the required safety stocks achieving 95% and 98% service levels. The authors present experimental comparisons of three lot sizing rules; namely, economic order quantity, part-period balancing and Wagner-Whitin techniques based on total inventory cost estimation. The assumption of deterministic demand is removed. Instead, the demand (the demand for end item) in their study is assumed to be stochastic.

They study a single stage MRP system (time phased order point) so that lumpiness and future planning mechanisms are applied. The reason behind using SLDR is because it makes lot sizing comparisons straight forward regardless of variety of service levels or stockout costs. Their conclusion is mostly based on the comparison on the total costs of applying different lot sizing techniques.

(Debodt and Van Wessenhove, 1983) present a case study at a company that adopts MRP systems in a highly variable demand environment. The authors utilize a simulation study to analyze the safety stock settings. However, they do not discuss how the safety stock should be determined to meet a desired service level. The experiments indicate the relationship between the average inventory levels and service level. They provide high level insight to management by showing that savings can be possible at the company.

Wemmerlöv and Whybark (Wemmerlöv and Whybark, 1984) also perform simulation experiments to compare single-stage lot-sizing rules by determining net requirements based on allowing backorders under demand uncertainty. The demand uncertainty is introduced to the lot sizing problem via forecast error logic. Fourteen different lot sizing rules were compared to each other based on the cost of keeping a certain level of safety stock to achieve nearly a 100% service level for fill rate. The safety stocks are determined by repeating the simulations until the target service levels are reached (i.e. a search routine through simulation). (Wemmerlöv, 1986)studied a similar problem by determining net requirements based on lost sales. The performance measure of fill rate (i.e. the fraction of demand satisfied directly from stock) is used in these two studies.

The methodology labeled, "Safety Stock Adjustment Procedure" (SSAP), in (Gudum and de Kok, 2002) is also motivated by the problem of comparing different lot-sizing rules. When comparing lot-sizing rules via total cost, it is important that the rules be compared under exactly the same service levels. Thus, decision makers can directly determine the better rule without resorting to more complicated analysis via a trade-off curve approach. By assuming a particular time phased order point policy (TPOP Orlicky, 1975), the authors are able to show that a simulation based procedure that estimates the empirical probability distribution of the net stock at the end of a period can be exploited to develop update formulas for the safety stock. That is, the procedure keeps track of the behavior of the net stock levels observed through the simulation run and builds an empirical probability distribution to determine the amount of safety stock to be adjusted so that the target service level is exactly achieved. The updated safety stock values can then be tested to see if they meet the target level via another simulation. The procedure in (Gudum and de Kok, 2002) constitutes a beginning for other related studies and practical applications. The objective of attaining the target service level may be pursued by developing a method through simulation approaches to determine the inventory policy parameters (e.g. safety stock) for various inventory systems. For example, (Boulaksil and Fransoo, 2009) adopts the procedure to determine the empirical probability distribution of the backorder quantities instead of the net stock levels. In their approach, net stock levels in a multi-stage inventory system are determined based on backorder quantities by solving the mathematical model repetitively in a rolling horizon. Other simulation based methodologies are also available in the literature on determining safety stocks in multi-stage inventory systems. The reader is referred to (Boulaksil and Fransoo, 2009) for further discussion and the literature.

A rich body of the literature including a variety of methods is available for the unconstrained version of the problem of determining the optimal parameters of a stochastic re-order type inventory control system. However, the methods including simulation optimization techniques are limited. (Fu and Healy, 1992) apply two simulation optimization techniques, namely, the gradient-based and the retrospective, for the periodic review (s, S) inventory system. In order to apply the gradient-based method, the demand is assumed to be continuous in their study. In addition, the inventory control system is assumed to receive demands in zero lead-time. The gradient-based optimization techniques are known to find only local optima. However, the authors point out that the local optima obtained by the gradient-based algorithm is also a global optima due to the convexity of the underlying cost function (ς) which is the linear combination of order, holding and backorder costs. As far as the gradient-based algorithm is concerned, the authors apply the perturbation analysis whose estimators are derived by (Fu, 1994). For an (s, S) inventory system, the optimization problem is to find s and S for which the variable $\triangle$ can be defined to represent $S - s$. Therefore, the optimization problem can also be defined as to determine s and $\triangle$. The authors present the perturbation analysis algorithm in order to estimate the corresponding gradients (i.e. $\partial\varsigma/\partial s$ and $\partial\varsigma/\partial\triangle$). The forgoing gradients are estimated for N periods. After every N periods, the policy parameters of s and $\triangle$ must be updated based on an optimization algorithm. In this respect, the authors adopt the two dimensional Robbins-Munro stochastic approximation algorithm introduced by (Kushner and Clarck, 1978). In the modified version of the approximation algorithm, the step size is reduced by one unit only if both gradient estimates chance sign. Otherwise, the step size is increased by one unit. The idea principally leads to a search routine to find better local optima. The half of the expected demand ($E[D]/2$) is selected for the initial starting point for both parameters of s and $\triangle$. Therefore, no particular methodology is applied to initialize these parameters in their study. In addition, no convergence proofs are given by the authors. The retrospective simulation technique in their study is predicated on the assumption that the realization of the demand distribution is independent of the decision parameters (i.e. policy parameters of s and S). Since the demand is known in the retrospective simulation technique, a search algorithm can be established in order to determine the optimal values of s and $\triangle$. The search algorithm exploits two facts: (1) orders are determined by $\triangle$ and (2) for fixed $\triangle$, the cost function ς is continuous, piecewise linear and convex with respect to the policy parameter S. Therefore, the search algorithm determines the optimal $\triangle$ by starting with $\triangle = 0$ and ending with $\triangle = \sum_{i=1}^{n} D_i$ where n is the horizon length. The former case implies that an order is placed in each of n periods whereas the latter case implies that no orders are placed in any periods. Then the optimization problem can be represented as the N-period sample path problem whose cost function is $\varsigma_n\left(\hat{S}_n, \triangle\right)$ where $\hat{S}_n$ is the corresponding order-up-to level for a given value of $\triangle$. A remarkable point related to the forgoing problem is that it is a deterministic problem and $\hat{S}_n$ is a piecewise constant function of $\triangle$. Thus, a finite number of subintervals can be determined to find the corresponding value of $\hat{S}_n$ that eventually minimizes the cost function of $\varsigma_n\left(\hat{S}_n, \triangle\right)$. The authors also propose a special technique to determine the subsequent intervals of $\triangle$. Although the implementation of the gradient-based method is relatively easier than the retrospective approach, the main difficulty lies in initializing the parameters of s, $\triangle$, N and initial step size a. The authors also point out that for a moderate sized planning period, the gradient-based technique is computationally less efficient as compared to the retrospective simulation technique. However, as the horizon length increases, the increase in the computational requirements of retrospective technique becomes excessively larger than the gradient-based technique. In their experiments the sampling variance delivered by the retrospective technique is lower than the gradient-based technique.

(Fu and Healy, 1997) continue this line of research by adding another search algorithm for the same optimization problem. The new search algorithm is the hybrid of previously introduced gradient-based and retrospective simulation optimization approaches. As discussed previously, the gradient-based method suffers from the high sampling variance (i.e. slow convergence) while the retrospective technique can be computationally inefficient for large horizon lengths. Thus, the objective of introducing the new search method is to alleviate the disadvantages of both approaches. The idea behind the hybrid approach is that a search routine is applied over subintervals of $\triangle$ with the gradient method, instead of enumerating the subintervals of $\triangle$ over which $\hat{S}_n$ is constant (i.e. instead of implementing the idea behind the pure retrospective technique introduced by (Fu and Healy, 1992)). The authors conclude that the hybrid approach yields fairly good results in the case of short and moderate sizes of horizons. However, the gradient-based technique still yields superior results in the case of long horizons.

(Lopez-Garcia and Posada-Bolivar, 1999) propose a simulation optimization procedure by employing a tabu search to approximate optimal solutions of stochastic inventory models. The optimal solution in their approach is determined by the lowest total cost out of a number of inventory policies, namely; (r, Q), (S, T), (r, NQ, T) and (s, S, T).

As far as simulation-based procedures are concerned, there is a scant literature on the constrained version of the defined optimization problem. From the standpoint of determining optimal inventory control policy parameters to meet a target service level via simulation, the inventory literature delivers the following key studies in the context of single-stage inventory systems: (Bashyam and Fu, 1998) consider the problem of minimizing total relevant costs (ordering and holding) subject to a service level (complement of the fill rate) for the periodic review (s, S) inventory systems under continuous demand, full backordering and random lead-times. The random lead-time, in essence, relaxes the no order crossing assumption, meaning that orders are allowed to cross in time. Therefore, the optimization problem becomes harder than what is originally defined. The analytical methods, such as those introduced by Tijms and (Tijms and Groenevelt, 1984) and (Schneider and Ringuest, 1990), yield poor results. The authors consider the following optimization model:

$$\min C(s, \triangle)$$

$$subject\,to\,\mathcal{F}(s, \triangle) \leq \beta$$

In the above optimization model, $C(s, \triangle)$ is the long-run average cost per period while $\mathcal{F}(s, \triangle)$ is the long-run estimation of the complement of the fill rate measure for a given s and $\triangle$. Also, β denotes the desired service level (e.g. 10%). The simulation optimization approach is applied to find the optimal (or near optimal) values of s and $\triangle$. In order to apply a simulation optimization algorithm, the authors consider perturbation analysis which requires the calculation of the estimators of $\partial \mathcal{C}/\partial s$, $\partial C/\partial \triangle$, $\partial \mathcal{F}/\partial s$ and $\partial \mathcal{F}/\partial \triangle$. The estimators of $\partial \mathcal{C}/\partial s$ and $\partial C/\partial \triangle$ were already derived by Fu and (Fu and Hu, 1994). The authors derive the estimators of $\partial \mathcal{F}/\partial s$ and $\partial \mathcal{F}/\partial \triangle$. These estimators are then used in a simulation optimization algorithm based on an adaptation of the feasible directions method which ensures the search in the feasible region defined by the optimization model. The authors point out that feeding the simulation optimization algorithm with a good starting point plays a key role for rapid convergence. In addition, a good starting point directly affects the quality of the solution found by the simulation optimization algorithm. Thus, before the simulation optimization algorithm, the procedure starts with two phases which find good initial values of s and $\triangle$. In the first phase, $\triangle$ is calculated based on the economic order quantity formula. In the second phase, a line search is performed to estimate s by keeping $\triangle$ constant. The authors do not provide any convergence proof for the proposed approach. However, they conduct extensive empirical experiments to exhibit the procedure gives promising results.

The brute force method is also applied in the literature (Bashyam and Fu, 1998; (Angün et al., 2006); Wan and Kleijnen, 2006) to estimate the optimal policy parameters. However, (Kleijnen and Wan, 2006) point out that these papers report different parameter values as optimal (i.e. s and S). (Kleijnen and Wan, 2006) apply a simulation optimization methodology based on a search technique which is composed of several metaheuristic such as Tabu Search, Neural Networks and Scatter Search. This search technique is implemented within OptQuest (provided by Opt-Tek System Inc.) which uses more than one heuristic during the search. The authors set a minimum 90% of service rate (fill rate) as a constraint for the optimization problem to determine s and S. In the given procedure, Karesh-Kuhn-Tucker (KKT) conditions are used to determine the stopping criteria of the search. In the experimental part, authors compare their results with the results of (Bashyam and Fu, 1998) who applied perturbation analysis and feasible directions techniques (PA and FD) and the results of (Angün et al., 2006) who applied the modified response surface method (modified RSM). According to the results, among others, OptQuest yields the minimum cost at the desired service level.

In the case where the inventory system is the continuous review (r, Q), the literature on the solution methods providing joint optimization of policy parameters can be classified into two groups. The literature in the first group provides techniques to solve the problem so that the solution has integer values of r and q. (Zipkin, 2000) points out that this is a hard problem in general and discusses some sort of the solution approaches. For fixed q, the smallest feasible value of r, which is easy to find, is the optimal solution for the problem. As he explains, finding q requires a full search. Zipkin proposes a very simple heuristic for which a backorder cost is defined in his heuristic based on the given value of fill rate (w). Order cost is also redefined so that fill rate can be expressed as w_= (backorder cost)/(backorder cost + holding cost). After these modifications, the algorithm proposed by (Federgruen and Zheng, 1992) is used to find the values of r and q. (Zipkin, 2000) claims that the resulting policy from the heuristic has close values of stockout probability $\overline{A}$ values to the (1-w_).

The literature in the second group offers approaches to solve the problem so that the solution has continuous values of r and q. (Axsäter, 2006) discusses solution techniques. First of all, (Axsäter, 2006) recommends the literature by (Rosling, 002b) for a pure optimal strategy. However, he reformulates the problem so that it has a fill rate constraint rather than a stock out probability. His formulation also depends on only single variable. This single variable is derived based on the given demand and cost parameters. His solution is based on linearly interpolated values of replenishment quantity. These values are provided as tables in (Axsäter, 2006). In (Axsäter, 2006), the author explains that these tables are created for normally distributed demand during lead time cases. Therefore, the solution approach in (Axsäter, 2006) is limited to this special demand case. (Rosling, 002b) proposes an optimization algorithm. Rosling's algorithm is known as square-root algorithm. In the square root algorithm, first an initial solution is determined. In the initial solution, the replenishment quantity is determined via the classic economic order quantity expression. Reorder point is determined so that the given constraint is satisfied. Next, the value of replenishment quantity is obtained via a square root expression which takes into account the Lagrange multipliers of the constraint. The reorder point satisfying the constraint is determined in the same way. This process is repeated until r and q converge. (Yano, 1985) also proposes an optimization algorithm. (Yano, 1985) derives two expressions to build the optimization algorithm. The first expression is to optimize based on a given reorder point. The second expression is responsible for optimizing reorder point for a given replenishment quantity. For an initial value of r or q, the algorithm given in (Yano, 1985) converges to the optimal solution via iterative optimization of r and q. A proof also provided in (Yano, 1985). In addition to an optimization algorithm, (Yano, 1985) provides an iterative heuristic that is proved to converge. However, the given heuristic works a for normally distributed lead time demand. (Platt et al., 1997) provide a comprehensive review of the procedures for different inventory systems with constrained service levels.

The single-stage inventory policy optimization problem was studied in different forms in the literature. It may be of interest to compare the optimization procedure in this paper and a procedure proposed in the literature. Table 4 provides the related literature.

Table 4: Literature: Solution Approaches

Literature	Approach	Optimality Guaranteed	Discrete Policy Variables	Service Level Considered	Explicit LTD Independence
Yano (1985)	Optimization procedure based on Lagrange multiplier	Yes	No	Yes	No
Federgruen and Zheng (F&Z) (1992)	Optimization algorithm	Yes	Yes	No*	No
Rosling (1999)	Square root algorithm	Yes	No	Yes	No
Zipkin (2000)	Heuristic based on F&Z (2000)	No	Yes	Yes	No
Agrawal and Seshadri (A&S) (2000)	Optimization algorithm based on bounded Q	Yes	Yes	Yes	No
Ünlü and Rossetti (2011)	Simulation optimization procedure based on SAA	Yes	Yes	Yes	Yes

Our approach differs from the ones proposed in the literature in the following aspects. First of all, the problem that we are concerned with in this study is defined under the cases where the explicit LTD model is not known (or not available in a closed mathematical form). The associated inventory policy optimization literature to date is only concerned with the cases where the LTD model is known or available based on an assumed distribution. Secondly, we are interested in the following inventory optimization problem: Find the integer policy parameters r and q in order to minimize the sum of ordering and holding costs subject to the constraint that the ready rate should be at least equal to γ. That is, it is the constrained version of the discrete policy optimization problems.

In Table 4, the listed approaches deal with the constrained problem except the algorithm proposed by (Federgruen and Zheng, 1992) (F&Z). However, with the help of Lagrange variables, F&Z algorithm is applicable to the constrained problem provided that the LTD is Poisson. Therefore, F&Z algorithm can be used for comparison only in the cases where the LTD follows the Poisson distribution. Notice that the algorithms proposed by (Yano, 1985) and (Rosling, 002b) cannot be used for comparison since these algorithms are applicable under continuous policy variables. Discrete case is a much harder problem as pointed out by (Zipkin, 2000, p 226). He proposes a heuristic (Zipkin, 2000, p 226) for which no extra work is presented in terms of its optimization quality. In addition, there is no guarantee that the heuristic finds optimal policy variables. However, it is applicable for solving discrete policy optimization problem.

The algorithm proposed by (Agrawal and Seshadri, 2000) (A&S) is the most promising algorithm in terms of addressing the majority of the criteria given in Table 4. As can be noted from the table, the characteristics of the algorithm can be given as follows:

1. Guarantees the optimality (with an error bound),
2. Gives discrete optimal policy variables,
3. Service level can be imposed.

**Remark - A*: The service level considered in A&S algorithm is given as the fill rate. The authors make a common mistake in formulating the service level. The formula defined in the paper represents the ready rate in the general form. It only represents the fill rate in the case of the

Poisson LTD. Therefore, the algorithm is applicable for comparison since the ready rate constraint is considered in our study.

**Remark - B*: The paper by (Agrawal and Seshadri, 2000) does not reveal that the algorithm can be used for compound demand processes. In addition, it is not discussed in the paper whether the bounds on Q are applicable for compound demand cases. This remark is pointed out since our SAA based optimization procedure is also applicable to the compound demand cases under the assumption of the unit demand processing. However, the A&S algorithm is applicable under an assumed LTD distribution regardless of continuous or discrete form.

4.3 Optimization Problem

4.3.1 System Description

The constrained optimization problem arises in the following inventory system. The inventory is reviewed continuously at a single stage for a single item and controlled by the (r, q) policy with the following mechanism. The items are replenished with a constant lead time (*L*). The net inventory ($IN(t)$) and inventory position ($IP(t)$) are defined as follows: $IN(t)$: A random variable that refers to the amount of items on hand minus the number of backordered items at time point t. The equilibrium net inventory is denoted by IN. $IP(t)$: A random variable that refers to the amount of the net inventory level plus the number of (outstanding) ordered items at time point t. The equilibrium inventory position is denoted by IP. If no orders are currently outstanding, then $IN(t)$ and $IP(t)$ are the same. Otherwise, their difference is the total amount of items ordered and outstanding at the present time point. Notice that at any time there can be more than one order outstanding. Whenever the inventory position $IP(t)$ drops to or below an integer value of re-order level (r), the amount of q units of items is issued to replenish the inventory, and the ordered items arrive at the inventory system after a constant time delay L. The inventory system faces a discrete compound demand process. Let λ be the mean of the demand epoch rate random variable and let $E[J]$ be the mean of demand quantity (i.e. demand size) random variable. The lead times are assumed to be independent of the demand process. Order crossings are not allowed, since lead times are fixed. Let D be the random variable representing the total demand during a unit time. Then $E[D] = \lambda * E[J]$. Let Y be total demand during lead time, which is a random variable. Then $Y = L * E[D]$. IP takes integer values on the set $\{r+1, r+2, ..., r+q\}$ while IN can take integer values on the set $\{-\infty, ..., -1, 0, 1, ..., r+q\}$. The inventory position is assumed to be uniformly distributed in the interval $[r+1, r+q]$. A customer demand can either be a batch of items or a single item. In case of batch demand, a customer agrees that some of the batch can be satisfied from on-hand stocks. If the available stock is not enough to fully meet the demand, then the batch can be split. That is, a demand splitting rule (Teunter and Dekker, 2008) is applied to customer demands. Unsatisfied demand is fully backordered. In addition, the inventory system processes demands based on a first-come-first-served fashion. The received (and backordered) demands can be partially or fully satisfied from the available stock on-hand. Therefore, based on these assumptions, a customer with a demand for multiple units can be regarded as multiple customers with demand of unit size. This allows the inventory system to face the compound demand process as if it faces a unit demand process. The foregoing mechanism prevents the undershoot of the re-order point, which validates the formulas used in this paper.

4.3.2 Optimization Model

Under the above described inventory system, the following exact formulations are used to compute the inventory performance measures of the ready rate ($E[RR]$), the expected number of inventory on-hand ($E[I]$) and the expected order frequency ($E[OF]$) (Zipkin, 2000).

$$E[RR] = 1 - \frac{1}{q}\left[G^1(r) - G^1(r+q)\right] \tag{64}$$

$$E[I] = \frac{1}{2}(q+1) + r - Y + \frac{1}{q}\left[G^2(r) - G^2(r+q)\right] \tag{65}$$

$$E[OF] = \frac{E[D]}{q} \tag{66}$$

where $G_F^1(.)$ and $G_F^2(.)$ are the first and second order loss functions of the lead time demand distribution F. Let $[\varkappa]^+$ denote $\max\{0, \varkappa\}$. Then it follows that $G_F^1(\varkappa) = E\left[[Y-\varkappa]^+\right]$ and $G_F^2(r) = 0.5E\left[[Y-r]^+[Y-r-1]^+\right]$.

In modeling the inventory optimization problem, a cost structure is imposed on $E[OF]$ and $E[I]$. The backordered demand is controlled by imposing a service level constraint in the model. Although the fill rate constraint is mostly applied in the literature, the ready rate is used as the service level in this study due to the existence of a tractable analytical formulation for the underlying inventory environment. The policy optimization of the continuous review (r, q) system is performed by solving the corresponding stochastic inventory problem. The goal is to obtain the optimal discrete policy parameters r and q which minimize the sum of ordering and holding costs subject to the constraint that the ready rate should be at least equal to γ. Let k be the fixed cost to place an order and h be the holding cost per unit per unit time. The cost measures are assumed to be positive in order for q to be finite positive integer value while policy parameter r takes any finite integer values on the set $\mathbb{Z}$. For a pair of (r, q), denote the expected total cost by $E[T(r, q)]$. Then the optimization problem is given as follows.

Optimization Problem P1:

$$\min E[TC(r, q)] = kE[OF] + hE[I] \tag{67}$$

subject to

$$E[RR] \geq \gamma \tag{68}$$

If the LTD follows the Poisson distribution (i.e. demand size is 1 and inter-arrivals are exponentially distributed), P1 can be solved through the algorithm proposed by (Federgruen and Zheng, 1992) with the help of Lagrange multipliers. The search of the optimal policy requires a full enumeration of q if the LTD follows a distribution other than Poisson (Zipkin, 2000, p 226). In the case where the LTD follows a known distribution (e.g. gamma), (Agrawal and Seshadri, 2000) propose an optimization algorithm for P1. This paper, on the other hand, is focused on the cases where the lead time demand distribution model is not known (or not available in closed mathematical form).

4.4 Solution Procedure

The solution procedure in this paper is predicated on the sample average approximation (SAA) technique. The SAA method (Ahmed and Shapiro, 2002; Kleywegt and Shapiro, 2001) is applied in order to estimate the expected costs in the problem. The motivation behind employing SAA is to exploit the theoretical fact that the solution to the approximation problem exponentially converges to the optimal solution as the number of scenarios increases (Kleywegt and Shapiro, 2001). The SAA method has been applied to many different stochastic problem domains in the literature. Examples include the stochastic bidding and stochastic scheduling problems (Greenwald, Guillemette, Naroditskiy, and Tschantz 2006), vehicle assignment, aircraft allocation, network design and cargo flight scheduling (Linderoth et al., 2006). (Shapiro and Philpott, 2007) present a tutorial that introduces some basic ideas of the stochastic programming in the contex of the SAA approach. The following section will provide an overview of the SAA approach.

4.4.1 An Overview of the SAA

The SAA provides many statistical tools that may be used to determine/optimize the number of the scenarios required to approximate the true problem. We provide the most generic version of the SAA applied in this paper. The following notation is used.

N: sample size used to build a single SAA problem (replication).

M: number of independent and identically distributed ($i.i.d.$) batches of random samples of size N.

N': sample size used to build independent SAA replications.

M': number of SAA replications based on independently generated sample size N'. The SAA replications based on independently generated sample size N' are solved to optimality M' times. Parameters N' and M' are set to estimate a lower bound for the candidate solution.

$\hat{\sigma}^2_{LB}$: an estimate of the lower bound variance

$\hat{\sigma}^2_{UB}$: an estimate of the upper bound variance

$\hat{TC}$: estimated total cost

$\hat{UB}$: estimated upper bound

$\hat{LB}$: estimated lower bound

$\hat{gap}$: estimated optimality gap obtained by $(\hat{UB} - \hat{LB})$

$\hat{\sigma}^2_{gap}$: estimated gap variance

Under the assumption that the problem is defined based on minimizing total costs, we are interested in solving optimization problems of the form:

$$v^* = \min_{x \in \chi} g(x) \tag{69}$$

where

$$g(x) = E[TC(x, \Upsilon)] \tag{70}$$

where $TC(.)$ is the (total cost) function of x and Υ. $x \in \chi$ is a vector of decision variables that take values in the finite set χ. Υ is a vector of discrete random variables with joint probability distribution f. Suppose that the distribution f has a domain ϖ with realized values w. Then,

$$E[TC(x, \Upsilon)] = \sum_{w \in \varpi} [f(X = w) TC(x, w)] \tag{71}$$

We call each realization w of Υ a scenario which is generated from the distribution f. There may be infinitely many possible scenarios. Hence, it may be prohibitively expensive to compute $E[TC(x, \Upsilon)]$. On the other hand, it is relatively less expensive to compute $TC(x, w)$. Therefore, the sample average approximation technique is applied to approximate $E[TC(x, \Upsilon)]$. That is, given a sample of size N (i.e. batch of a random sample), $E[TC(x, \Upsilon)]$ is approximated by the following

$$\hat{q}_N(x) = \frac{1}{N} \sum_{i=1}^{N} TC\left(x, w^i\right) \tag{72}$$

As commonly known, the sample average approximation method is a numerical means of approximating a solution to the true problem ($v^* = \min_{x \in \chi} g(x)$) via Monte Carlo simulation. Monte

Carlo simulation technique is used to generate a batch of a random sample and, accordingly, create an SAA problem. The SAA problem can be expressed by

$$\min_{x \in \chi} \left[\hat{q}_N(x) = \frac{1}{N} \sum_{i=1}^{N} TC\left(x, w^i\right) \right] \tag{73}$$

Suppose that we are given a feasible point $\hat{x} \in \chi$ gained through solving 73. It is of iterest to see if this feasiable point can be used as a candidate solution to solve the true problem. One way to evaluate the quality of this feasible point is to estimate a probabilistic optimization gap. An optimization gap can be built based on the difference between an estimate of the upper and lower bounds.

Note that for a given feasible point $\hat{x}$,

$$g(\hat{x}) \quad \geq \quad v^*$$

Therefore, in order to estimate an upper bound, we first construct an unbiased estimator of $g(\hat{x})$. We generate M independent and identically distributed (i.i.d.) batches of random samples. In addition, each batch consists of N random elements. Let the generated batches be denoted by i.i.d. random elements of $w^{1,j}$, $w^{2,j}$, ..., $w^{N,j}$ $j = 1, 2, ..., M$. Then, the unbiased property of each batch can be denoted by

$$E\left[\hat{q}_N^j(\hat{x}) = \frac{1}{N} \sum_{i=1}^{N} TC\left(\hat{x}, w^{i,j}\right) \right] = g(\hat{x}) \tag{74}$$

Let $\overline{g(\hat{x})}_{N,M}$ be an estimate for $g(\hat{x})$. Then

$$\overline{g(\hat{x})}_{N,M} = \frac{1}{M}\sum_{j=1}^{M}\left[\hat{q}^{j}(\hat{x})\right] \quad (75)$$

is an unbiased estimate of $g(\hat{x})$. The associated sample variance estimator is obtained by

$$\hat{\sigma}_{UB}^{2} = \frac{1}{M-1}\sum_{j=1}^{M}\left[\hat{q}^{j}(\hat{x}) - \overline{g(\hat{x})}_{N,M}\right]^{2} \quad (76)$$

Since $g(\hat{x}) \geq v^{*}$ (for minimization type problems), an approximate $100(1-\alpha)\%$ upper bound estimate is given by

$$U\hat{B}_{N,M} = \overline{g(\hat{x})}_{N,M} + \frac{t_{\alpha,\nu}\hat{\sigma}_{UB}}{\sqrt{M}} \quad (77)$$

where $\nu = M-1$ and $t_{\alpha,\nu}$ is the α-critical value of the t-distribution with ν degrees of freedom.

Let $\hat{v}_{N'}$ be the optimal value of an SAA problem based on sample size N'. Then, a lower bound can be estimated based on the fact that

$$v^{*} \quad \geq \quad E\left[\hat{v}_{N'}\right]$$

We estimate a lower bound for $E\left[\hat{v}_{N'}\right]$. $E\left[\hat{v}_{N'}\right]$ can be estimated by solving SAA problems several times and averaging the calculated optimal values. M' SAA problems are created based on generated i.i.d. batches which contain i.i.d. random elements of $w^{1,j}$, $w^{2,j}$, ..., $w^{N',j}$ $j = 1,2,...,M'$. Let $\hat{v}_{N'}^{1}, \hat{v}_{N'}^{2}, ..., \hat{v}_{N'}^{M'}$ be the computed optimal values of the SAA problems. Then

$$\overline{v}_{N',M'} = \frac{1}{M'}\sum_{i=1}^{M'}\hat{v}_{N'}^{i} \quad (78)$$

is an unbiased estimator of $E\left[\hat{v}_{N'}\right]$. We can estimate the variance of $\overline{v}_{N',M'}$ as follows

$$\hat{\sigma}_{LB}^{2} = \frac{1}{(M'-1)}\sum_{i=1}^{M'}\left(\overline{v}_{N',M'} - v_{N'}^{i}\right)^{2} \tag{79}$$

An approximate 100(1−α)% lower bound for $E\left[\hat{v}_{N'}\right]$ is then given by

$$\hat{LB}_{N',M'} = \overline{v}_{N',M'} - \frac{t_{\alpha,\nu}\hat{\sigma}_{LB}}{\sqrt{M'}} \tag{80}$$

where $\nu = M'-1$ and $t_{\alpha,\nu}$ is the α-critical value of the t-distribution with ν degrees of freedom. The quality of $\hat{x}$ can be measured by the optimality gap

$$gap\left(\hat{x}\right) = g\left(\hat{x}\right) - v^{*} \tag{81}$$

We outline a statistical procedure for estimating this optimality gap via upper bound ($\hat{UB}$) and lower bound ($\hat{LB}$) analysis. Thus,

$$g\hat{a}p\left(\hat{x}\right) = \hat{UB} - \hat{LB} \tag{82}$$

and

$$\sigma_{gap}^{2} = \hat{\sigma}_{UB}^{2} + \hat{\sigma}_{LB}^{2} \tag{83}$$

In what follows, we present a pictorial representation of the generic SAA based optimization procedure in Figure 9.

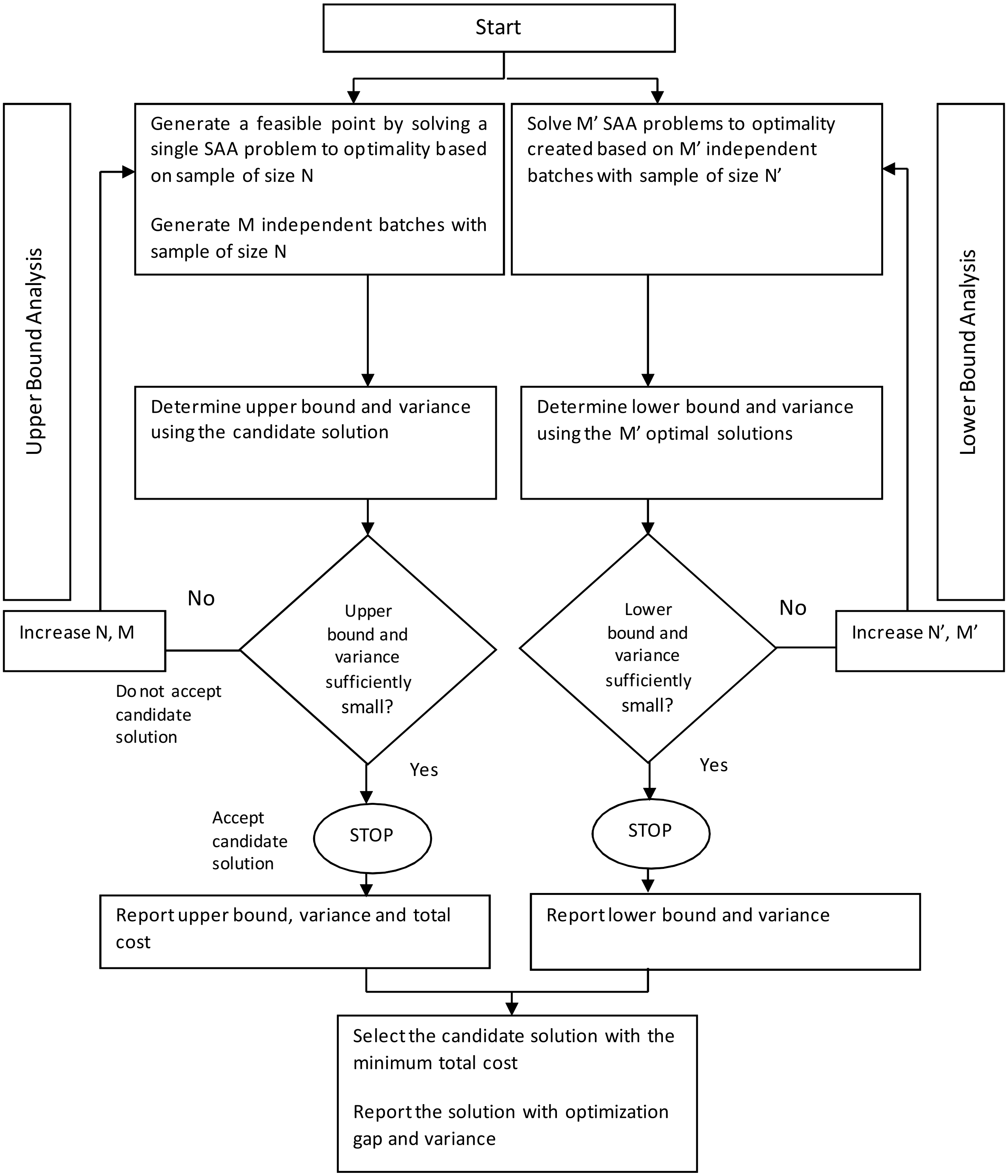


Figure 9: SAA based Optimization Procedure

We apply the SAA approach to the classic constrained stochastic inventory policy optimization problem which is to minimize the total expected relevant inventory costs subject to a service level constraint. The foregoing problem arises in the context of the continuous review (r, q) inventory system where policy parameters are determined by discrete variables of r and q. We consider the ready rate as the underlying service level constraint. The ready rate is known as the fraction of time with positive stock on-hand (Axsäter, 2006, p 94). Even though the problem contains a service level contraint, we propose a simple approach to represent it in a single objective function within an expected form. The details of the problem will be introduced in the next section where we also discuss how it can be solved via the SAA method.

The total expected costs expressed in (67) can be approximated using the sample of LTD values or the sample of net inventory values. Thus, based on the sample, two types of solution methods are proposed in this paper. 1) LTD bootstrapping method: lead-time demands are sampled by performing bootstrapping randomly generated demand values. 2) Net inventory (IN) generation procedure through discrete-event simulation: IN values are sampled from an empirical probability mass function which is built through a single simulation run of the underlying inventory system.

4.4.2 LTD Bootstrapping Method

Let Y be the total demand during lead time, which is a random variable with the expected value $L * E[D]$. A bootstrapping method is of interest in generating lead time demand values which will be utilized in the SAA procedure. Since the lead times are assumed to be independent of the demand process, the following procedure can be used to generate Y. In most situations, even if the inventory system faces a compound demand process, the information related to the demand during a unit time (per day, per week, etc.) is available. By using the given mean (μ_X) and variance

(σ_X^2) parameter information of the demand during a unit time, demand amounts are generated for discrete unit time points (t). The total sum value of the demand observed over a fixed lead time (L) gives independent and identically distributed Y values. This procedure is independently repeated to produce a sample of Y. We generate the unit time demand component from a distribution. Let X_t be the generated demand amount during a time unit and $X_t \sim f_X(.)$, then clearly,

$$Y = \sum_{t=1}^{L} X_t \tag{84}$$

Reformulation of P1:

The optimization problem P1 can be represented as

$$\begin{aligned} \min E\left[TC(r,q)\right] \;=\; & E\left[\frac{kE\left[D\right]}{q} + h\left(\frac{q+1}{2} + r - Y\right)\right. \\ & + \frac{h}{2q}\left(\max\{Y-r,0\}\max\{Y-r-1,0\}\right. \\ & \left.\left. - \max\{Y-(r+q),0\}\max\{Y-(r+q)-1,0\}\right)\right] \end{aligned} \tag{85}$$

$$\text{subject to } 1 - \frac{1}{q}E\left[\max\{Y-r,0\} - \max\{Y-(r+q),0\}\right] \geq \gamma \tag{86}$$

For a given q, let $r(q)$ be the re-order point that satisfies (86). Let $S = \{r : r \geq r(q)\}$. Thus, S represents the set of possible r that satisfy the service level. For a given q, let $r^*(q)$ be the optimal re-order point to the optimization problem defined above. That is, $r^*(q) = \arg\min_{r\in S} E\left[TC(r,q)\right]$. Then the optimization problem can be rewritten as a single objective function as follows:

Optimization Problem P2:

$$
\begin{aligned}
\min E\left[TC(q, r^*(q))\right] &= E\left[\frac{kE[D]}{q} + h\left(\frac{q+1}{2} + r^*(q) - Y\right)\right. \\
&+ \frac{h}{2q}\left(\max\{Y - r^*(q), 0\}\max\{Y - r^*(q) - 1, 0\}\right. \\
&- \left.\left.\max\{Y - (r^*(q) + q), 0\}\max\{Y - (r^*(q) + q) - 1, 0\}\right)\right] \qquad (87)
\end{aligned}
$$

If Y follows the Poisson distribution (i.e. demand size is 1 and inter-arrivals are exponentially distributed), then (87) is convex since $r(q)$ is unimodal. In this case, P1 can be solved through the algorithm proposed by (Federgruen and Zheng, 1992) with the help of Lagrange multipliers. Unfortunately (87) is generally not convex. The search of the optimal policy requires a full enumeration of q if Y follows a distribution other than Poisson (Zipkin, 2000, p 226). In the case where Y follows a known distribution (e.g. gamma), (Agrawal and Seshadri, 2000) propose an optimization algorithm for P1. This paper, on the other hand, is focused on the cases where the lead time demand distribution model is not known (or not available in the closed mathematical form). The random lead time demand variable Y is bootstrapped by generating random demand values over a lead time. We perform the full enumeration over a finite set of q which is determined by bounds applied on q from (Agrawal and Seshadri, 2000). Each possible q value creates a candidate solution. These candidate solutions are evaluated and the best solution is selected from the set. We evaluate candidate solutions by using the sample average approximation (SAA) technique, which allows the estimation of the expected value in (87). The next section gives the details related to the solution procedure including the SAA method.

4.4.3 SAA Problem and Obtaining a Candidate Solution

The evaluation procedure involves constructing the optimization gap for each candidate solution $\hat{x}$. The candidate solutions (i.e. pair of $(q, r^*(q))$) are obtained follows. Let $x = (q, r)$ where $r \in S$, and $S = \{r : r \geq r(q)\}$, and $r(q)$ is the re-order point that satisfies the desired service level. Then, for any given $q \geq 1$, the value of $\hat{q}_N(x) = \frac{1}{N}\sum_{i=1}^{N} TC(x, w^i)$ increases in $r \in S$ where $w^i \sim \hat{F}_{LTD}(.)$. In addition, there exists $r^*(q) \in S$ such that $r^*(q) = \arg\min_{r \in S} \hat{q}_N(x)$. Thus, for a fixed q, the optimal value of (87) can be obtained by the minimum feasible value of r. This will provide $r^*(q)$. The solution is performed satisfying the constraint (86) in the sample average sense. That is, for a given q and a sample of Y, $r^*(q)$ is the minimum value of r that satisfies the following:

$$1 - \frac{1}{q}\left\{\frac{1}{N}\left(\sum_{i=1}^{N}\left[\max\left\{Y^i - r, 0\right\} - \max\left\{Y^i - (r+q), 0\right\}\right]\right)\right\} \geq \gamma \tag{88}$$

For a given value of q, it is trivial to obtain $r^*(q)$ through a line search. The candidate solution is denoted by $\hat{x} = (q, r^*(q))$. Clearly, the pair $(q, r^*(q))$ minimizes the approximation $\hat{q}_N(\hat{x})$. This refers to the fact that the candidate solution $\hat{x} = (q, r^*(q))$ is generated by "solving the corresponding SAA problem to optimality." However, this solution should be evaluated to see its quality viewed as a candidate for solving the true problem. Notice that the true objective function value of this solution is different from the approximated one. For a given q, we apply statistical methods to estimate bounds for the true objective value. The details of constructing the optimization gap based on upper and lower bounds will be discussed in Section 4.5.

4.4.4 IN Generation Method

The optimization model P1 can also be represented using the random variable IN which is generated from an estimated probability mass function, $\hat{f}_{IN}(.)$. The empirical discrete distribution of IN is estimated through a discrete event simulation run. The empirical probability mass function is built based on the observed net inventory values during a single simulation run after warm-up period. Suppose that the standard continuous review (r, q) inventory system creates an ergodic IN process. Then the empirical probability mass function is built based on the observed IN values as follows. By using the path of observed IN values, the probability of each observed value can be estimated after warm-up period by the ratio of the total time where IN observed during simulation and the total time where all observed IN. Thus, for i different IN observations

$$\hat{f}_{IN}(IN = 0) = \frac{\text{total time where } IN = 0}{\text{total time}}$$

$$\hat{f}_{IN}(IN = 1) = \frac{\text{total time where } IN = 1}{\text{total time}}$$

$$\ldots$$

$$\hat{f}_{IN}(IN = i) = \frac{\text{total time where } IN = i}{\text{total time}}$$

Example:

During a single discrete event simulation run, let the observed IN values after warm-up period be depicted in Figure 10.

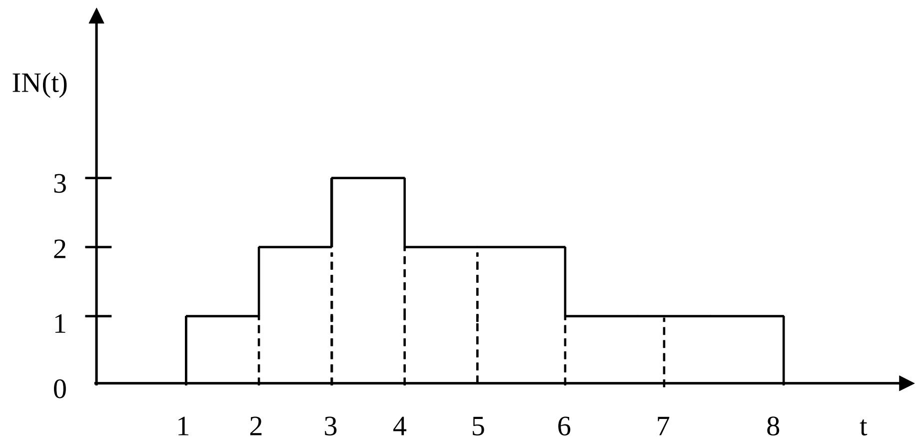


Figure 10: Observed Net Inventory After Warm-up Period

As can be seen from the figure, the simulation run length (after warm-up period) is 8 time units. During the simulation, 4 different *IN* values are observed; 0, 1, 2 and 3. Thus, the estimated probability values are

$$\begin{aligned}\hat{f}_{IN}(IN=0) &= \frac{1}{8}\\ \hat{f}_{IN}(IN=1) &= \frac{3}{8}\\ \hat{f}_{IN}(IN=2) &= \frac{3}{8}\\ \hat{f}_{IN}(IN=3) &= \frac{1}{8}\end{aligned}$$

<u>Reformulation of P1</u>:

The optimization problem P1 can be represented as

$$\min E[TC(r,q)] = kE[OF] + hE\left[[IN]^{+}\right] \quad (89)$$

s.t

$$E[RR] \geq \gamma \quad (90)$$

where $RR = \begin{cases} 1, & if\, IN > 0 \\ 0, & otherwise \end{cases}$ and $E\left[[IN]^{+}\right]$ is the expected number of inventory on-hand. Since $[IN]^{+}$= $\max\{0, IN\}$, $E[I] = E[\max\{0, IN\}]$. Then, based on q and $r^*(q)$, the optimization problem can be rewritten as a single objective function as follows:

Optimization Problem P3:

$$\min E\left[TC(q, r^*(q))\right] = E\left[\frac{kE[D]}{q} + h\max\{0, IN\}\right] \tag{91}$$

4.4.5 SAA Problem and Obtaining a Candidate Solution

The optimization problem P3 can be approximated via a sample average approximation problem. Suppose that the true net inventory distribution (i.e. $f_{IN}(.)$) has a domain Ω with realized values ω. Then,

$$E[TC(x, Y)] = \sum_{\omega \in \Omega} f_{IN}(\omega)\, TC(x, \omega)$$

Given a sample, we can approximate $E[TC(x, Y)]$ by the following

$$\hat{q}_N(x) = \frac{1}{N}\sum_{i=1}^{N} TC\left(x, \omega^i\right) \tag{92}$$

Since $f_{IN}(.)$ is estimated by building the associated discrete empirical probability distribution function $\hat{f}_{IN}(.)$,

$$E[TC(x, Y)] \approx \sum_{w \in \Omega} \hat{f}_{IN}(IN = w)\, TC(x, w)$$

and

$$\hat{q}_N(x) = \frac{1}{N}\sum_{i=1}^{N} TC\left(x, w^i\right)$$

where w is the independent and identically distributed IN values generated from $\hat{f}_{IN}(.)$.

The evaluation procedure involves constructing the optimization gap for each candidate solution $\hat{x}$. The candidate solutions (i.e. pair of $(q, r^*(q))$) are obtained as follows. Let $x = (q, r)$ where $r \in S$, and $S = \{r : r \geq r(q)\}$, and $r(q)$ is the re-order point that satisfies the desired service level. Then, for any given $q \geq 1$, the value of $\hat{q}_N(x) = \frac{1}{N}\sum_{i=1}^{N} TC\left(x, w^i\right)$ increases in $r \in S$ where $w^i \sim \hat{f}_{IN}(.)$. In addition, there exists $r^*(q) \in S$ such that $r^*(q) = \arg\min_{r \in S} \hat{q}_N(x)$. For a given q, let $\vartheta^* = \min TC(\{r^*(q), q\}, Y)$ be the optimal solution to the true problem. For a given q, let χ^* be the set of all possible pairs of $(q, r(q))$. A set of scenarios (IN values) can be generated from the empirical probability distribution $\hat{f}_{IN}(.)$ which is built based on a given q, an arbitrary initial r and a demand distribution $F_D(.)$. In our analysis, we set initial r equal to 0. The generated set of IN values yields a sample path which can be regarded as a set of scenarios. Based on this set of scenarios, we solve the corresponding SAA problem to optimality to get the pair of $(q, r^*(q))$. This solution is performed through the re-order point adjustment procedure. Then $\hat{x} = (q, r^*(q))$. The following theorem introduces the re-order point adjustment procedure within the context of the translation invariance property of net inventory process.

Theorem: Let $IN(t)$ be a random variable that refers to the amount of the net inventory at time t. Let IN_t be the realized set of random variables through time t such that $IN_t = \{IN(t) : t \geq 0\}$. We call IN_t as the sample path of the net inventory process. Let r be the re-order point for this sample path. Let $v = IN(t = 0)$ and denote $IN_t(r, v)$ the sample path of the net inventory process given

r and v and a possible sequence of demand realizations after time t. Suppose that the continuous review (r, Q) inventory system functions under the following assumptions:

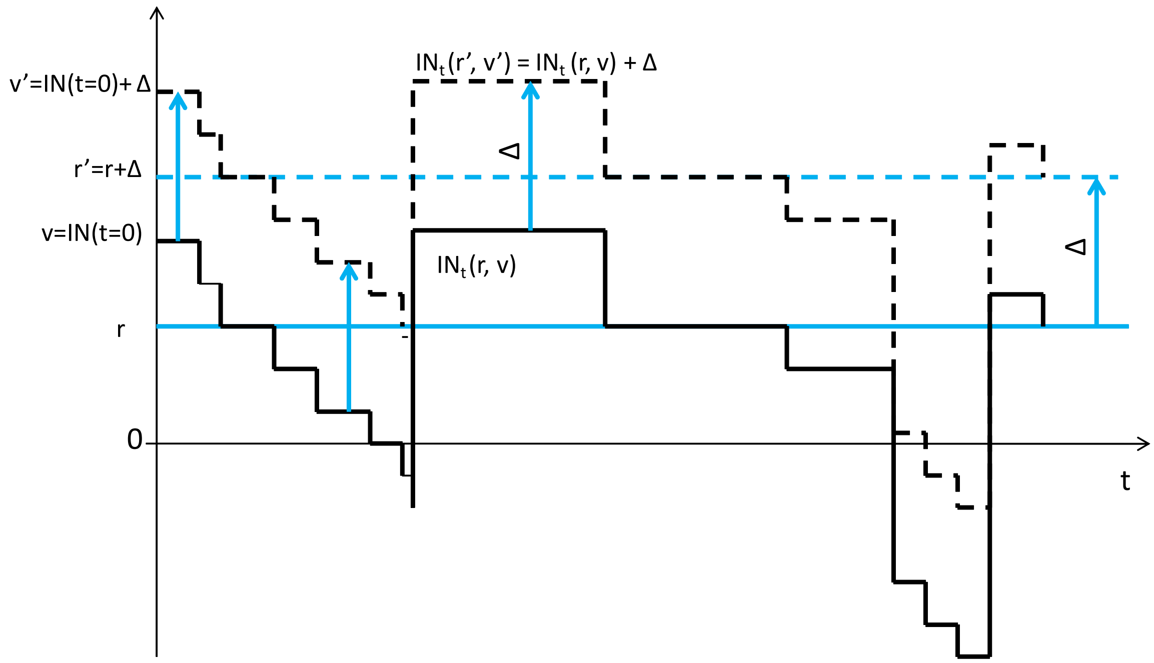
1. The realization of the demand process is independent of the re-order point and the net inventory process.
2. The excess demand is fully backordered.
3. Let $IN_t(r', v')$ be a net inventory process which experiences the same realized demand and $r' = r + \triangle$, $v' = v + \triangle$ and $v' = IN'(t = 0)$

Then, (93) holds.

$$IN_t\left(r', v'\right) = IN_t\left(r + \triangle, v + \triangle\right) = IN_t\left(r, v\right) + \triangle \quad \forall \triangle \in \mathbb{Z} \tag{93}$$

When (93) is true, we say that the process $IN(t)$ is translation invariant to the re-order point (r). The proof of the theorem is given in Appendix C. The theorem states that sliding the sample path by $\triangle$ units to any direction on the y axis of the 2-dimensional axis does not deteriorate of the previous sample path under the aforementioned assumptions (Figure 11). The re-order point adjustment procedure is the shifting procedure of the given net inventory process, which enables that the adjusted r actually attains the desired service level.

Figure 11: Original and Shifted Net Inventory Sample Paths



Let $x = (q, r)$ where $r \in S$, and $S = \{r : r \geq r(q)\}$, and $r(q)$ is the re-order point that satisfies the desired service level. Then, for any given $q \geq 1$ and any given initial value of $r \in \mathbb{Z}$, re-order point adjustment procedure provides the optimal solution for the following SAA problem.

$$\min \hat{q}_N(x) = \frac{1}{N}\sum_{i=1}^{N} TC\left(x, w^i\right)$$

where $w^i \sim \hat{f}_{IN}(.)$. Notice the re-order point adjustment procedure only promises the optimal solution to the approximation problem not to the true problem $\min_{x\in\chi} g(x)$ where χ is a set including all q and $r(q)$ values. Since net inventory process is translation invariant, new sample of IN values will be $w^1 + r^*(q), w^2 + r^*(q), w^3 + r^*(q), \ldots, w^N + r^*(q)$. All statistical evaluation is done with this new set of net inventory scenarios.

In what follows, we introduce sampling methods that use either the LTD bootstrapping method or IN generation method to solve the introduced SAA problems. It is also possible to reduce the estimated variance through different sampling strategies. The following section will discuss these ideas.

4.5 Sampling Methods

4.5.1 Independent Sampling (CMC: Crude Monte Carlo Method)

The sample set $\overline{g(\hat{x})}_{N,M}$ (or $\bar{v}_{N',M'}$) is obtained based on M (or M') independent batches, respectively. In addition, the elements in each batch are independent and identically distributed. Therefore, the method can also be called Crude Monte Carlo Method. Then the same expressions introduced in Section 4.4.1 are used to obtain an optimality gap for the candidate solution.

We determine the candidate solution ($\hat{x}$) during the upper bound estimation procedure. The candidate solution, the optimality gap and gap variance are subject to change according to different sample sizes. Although a large sample size gives a better estimate, it increases the computational time of the evaluation procedure. Therefore, the SAA parameters N (batch size for UB estimation), N' (batch size for LB estimation), M (number of batches for UB estimation) and M'(number of batches for LB estimation) should be wisely determined in the optimization algorithm development phase. The optimality gap and gap variance are considered as major precision criteria in the development of an optimization procedure that evaluates the candidate solutions. It should be noted that the variance observed during the evaluation procedure also effects the precision of the optimization gap. In this respect, reducing variance is the key to an efficient SAA based optimization algorithm. Note that the random elements within a batch need not be i.i.d provided that the statistical bounds are constructed based on the $i.i.d.$ batches. The reader is referred to (Mak et al.,

1999) where the underlying theory is discussed in detail. We now present a number of variance reduction techniques by utilizing the foregoing theory.

4.5.2 Antithetic Variates (AV)

In independent sampling method, $\overline{g(\hat{x})}_{N,M}$ (or $\overline{v}_{N',M'}$) is obtained based on M (or M') independent batches, respectively. As far as antithetic variates are concerned, independent batches can be generated as follows. We first find an estimate for $g(\hat{x})$ (or $E[\hat{v}_{N'}]$) based on a batch of random sample of size N (or N'). Next, we find another estimate for $g(\hat{x})$ (or $E[\hat{v}_{N'}]$) based on the same sample size. However, the second batch contains the antithetics of the first batch. In order to reach an independent estimate of $g(\hat{x})$ (or $E[\hat{v}_{N'}]$), the average of those two estimates is obtained. Therefore, $\overline{g(\hat{x})}_{N,M}$ (or $\overline{v}_{N',M'}$) is obtained based on $M/2$ (or $M'/2$) i.i.d. batches of random samples. Then the same expressions introduced in Section 4.4.1 are used to obtain an optimality gap for the candidate solution. AV is one of the most applied variance reduction techniques. An application of the AV to the newsvendor problem is studied by (Freimer et al., 2010).

4.5.3 Latin Hypercube Sampling (LHS)

We generate random elements in each batch of samples via the Latin hypercube sampling method. If the lead time is given as 1 time unit (i.e. $L = 1$), then in this one-dimensional sampling, we divide the interval [0, 1] into N (sample size) equal segments. The lead time demand value is drawn uniformly from the i^{th} segment. That is, the lead time demand value under LHS is uniformly distributed on $[(i-1)/N, i/N]$. If the lead time is greater than 1 unit, then in this multi-dimensional sampling, the range of [0, 1] is portioned into N non-overlapping intervals of equal probability $1/N$. From each interval one demand value is selected randomly according to the probability density of the interval. The N values of D_1 are paired in a random manner with values of D_2, these pairs are then paired similarly with values of D_3 and so on, until N samples of L time units are formed. Then the corresponding lead time demand value (Y) is obtained by the total sum value of the demand observed over L time units (i.e. $Y = D_1 + D_2 + ... + D_L$). Note that the foregoing strategy allows the generation of i.i.d. batches of random samples. Then the same expressions introduced in Section 4.4.1 are used to obtain an optimality gap for the candidate solution. The reader is referred to (Matala, 2008) for the accuracy of LHS method and the simple strategy to evaluate N for general problem domains.

4.5.4 Common Random Numbers (CRN)

The common random numbers method within the sample average approximation is proposed by (Mak et al., 1999). The idea is to use the batch means approach to directly estimate the optimization gap for the candidate solution. Based on a batch of sample of size N, an estimate for $gap\left(\hat{x}\right)$ can be obtained by

$$E\left[\frac{1}{N}\sum_{i=1}^{N} TC\left(\hat{x}, w^i\right) - \min_{x\in\chi}\frac{1}{N}\sum_{i=1}^{N} TC\left(x, w^i\right)\right] \tag{94}$$

Note that the upper and lower bounds are estimated by using the same batch, which can be considered as an application of common random numbers. The optimization gap and gap variance can be estimated based on separately estimated upper and lower bounds by using the formulas introduced in Section 4.4.1.

The above discussed sampling methods are applied to the problem domain studied in this paper. In Section 4.7, we evaluate these sampling techniques by using the LTD bootstrapping method. The next section will discuss the optimization algorithms that utilize the SAA approach.

4.6 Optimization Algorithms

This section introduces a number of optimization algorithms that eventually build the optimization procedure to solve the problem P1 based on SAA technique. The optimization algorithms utilize the solution of an individual SAA problem. Algorithm 8 and Algorithm 9 are developed for each solution procedure of the LTD bootstrapping method and IN generation method, respectively. Both algorithms are named "*SolveSAA* (.)" since the joint optimization algorithm (introduced next section) will use any of these methods depending on the method selected.

Algorithm 8 *SolveSAA* (.): SAA Solution based on *LTD* Bootstrapping Method

Initialize N, q, $r = -q$ and let $achieved = false$, $countINpositive = 0$, $sum = 0$, $\bar{\gamma} = 0$, $(\delta = 1 - \gamma)$ ←disservice level
Generate $W_{LTD} \sim \hat{f}_{LTD}(.) \leftarrow set\,of\,LTD$ with sample size of N
While$(!achieved)$
    For LTD in W_{LTD} Do
        $sum = sum + max\{LTD - r, 0\} - max\{LTD - (r+q), 0\}$ and $\delta = \frac{1}{Q} * \frac{1}{N} * sum$
        $sum2 = k * ((q+1)/2 + r - LTD)$
        $sum3 = max\{LTD - r, 0\} * max\{LTD - r - 1, 0\}$
        $sum4 = max\{LTD - (r+q), 0\} * max\{LTD - (r+q) - 1, 0\}$
        $LTDcost = sum2 + (k/(2q)) * (sum3 - sum4)$ and S =CollectStatistics($LTDcost$)
    End-Do
    $\bar{\gamma} = 1 - \delta$ ←achieved service level
    If $\bar{\gamma} \geq \gamma$ then
        $achieved = true$
    End-if
    $r{+}{+}$
Loop
Report $r, \bar{\gamma}$
Return S

Algorithm 9 *SolveSAA* $(.)$: SAA Solution based on *IN* Generation Method

Initialize N, q, $r = -q$ and let $achieved = false$, $countINpositive = 0$, $\bar{\gamma} = 0, \gamma \leftarrow$desired service level
Generate $W_{IN} \sim \hat{f}_{IN}(.) \leftarrow set\,of\,IN$ with sample size of N
While$(!achieved)$
    For IN in W_{IN} Do
        $IN = IN + r$
        If $IN > 0$ then
            $countINpositive++$
        End-if
        $INcost = \max\{0, IN\} * HoldingCost$ and $S =$CollectStatistics($INcost$)
    End-Do
    $\bar{\gamma} = countINpositive/N \leftarrow$achieved service level
    If $\bar{\gamma} \geq \gamma$ then
        $achieved = true$
    End-if
    $r++$
Loop
Report $r, \bar{\gamma}$
Return S

Note that the algorithm *SolveSAA* $(.)$ for the IN generation method performs the so-called method "reorder point adjustment procedure." For a given q, the procedure initially sets r to the minimum possible value $-q$. Next, the sample path of *IN* values (i.e. given sample set of *IN*) are shifted upward by increased values of r each time until the desired service level is achieved. The solution of the SAA problem is the minimum value of r that satisfies the service level. On the other hand, for a given q the algorithm *SolveSAA* $(.)$ for the LTD bootstrapping method performs a line search and estimates the service level based on the given sample set of LTD by increasing r every time until the desired service level is achieved. The algorithms report the achieved service level and the statistics on the estimated costs which will be used for the optimization gap construction algorithm.

According to the method used for constructing the optimization gaps, two approaches are considered: naïve approach and direct approach. These approaches will be discussed in the following sections.

4.6.1 Naïve Approach

Naïve approach is concerned with developing an optimization gap by estimating upper and lower bounds separately. Thus, an independent sampling is performed for each bound. For each of the lower bound and upper bound an independent sampling is performed. The optimization gap is obtained by the difference of these bounds. The following notation is used:

$\overline{UB}$: desired upper bound estimate

$\hat{LB}$: estimated lower bound

$\overline{LB}$: desired lower bound estimate

W_q: finite set of q determined by the bounds applied on q

$|W_q|$: number of elements in W_q

ε: optimality gap tolerance

ρ_{LB}: maximum tolerable variance value for the estimate of the lower bound

ρ_{UB}: maximum tolerable variance value for the estimate of the upper bound

ρ_{gap}: maximum tolerable variance value for the estimate of the optimization gap

γ: desired service level

$\bar{\gamma}$: achieved service level

As discussed in Section 4.5.1, the SAA parameters (N, M, N' and M') should be wisely determined in the optimization algorithm development phase. Although setting these parameters to large values results in a more precise solution, it causes a substantial increase in the computational time. Therefore, in the optimization algorithm development phase, the strategy for an SAA parameter is to start with an initial value and increase it based on a rule. This strategy is applicable for both LB and UB estimation procedures. As can be seen from Algorithm 10 and Algorithm 11, number of batches (M, M') are increased by 1 while batch sizes (N, N') are increased by half of the previous size. The motivation behind this updating strategy is to achieve an economic sampling plan. A new estimate of the bound is obtained at each updating step. The updating procedure lasts until the desired precision is gained. The desired precision is determined based on the desired bound ($\overline{LB}, \overline{UB}$) and the desired variance values (ρ_{LB}, ρ_{UB}). The major difference between the two algorithms is the number of SAA problems solved to optimality at each updating step. As Algorithm 10 indicates, a single SAA problem is solved to optimality at each step. The solution is gained from either Algorithm 8 or Algorithm 9 depending on the underlying solution method (i.e. LTD bootstrapping or IN generation). The solution is the so-called "candidate solution." The candidate solution is used to estimate the upper bound based on M independent sets of batches. Note that the candidate solution is subject to change at each updating step due to the size of the batch (N). This, in principle, leads to more reliable candidate solution as the precision of the solution increases. As Algorithm 11 indicates, an estimate of the lower bound is obtained by solving M' independent SAA problems to optimality based on M' independent batches of samples of size N'.

Algorithm 10 Upper Bound Estimation Algorithm

Initialize N, M and let $converged = false$
Do
  Candidate Solution $(q, r^*(q))$: Solve 1 SAA problem with sample size of N and $S = SolveSAA(.)$
  For $i = 1$ to M Do
    Using $(q, r^*(q))$, $\hat{TC} = S.getAverage(.) + (k * E[D])/q \leftarrow$ an estimate using an independent batch of sample of size N
    s =CollectStatistics($\hat{TC}$)
  End-Do
  $\hat{\sigma}^2_{UB} = S.getVariance(.)$ and $\hat{UB} = \hat{TC} + t_{\alpha,\upsilon}\sqrt{\hat{\sigma}^2_{UB}/M}$
  If $\hat{\sigma}^2_{UB} \leq \rho_{UB}$ and $\hat{UB} \leq \overline{UB}$ Then
    Record current solution
    Let $converged = true$
  End-If
  $M = M + 1$ and $N = N + \left\lfloor \frac{N}{2} \right\rfloor$ ←update SAA parameters
While $(!converged)$
Report Candidate Solution, $\hat{\sigma}^2_{UB}$, $\hat{TC}$ and $\hat{UB}$

Algorithm 11 Lower Bound Estimation Algorithm

Initialize N', M' and let $converged = false$
Do
  For $i = 1$ to M' Do
    Solve 1 SAA problem with sample size of N' and $S = SolveSAA(.)$
    Get $\hat{TC} = S.getAverage(.) + (k * E[D])/q$ and s =CollectStatistics($\hat{TC}$)
  End-Do
  $\hat{\sigma}^2_{LB} = S.getVariance(.)$ and $\hat{LB} = \hat{TC} - t_{\alpha,\upsilon}\sqrt{\hat{\sigma}^2_{LB}/M'}$
  If $\hat{\sigma}^2_{LB} \leq \rho_{UB}$ and $\hat{LB} \leq \overline{LB}$ Then
    Record current solution
    Let $converged = true$
  End-If
  $M' = M' + 1$ and $N' = N' + \left\lfloor \frac{N'}{2} \right\rfloor$ ←update SAA parameters
While $(!converged)$
Report current solution, $\hat{\sigma}^2_{LB}$ and $\hat{LB}$

The pictorial representation of the estimation procedures of LB and UB for the LTD bootstrapping and IN generation methods are given in Figure 12 and Figure 13, respectively.

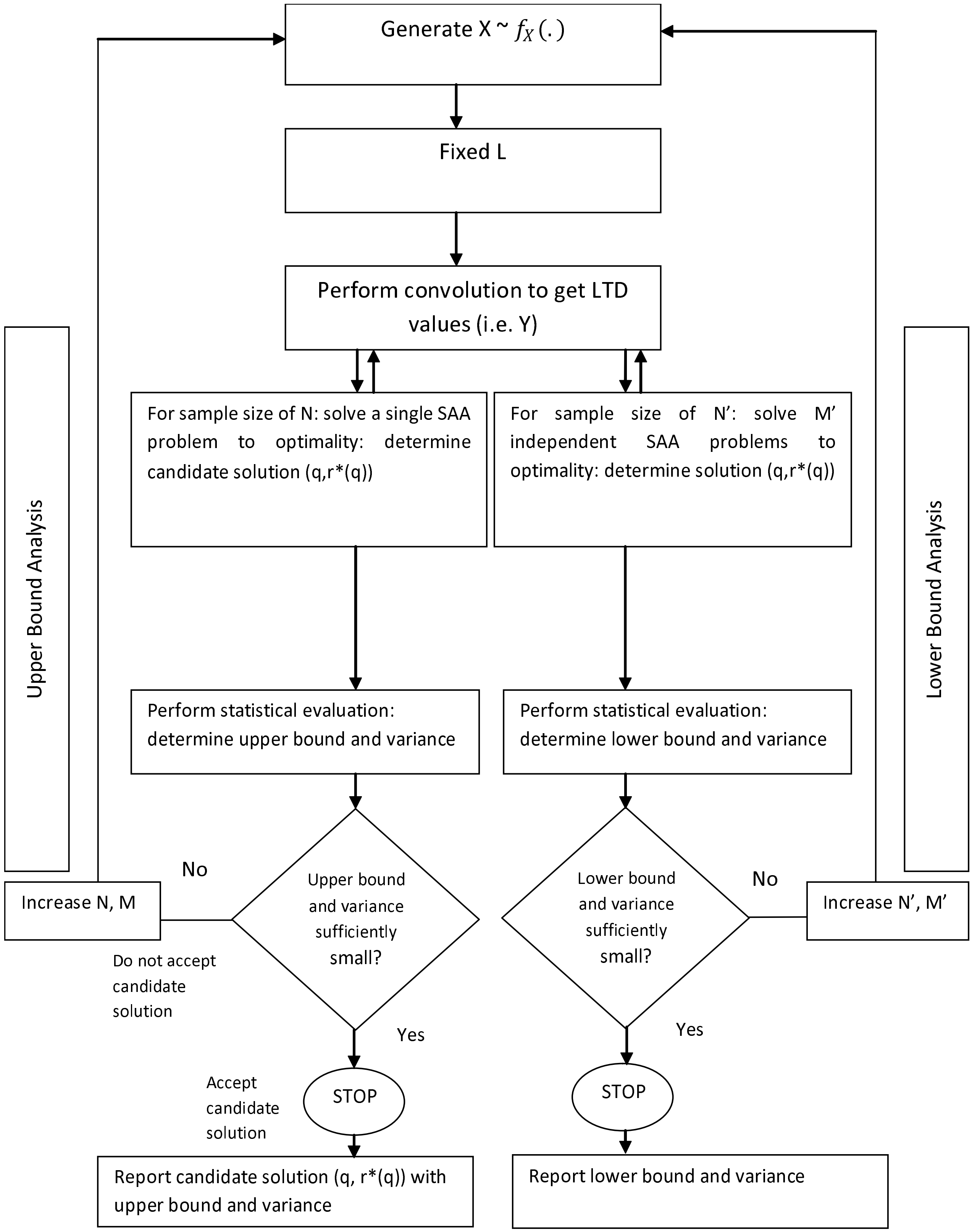


Figure 12: LTD Bootstrapping Method for a Given q

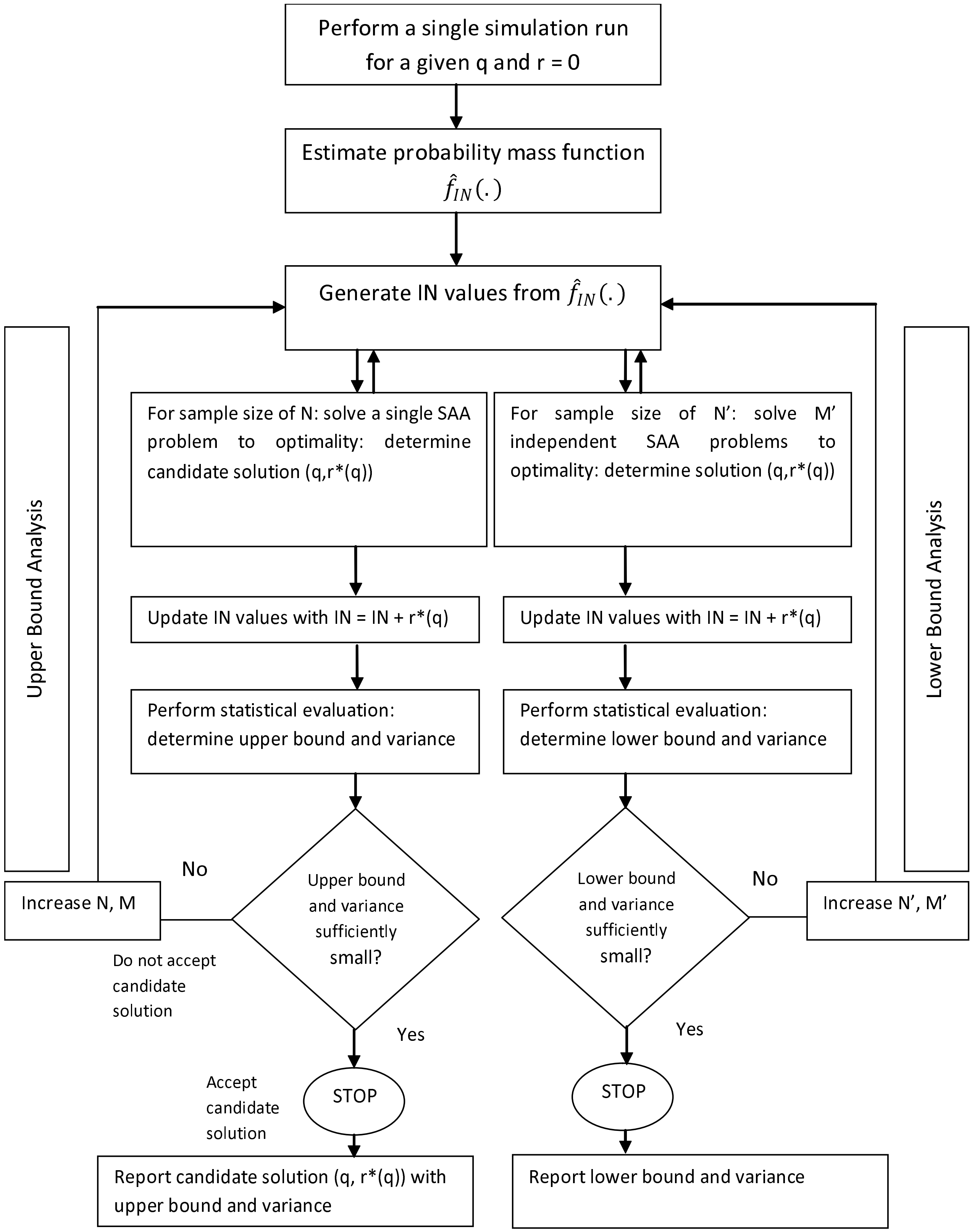


Figure 13: IN Generation Method for a Given q

Note that as Algorithm 13 indicates, only a single discrete-event simulation run is performed for a given q. All the SAA procedure is carried out by using the empirical probability distribution function which is built based on this single simulation run. For a given q, Algorithm 10 provides the estimated upper bound $\left(\hat{UB}\right)$, total cost $\left(\hat{TC}\right)$ and the solution (r, q) while Algorithm 11 provides the estimated lower bound. The joint optimization of policy parameters (r, q) is performed by enumerating each possible value of q within a finite set (W_q) which is determined by the bounds applied on q (Agrawal and Seshadri, 2000). The finite set W_q can be determined through Algorithm 12.

Algorithm 12 Determining Finite Set of q (W_q)

1: Determine $EOQ = \sqrt{\frac{2kE[D]}{h}}$ and $Q_r = q$ where $r(q) = q$
2: If $EOQ < Q_r$ then
3: $\quad W_q = [EOQ, 4Q_r]$
4: Else (i.e. $EOQ \geq Q_r$)
5: $\quad W_q = [EOQ, 4EOQ]$
6: End-if
7: Return W_q

In what follows, a joint optimization algorithm is developed to bring together the ideas from Algorithm 10 and Algorithm 11. The joint optimization algorithm employs a single SAA parameter for the number of batches (M'') and batch size (N''). The steps of the algorithm are depicted in Algorithm 13.

Algorithm 13 Joint Optimization Algorithm

Initialize N'', M'', determine $W_q \leftarrow$finite set of q
For q in W_q Do
    let $converged = false$
    Do
        Solve 1 SAA problem with a sample size of N'' and $S1 = SolveSAA\,(.)$
        For $i = 1$ to M Do
            Using $(q,\, r^*\,(q))$, $\hat{TC}1 = S1.getAverage\,(.) + (k * E\,[D])/q$
            Solve 1 SAA problem with an independent sample size of N'' and $S2 = SolveSAA\,(.)$
            Get $T\hat{C}2 = S2.getAverage\,(.) + (k * E\,[D])/q$
            $gap = \left|\hat{TC}1 - \hat{TC}2\right|$ and $S3 =$CollectStatistic(gap)
        End-Do
        $\hat{\sigma}^2_{gap} = S3.getVariance\,(.)$ and $g\hat{a}p = S3.getAverage\,(.) + t_{\alpha,\upsilon}\sqrt{\hat{\sigma}^2_{gap}/M''}$
        If $g\hat{a}p \leq \varepsilon$ and $\hat{\sigma}^2_{gap} \leq \rho_{gap}$ Then
            Record solution $(q,\, r^*\,(q))$, $\hat{TC}1$ and $g\hat{a}p$
            Let $converged = true$
        End-If
    $M'' = M'' + 1$ and $N'' = N'' + \left\lfloor \frac{N}{2} \right\rfloor$ $\leftarrow$update SAA parameters
    While $(!converged)$
End-Do
Select Solution with minimum $\hat{TC}1$ and report optimal solution $(q*,\, r^*\,(q))$, $\hat{TC}1$ and $g\hat{a}p$

In order to obtain an estimate for UB and LB by using any of the sampling methods including the independent sampling, antithetic variables and Latin hypercube sampling methods, each batch has to be independent and identically distributed. Clearly, Algorithm 13 can be used for any of the foregoing sampling methods.

4.6.2 Direct Approach

Instead of developing an optimality gap by estimating $\hat{UB}$ and $\hat{LB}$ separately, as in the naïve approach, observe the optimality gap expression (94) that uses the same batch samples for both bounds. Thus, for the direct approach the same steps are used as depicted in Algorithm 13 except that the batch samples used at step 5 and 8 are the same. The resulting algorithm can be used for estimating the optimization gap based on the common random numbers sampling method.

4.6.3 An Extension: Lookahead Approach

The sequential sampling method (Law and Kelton, 1999) is a statistical means of stopping rule for simulation replications. The sequential cumulative mean of the interested variable is approximated via a number of replications during which confidence intervals are constructed. The replications are terminated when the user defined desired precision is achieved. (Hoad et al., 2010) state that the sequential sampling method is very useful since it uses output data from the model. In addition, a statistical precision can be gained from the procedure. However, the same authors point out that the procedure may suffer from early convergence. Hence, they bring a revision to the previously defined sequential method by suggesting an idea to avoid the early convergence. The idea is based on using the sequential method to calculate the confidence interval. The difference in their method lies in determining the stopping criteria. The procedure does not stop as soon as the desired confidence interval is achieved. At the number of replications (Nsol) where the desired confidence interval is achieved, it rather looks a number of replications ahead to see if the desired confidence interval is still preserved. If so, the procedure recommends Nsol as the number of replications to achieve the desired confidence interval. Otherwise, the procedure continues with the updated number of replications.

The idea behind the revised sequential sampling method (Hoad et al., 2010) is applicable to determine the parameters M, M' or M''. We will describe the method for only M'' in this section. Likewise, M and M' can be determined in a similar way. The parameter M'' can be determined by increasing by one unit every time as in depicted in Algorithm 14 until the desired optimization gap and variance are satisfied. So the potential for overestimating the sample size is much reduced. However, the user has to provide a stopping criteria for the desired convergence. The optimization gap and its variance are used for the stopping criteria. Then the number of replications (i.e. batches) M'' is determined via the revised sequential sampling method. It should be noted that every increase in the parameters should result in an independent set of random variables. This will guarantee valid statistical estimation. However, the user still needs to determine the initial values of the parameters. (Hoad et al., 2010) define $f(\kappa Limit, P) = \left\lfloor \frac{\kappa Limit}{\kappa} * \max(M'', K) \right\rfloor$ which is the actual number of replications checked ahead. The authors recommend $K = 100$ and $\kappa Limit = 5$ for simulation based experiments. Since the optimization procedure in this study is also predicated on simulation, we also recommend the same parameter values for convenience.

Algorithm 14 Joint Optimization Algorithm based on Lookahead Approach

Initialize N'', M'', determine W_q ←finite set of q
For q in W_q Do
let $converged = false$
Do
Solve 1 SAA problem with a sample size of N'' and $S1 = SolveSAA\,(.)$
For $i = 1$ to M Do
Using $(q, r^*(q))$, $\hat{TC}1 = S1.getAverage\,(.) + (k * E\,[D])/q$
Solve 1 SAA problem with an independent sample size of N'' and $S2 = SolveSAA\,(.)$
Get $\hat{TC}2 = S2.getAverage\,(.) + (k * E\,[D])/q$
$gap = \left|\hat{TC}1 - \hat{TC}2\right|$ and $S3 =$CollectStatistic(gap)
End-Do
$\hat{\sigma}^2_{gap} = S3.getVariance\,(.)$ and $\hat{gap} = S3.getAverage\,(.) + t_{\alpha,\upsilon}\sqrt{\hat{\sigma}^2_{gap}/M''}$
If $\hat{gap} \leq \varepsilon$ and $\hat{\sigma}^2_{gap} \leq \rho_{gap}$ Then
Record solution $(q, r^*(q))$, $\hat{TC}1$ and $\hat{gap}$ and let $converged = true$
Let $\kappa' = f\,(\kappa Limit, M'')$, set $\kappa = 1$ and $Converged = true$
While $converged$ and $\kappa \leq \kappa'$ Do
Using $(q, r^*(q))$, $\hat{TC}1 = S1.getAverage\,(.) + (k * E\,[D])/q$
Solve 1 SAA problem with an independent sample size of N'' and $S2 = SolveSAA\,(.)$
Get $\hat{TC}2 = S2.getAverage\,(.) + (k * E\,[D])/q$
$gap = \left|\hat{TC}1 - \hat{TC}2\right|$ and $S3 =$CollectStatistic(gap)
$\hat{\sigma}^2_{gap} = S3.getVariance\,(.)$ and $\hat{gap} = S3.getAverage\,(.) + t_{\alpha,\upsilon}\sqrt{\hat{\sigma}^2_{gap}/M''}$
If $\hat{gap} \geq \varepsilon$ or $\hat{\sigma}^2_{gap} \geq \rho_{gap}$ Then
$Converged = false$ and $M'' = M'' + \kappa$
End-if
$\kappa = \kappa + 1$
Loop
End-If
$M'' = M'' + 1$ and $N'' = N'' + \left\lfloor \frac{N}{2} \right\rfloor$ ←update SAA parameters
While $(!converged)$
End-Do
Select Solution with minimum $\hat{TC}1$ and report optimal solution $(q*, r^*(q))$, $\hat{TC}1$ and $\hat{gap}$

Note that the algorithm applies lookahead approach when the convergence is achieved. The lookahead approach checks ahead if the convergence is still preserved for a certain number of replications (κ'). If the convergence is still preserved during lookahead steps, then the algorithm proceeds with evaluating another candidate solution with a different value of q. Otherwise, the evaluation process is continued by advancing the number of batches (M'') and batch size (N'').

4.7 Experimental Analysis – Evaluating the Sampling Methods

By using the LTD bootsrapping method, the quality of the variance reduction techniques is computationally investigated on the estimated optimality gap and gap variance results across a large set of test cases. The results are collected under different demand models; namely, Poisson, negative binomial and gamma. The test cases are generated based on the combination of the low and high values of a number of experimental factors. These factors are given in Table 5.

Table 5: Experimental Factors

Level	Target Service Level	Lead Time	Mean LTD	Variance LTD	Ordering Cost	Holding Cost
Low	0.90	1	1.8	4	50	1
High	0.95	4	3.6	8	100	10

Based on the given experimental factors, the algorithm proposed by Agrawal and Seshadri (2000) indicates that the optimal reorder quantity can take values over the range between 1 and 30. Therefore, test cases are generated based on the combination of the given values from Table 5 and reorder quantity enumerated over the set $\{1,2,3,...,30\}$. This creates $2^6 x30 = 1920$ different test cases. For each test case, the candidate solution (i.e. $\hat{x} = (q, r^*(q))$) is obtained based on a given sample size. Then an estimate is obtained for the upper bound, lower bound, (accordingly, optimization gap on the candidate solution) and their variances estimates. The average values of these estimates are tabulated in Table 6 and Table 7 for different total sample size values. Notice that tables show blank cells for upper and lower bounds of the sampling technique CRN whose optimization gap and gap variance are directly estimated.

Table 6: Optimization Gap and Variance Results for Total Sampling 1000

	Sampling Technique	Upper Bound			Lower Bound			Total Sampling	Optimization Gap	Total Gap Variance	Variance Reduction
		N	M	UB Variance	N'	M'	LB Variance				
Pure Poisson	CMC	50	10	0.0302	50	10	0.4721	1000	2.8488	0.5023	–
	AV	50	5	0.0399	50	5	0.0881	1000	2.5540	0.1280	75%
	CRN	100	10	-	-	-	-	1000	1.9162	0.0219	96%
	LHS	50	10	0.0006	50	10	0.2178	1000	1.0169	0.2184	57%
Negative Binomial	CMC	50	10	0.0559	50	10	1.6389	1000	5.8829	1.6948	–
	AV	50	5	0.0618	50	5	0.2615	1000	4.0354	0.3233	81%
	CRN	100	10	-	-	-	-	1000	3.1389	0.0501	97%
	LHS	50	10	0.0013	50	10	0.5932	1000	2.1755	0.5945	65%
Gamma	CMC	50	10	0.0529	50	10	1.9523	1000	6.4134	2.0052	–
	AV	50	5	0.0508	50	5	0.2930	1000	4.3434	0.3437	83%
	CRN	100	10	-	-	-	-	1000	3.2256	0.0448	98%
	LHS	50	10	0.0011	50	10	0.6269	1000	2.2665	0.6280	69%

Table 7: Optimization Gap and Variance Results for Total Sampling 2000

	Sampling Technique	Upper Bound			Lower Bound			Total Sampling	Optimization Gap	Total Gap Variance	Variance Reduction
		N	M	UB Variance	N'	M'	LB Variance				
Pure Poisson	CMC	100	10	0.0156	100	10	0.3843	2000	2.2486	0.3999	–
	AV	100	5	0.0127	100	5	0.0668	2000	1.8927	0.0795	80%
	CRN	200	10	-	-	-	-	2000	1.4477	0.0173	96%
	LHS	100	10	0.0002	100	10	0.1392	2000	0.6324	0.1393	65%
Negative Binomial	CMC	100	10	0.0282	100	10	0.9693	2000	4.1098	0.9975	–
	AV	100	5	0.0184	100	5	0.1655	2000	3.1790	0.1839	82%
	CRN	200	10	-	-	-	-	2000	2.2823	0.0265	97%
	LHS	100	10	0.0004	100	10	0.2869	2000	1.2596	0.2873	71%
Gamma	CMC	100	10	0.0267	100	10	1.1145	2000	4.4636	1.1413	–
	AV	100	5	0.0178	100	5	0.1893	2000	3.3281	0.2071	82%
	CRN	200	10	-	-	-	-	2000	2.3415	0.0258	98%
	LHS	100	10	0.0003	100	10	0.2769	2000	1.1935	0.2772	76%

As can be noted from both tables, the VRTs are effective in terms of reducing the optimization gap variance estimated through the crude Monte Carlo sampling method. In addition, the value of the optimization gap is inclined to be smaller under the applied VRTs. For example, as can be seen from Table 6, for the negative binomial model with sample size 1000, CMC sampling method yields 5.8829 for optimization gap and 1.6948 for optimization gap variance while the optimization gap and gap variance under CRN are observed 3.1389 and 0.0501, respectively. CRN is able to reduce the optimization gap variance by 97%. The minimum optimization gap value is always estimated by the Latin hypercube sampling method. Common random numbers yield the minimum estimated optimization gap results among all considered sampling methods. By comparing the observed variance results in Table 6 and Table 7, one can note that the VRTs are more effective under a larger amount of sampling since VRTs are able to reduce the observed variance by similar percentages. For example, for the gamma distribution CRN is able to reduce the observed optimization gap variance by approximately 98% for each sample size 1000 and 2000. Therefore, VRTs yield much smaller variance values for large sample sizes.

Table 8, Table 9 and Table 10 tabulate the results of the "Tukey-Kramer HSD" procedure for the Poisson, negative binomial and gamma models, respectively with 95% confidence level and with sample size 2000. If two sampling techniques share a letter, then they are regarded as not significantly different from each other. In this respect, the performance if the VRTs is sorted in descending order as follows: $\{CRN\} > \{AV\} > \{LHS\} > \{CMC\}$ for the Poisson and negative binomial models and $\{CRN\} > \{AV, LHS\} > \{CMC\}$ for the gamma model. For all models, *AV*, *LHS* are the two VRTs whose performances are significantly higher than *CMC* while the performance of *CRN* is significantly higher than other VRTs.

Table 8: Comparisons for all Sampling Techniques using Tukey-Kramer HSD for the Poisson Model

Models	Category 1	Category 2	Category 3	Category 4	Mean
CMC	D				0.3999
AV		B			0.0795
CRN			A		0.0173
LHS				C	0.1393

Table 9: Comparisons for all Sampling Techniques using Tukey-Kramer HSD for the Negative Binomial Model

Models	Category 1	Category 2	Category 3	Category 4	Mean
CMC	D				0.9975
AV		B			0.1839
CRN			A		0.0265
LHS				C	0.2873

Table 10: Comparisons for all Sampling Techniques using Tukey-Kramer HSD for the Gamma Model

Models	Category 1	Category 2	Category 3	Category 4	Mean
CMC	D				1.1413
AV		B			0.2071
CRN			A		0.0258
LHS				B	0.2772

Exhibit 14: For the Poisson Model, Hsu's MCB Results

```
Hsu Individual 95% CIs For Mean Based on Pooled StDev
Level     N    Mean   StDev   +---------+---------+---------+---------
CMC    1920  0.3999  0.5749                                  (*-)
AV     1920  0.0795  0.1700        (-*)
CRN    1920  0.0173  0.0411   (*-)
LHS    1920  0.1393  0.4863             (-*)
                              +---------+---------+---------+---------
                            0.00      0.12      0.24      0.36
```

```
Hsu Intervals for Level Mean Minus Smallest of Other Level Means
Level    Lower   Center   Upper  ------+---------+---------+---------+---
CMC     0.0000   0.3825  0.4083        (------------------------*)
AV      0.0000   0.0621  0.0879        (---*-)
CRN    -0.0879  -0.0621  0.0000  (-*---)
LHS     0.0000   0.1220  0.1477        (-------*-)
                                 ------+---------+---------+---------+---
                                     0.00      0.15      0.30      0.45
```

Exhibit 15: For the Negative Binomial Model, Hsu's MCB Results

```
Hsu Individual 95% CIs For Mean Based on Pooled StDev
Level     N   Mean  StDev   -+---------+---------+---------+--------
CMC    1920  0.998  1.984                                  (*-)
AV     1920  0.184  0.597        (-*-)
CRN    1920  0.026  0.064   (-*-)
LHS    1920  0.287  0.779            (-*)
                            -+---------+---------+---------+--------
                           0.00      0.30      0.60      0.90
```

```
Hsu Intervals for Level Mean Minus Smallest of Other Level Means
Level   Lower  Center  Upper  -------+---------+---------+---------+--
CMC     0.000   0.971  1.045         (---------------------------*-)
AV      0.000   0.157  0.231         (---*--)
CRN    -0.231  -0.157  0.000  (--*---)
LHS     0.000   0.261  0.335         (------*--)
                              -------+---------+---------+---------+--
                                   0.00      0.35      0.70      1.05
```

Exhibit 16: For the Gamma Model, Hsu's MCB Results

```
Hsu Individual 95% CIs For Mean Based on Pooled StDev
Level     N   Mean  StDev   -+---------+---------+---------+--------
CMC    1920  1.141  2.390                                 (-*)
AV     1920  0.207  0.741        (-*-)
CRN    1920  0.026  0.060   (-*)
LHS    1920  0.277  0.851          (-*-)
                            -+---------+---------+---------+--------
                           0.00      0.35      0.70      1.05
```

```
Hsu Intervals for Level Mean Minus Smallest of Other Level Means
Level   Lower  Center  Upper  -------+---------+---------+---------+--
CMC     0.000   1.116  1.203         (---------------------------*-)
AV      0.000   0.181  0.269         (----*-)
CRN    -0.269  -0.181  0.000  (-*----)
LHS     0.000   0.251  0.339         (-----*-)
                              -------+---------+---------+---------+--
                                   0.00      0.40      0.80      1.20
```

In order to recommend the best VRT with a confidence, we apply the multiple comparison procedure so-called Hsu's multiple comparisons with the best (Hsu's MCB). The method tests whether means are greater than the unknown minimum in case of determining the minimum. A statistically significant difference can only be observed between corresponding means if an interval contains zero as an end point. The results that are collected via the statistical package MINITAB by setting the default options are depicted in Exhibit 14, Exhibit 15 and Exhibit 16 under 95% confidence level for the Poisson, negative binomial and gamma models, respectively, with sample size 2000. As can be noted, *CRN* is the VRT whose performance is significantly different from others across all the demand models.

The experiment results indicate that all three VRTs are effective at reducing the total gap variance, with the CRN outperforming other VRTs. In what follows, we evaluate the efficiency of the optimization algorithm within the proposed sampling methods.

4.8 Experimental Analysis – Evaluating the Optimization Algorithm

The experimental analysis is two fold for evaluating the optimization algorithms. First of all, it is of interest to show that the SAA based optimization procedure reveals more quality solutions as compared to an optimization algorithm proposed in the literature. A&S algorithm addresses the same optimization problem (i.e. constrained policy optimization with discrete policy parameters). In this respect, we perform the experimental analysis by comparing the results of A&S algorithm and the SAA based optimization procedure. The SAA based optimization procedure promises joint optimization of discrete policy parameters (r, q) for any type of demand process. For the sake of showing the solution quality, a known demand process is employed so that the comparison is possible. The inventory system is assumed to face the compound Poisson demand process with logarithmic demand sizes. The underlying demand process creates the negative binomial LTD process (Axsäter, 2006). Thus, the true LTD process is available. In addition, the true optimal solution is obtained through the A&S algorithm since the LTD process is known.

The quality of the SAA based joint optimization procedure is computationally investigated across a number of test cases. The test cases are generated based on the combination of the low and high values of a number of experimental factors. These factors are given in Table 5. The given factors create 64 different test cases.

In order to use A&S algorithm, a LTD distribution must be given. The following distributions are assumed: Normal (N), Gamma (G), Poisson (P), Negative Binomial (NB) and Lognormal (LN). Of these assumed LTD models, NB provides the optimal solution through A&S algorithm. Table 11 provides the number and percentage of test cases solved to optimality based on the underlying optimization procedure. A&S algorithm assuming a LTD model is run for each assumed LTD model. For example, A&S(LN) indicates that A&S algorithm is run under the assumption that the LTD model is LN. For the SAA based optimization procedure Algorithm 13 is used for the joint optimization of policy parameters. The SAA based optimization procedure utilizes only the independent sampling strategy (CMC). The LTD bootstrapping and IN generation methods are denoted by LTD(CMC) and IN(CMC), respectively. For these methods both the total gap variance tolerance and total gap value tolerance are chosen as 0.1. The IN generation method requires the discrete event simulation of (r, Q) inventory system. The simulation parameters (warm-up and replication length) are set based on an initial experimental investigation. Using larger simulation lengths result a significant increase in the computational time while yielding more true optimal solutions. On the other hand, using lesser simulation length increases the efficiency while yielding less true optimal solutions. Thus, by tuning the parameters, the empirical probability distribution is built based on a single simulation run with 10,000 time units of warm-up and 100,000 time units of run-time.

Table 11: Comparison of A&S and SAA based Optimization Procedure

	A&S(NB)	LTD(CMC)	IN(CMC)	A&S(G)	A&S(P)	A&S(N)	A&S(LN)
Number of Test Cases	64	64	64	64	64	64	64
Number of Cases Solved to Optimality	64	46	45	45	16	17	19
% of Cases Solved to Optimality	100%	71.88%	70.31%	70.31%	25.00%	26.56%	29.69%

The results indicate that the SAA based optimization procedure is able to produce as many optimal solutions as the best assumed LTD model (G). As can be seen from Table 11, A&S(G) solves 45 test cases to optimality while LTD(CMC) and IN(CMC) produce 45 and 46 optimal solutions, respectively.

The performance of the SAA based optimization procedure can be improved by setting a much tighter desired optimization gap value and its variance value. However, this will affect the computational time spent when evaluating each candidate solution. Thus, the second experimental analysis lies in tuning the associated parameters of the SAA based optimization algorithm. In addition, it is of interest to compare the performance of the the sampling methods under naïve and direct approaches. Therefore, along with the independent sampling (CMC), the second experimental analysis compares the performance of the SAA based optimization procedure using the antithetic variables (AV) and common random numbers (CRN). For each sampling technique, the SAA based optimization algorithm is run by using two different values (0.1 and 0.001) for each of the optimization gap value and optimization gap variance value. The value of 0.001 is quite tight for the optimization gap, which might cause computational issues for some test cases. Therefore, a time limit is imposed on the evaluated candidate solution. The joint optimization algorithm is run under the constraint that the time spent on each candidate solution is at most 300 seconds. Therefore, for the joint optimization procedure, the worst computational time will be at most $300 * |W_q|$ seconds. After the given time limit for a candidate solution, the solution with the gained precision is used (i.e. optimization gap value and optimization gap variance value).

The experimental results are collected across the same 64 test cases. The results are tabulated in Table 12 and Table 13 for each tolerance value of 0.1 and 0.001, respectively. The results of LTD(CMC), LTD(AV), IN(CMC) and IN(AV) are gained through Algorithm 13 while the results of LTD(CRN) and IN(CRN) are gained through the same algorithm after the foregoing modifications discussed in Section 4.6.2 are carried out. For the IN generation method the same simulation parameters are used.

Table 12: Results with Total Gap Variance 0.1 and Total Gap Value 0.1

Approach	# of Test Cases	# of Test Cases Solved to Optimality	% of Test Cases Solved to Optimality	Average Computational Time (sec)
A&S(NB)	64	64	100%	1
A&S(G)	64	45	70.31%	1
LTD(CMC)	64	46	71.88%	87.07
LTD(AV)	64	57	89.06%	11.22
LTD(CRN)	64	59	92.19%	10.76
IN(CMC)	64	45	70.31%	185.28
IN(AV)	64	49	76.56%	250.08
IN(CRN)	64	50	78.13%	426.01

Table 13: Results with Total Gap Variance 0.001 and Total Gap Value 0.001

Approach	# of Test Cases	# of Test Cases Solved to Optimality	% of Test Cases Solved to Optimality	Average Computational Time (sec)
A&S(NB)	64	64	100%	1
A&S(G)	64	45	70.31%	1
LTD(CMC)	64	59	92.19%	227.71
LTD(AV)	64	63	98.44%	111.05
LTD(CRN)	64	61	95.31%	107.54
IN(CMC)	64	52	81.25%	896.42
IN(AV)	64	58	90.63%	632.76
IN(CRN)	64	52	81.25%	648.26

In the tables, the true optimal solution is gained through A&S(NB). A&S(G) results are also provided since A&S algorithm produces the best results only with the assumption that the LTD model is G among others. In this case, A&S produces only 45 true optimal solutions out of 64 test cases. In terms of solution quality, both LTD bootstrapping and IN generation methods yield better results than the results with A&S algorithm. The performance of A&S is low because the underlying LTD model (i.e. NB) is different from what is assumed for A&S algorithm (i.e. G).

In the case where the total gap variance tolerance and total gap value tolerance are set equal to 0.1 (Table 12), the best performance is observed by LTD(CRN) yielding 59 true optimal solutions out of 64 test cases. One can observe that the LTD generation method produces more true optimal solutions as compared to the IN generation method. The IN generation method can produce 50 optimal solutions through IN(CRN) for the same tolerance values. When the tolerance values are decreased to 0.001 (Table 13), the performance of both LTD bootstrapping and IN generation methods increases in terms of yielding more true optimal solutions. The best results are gained through LTD(AV) yielding 63 true optimal solutions out of 64 test cases. The IN generation method can produce 58 optimal solutions through IN(AV) for the same tolerance values.

As far as the computational time is concerned, A&S algorithm works very efficient. It produces solutions in a matter of seconds. The computational time of the LTD bootstrapping and IN generation methods are much higher than A&S. Even though more true optimal solutions are gained, their computational time increases when using a smaller tolerance value. For example, as can be seen from Table 12, when the tolerance values are set equal to 0.1, IN(CMC) produces 45 true optimal solutions out of 64 test cases within an average computational time of 185.28 seconds. The same approach yields 52 true optimal solutions within an average computational time of 896.42 seconds. The LTD bootstrapping method is a more efficient method as compared to the IN generation method. As Table 13 shows, LTD(AV) is able to produce 63 true optimal solutions out of 64 test cases within an average computational time of 111.05 seconds. IN(AV) can produce 58 true optimal solutions within an average computational time of 632.76 seconds.

Using a variance reduction technique reduces the overall average computational time for each of the LTD bootstrapping for each different tolerance value and IN generation methods for the tolerance value of 0.001. For example, Table 13 shows that when no variance reduction technique is used, the LTD bootstrapping method (i.e. LTD(CMC)) gives solutions within an average computational time of 227.71 seconds. Along with employing a variance reduction technique, the average computational times are decreased to 111.05 seconds for LTD(AV) and 107.54 for LTD(CRN). When a variance reduction technique is used, the average computational time reduction is much smaller for the LTD bootstrapping method as compared to the IN generation method.

4.9 Extension to Other Inventory Systems

The sample average approximation technique was previously shown to be applicable for the optimization of the continuous review (r, q) inventory system. Two approaches were proposed to optimize r for a given q: 1) LTD bootstrapping and 2) IN generation. The proposed approaches lie in creating candidate solutions by enumerating possible values of q from a finite set. This set can be determined by appropriately setting bounds on q. The optimization can then be performed by selecting the policy (candidate solution) which provides the minimum total expected inventory costs. In this section, the applicability of these approaches to the other inventory models will be discussed. In addition, model specific approaches will also be discussed for the sake of the use of SAA. In addition, lead time bootstrapping method will be discussed in the context of application of SAA technique in optimizing policy parameters.

As far as the periodic review (s, S) inventory system is concerned inventory position is checked in constant intervals of length R. If inventory position is less than or equal to s then an order is placed in order to bring the inventory position to S. The periodic review (s, S) inventory model is often denoted by (R, s, S) where R is the review period. In this study, R is assumed to be given and can be discrete or continuous. The IN generation method is applicable for the periodic review (s, S) inventory system since an optimal s can be set for (R, s, S) by using the re-order point adjustment procedure, if we are given a fixed $\triangle = S - s$. The same optimization procedure introduced for the continuous review (r, Q) inventory model can be applied for the periodic review (s, S) inventory model. However, the cost function should appropriately be set according to the inventory model. Although the cost function associated with the expected inventory on-hand remains the same, the cost function related to the ordering cost should be updated. The following holds for the net inventory inventory position processes: $E[IN] = E[IP] - E[X^*]$ where X^* is the total demand during lead time and review period. The cost function for the periodic review (s, S) inventory model (for non-zero lead times) can be defined as follows.

$$TC\left(s\left(\triangle\right),S\right)=E\left[I\left(\left\{IN+X^{*}\right\}\in A\right)\left\{k+c\left(S-\left(IN+X^{*}\right)\right)\right\}+h\left[IN\right]^{+}\right] \tag{95}$$

where $I\left(x\in A\right)$ is the indicator function of set A. If x represents $IN+X^{*}$, then we define $A=\left\{IN+X^{*}<s\right\}$. The LTD bootstrapping method can not be applied for the periodic review (s, S) inventory system due to the undershoot occurrence. The available optimization models are based on modeling undershoot in the approximation sense. Hence, a proposed LTD based optimization model may not reveal the optimal solution in the long run. Thus, the LTD bootstrapping method can not be applied under the periodic review (s, S) inventory model in the context of the proposed methodology.

In what follows, the corresponding inventory optimization problem is defined. We first assume that the lead times are zero so that the net inventory and inventory position processes coincide. This assumption reveals the following optimization problem.

The continuous review (s, S) inventory model is a special case of the (R, s, S) inventory model if R is set equal to 0. Since we do not impose any restriction on the parameter R, the optimization procedure given for the (R, s, S) inventory model is directly applicable for the continuous review (s, S) inventory model. Therefore, no further explanation is given here. The objective function of P1 can be modified with the updated ordering cost expression. Then the corresponding objective function value can be represented as follows:

$$TC\left(s\left(\triangle\right),S\right)=\frac{kE[D]}{S-s}+hE\left[I\right] \tag{96}$$

where $s(\triangle)$ is the re-order point that satisfies the service level for a given fixed $\triangle$. The *IN* generation method is directly applicable based on the above defined objective function. In order to apply the LTD generation method, one needs to derive exact expressions for the performance measures of the expected value of on-hand inventory and ready ready. Based on the objective function (96) and new expressions for the performance measures P1 is redefined. Then the same optimization procedure is applied in the context of the LTD generation method.

For the (R, r, Q) inventory model, every R time units an order Q is given to bring the inventory position above r (if inventory position hits or falls below r). For the (R, r, NQ) inventory model, every R time units a multiple N of the order size Q is given to bring the inventory position above r (if inventory position falls below r). In order to apply the *IN* generation method, the objective function of the optimization problem can be expressed as follows:

$$TC(r(Q), Q) = E\left[I(\{IN + X^*\} \in A)\{k + c(S - (IN + X^*))\} + h[IN]^+\right] \tag{97}$$

where $I(x \in A)$ is the indicator function of set A and $A = \{x : x < r\}$. For the *IN* generation method, we follow the same optimization procedure given for the case of the continuous review (r, Q) inventory model.

As far as the continuous review (r, NQ) systems concerned, the inventory position is reviewed continuously. A multiple N of the order size Q is given to bring the inventory position above r whenever inventory position falls below r. In order to apply the *IN* generation method, the objective function of the optimization problem can be expressed as follows:

$$TC(r(Q), Q) = E\left[I(\{IN + X\} \in A)\{k + c(S - (IN + X))\} + h[IN]^+\right] \tag{98}$$

For the *IN* generation method, we follow the same optimization procedure given for the case of the continuous review (r, Q) inventory model.

The LTD bootstrapping approach can not be applied under the (R, r, Q), (r, NQ) and (R, r, NQ) inventory models in the context of the proposed methodology. The available optimization models are based on modeling undershoot in the approximation sense. Hence, a proposed LTD based optimization model may not reveal the optimal solution in the long run.

4.10 Conclusion and Future Research

Setting policy parameters of an inventory system plays a major role in an effective inventory management. In practice, an inventory manager first attempts to tackle with the uncertainty by assuming a distributional model for the stochastic lead time demand. Next, the inventory manager sets optimal policy parameters with respect to a desired customer service level. Setting optimal policy parameters often takes place in the literature as the solution of the stochastic constrained policy optimization problem. There are far too many example solution methods available in the literature. Most of the solution methods are predicated on analytical LTD models that are capable of capturing the uncertaintity to some extent. Unfortunately, such optimization procedures may not provide true optimal solutions for the cases where the true LTD model is different from the assumed one. Hence, the policy parameters are set inaccurately.

In this study, we propose an optimization procedure that does not rely on an explicit LTD model. The problem is to jointly optimize policy parameters of the continuous review (r, Q) inventory system with a service level constraint. The problem is modeled by describing the minimization of the total expected ordering and inventory holding costs for the inventory system where backorders are allowed. Since backorder costs are often hard to estimate, a service level constraint is imposed to control the number of backorders. The proposed optimization procedure utilizes the sample average approximation technique to estimate the expected total costs. In addition, the SAA technique provides many statistical tools that facilitate developing the optimization algorithm. We apply the SAA method to evaluate each candidate solution which is determined by a finite set of Q. This set is obtained by applying the distribution free bounds proposed in the literature. Therefore, the optimization procedure enumerates each Q in the finite set by using the SAA technique. Clearly, the efficiency of the proposed procedure is more or less dependent on the size of the foregoing finite set.

The LTD bootsrapping and IN generation are the two methods evaluated throughout the experiments in order to investigate their efficiency and quality in terms of producing true optimal solutions. The experiment results based on 64 test cases indicate that these two methods are able to produce more true optimal solutions than the A&S algorithm which can be used by assuming a LTD model. However, A&S algorithm is much more efficient than the LTD bootsrapping or IN generation method. The experiment results also state that LTD bootstrapping is a more efficient method than IN generation method. This is because the generation of IN requires a discrete event simulation of the underlying inventory system, which often takes more computational time than generating LTD values that requires an efficient bootstrapping method. In addition, using a variance reduction technique reduces the total computational time of the proposed optimization algorithm for each method. For the given 64 test cases, the average computation time for LTD bootstrapping method can be decreased to less than 1 minute when employing a variance reduction technique.

In terms of applicability to the different inventory systems, the proposed IN generation method is a much more generic method as compared to LTD bootstrapping method. In addition, from a simulation optimization perspective, it should be considered as one of the efficient methods due to the following reason. Only a single discrete-event simulation run is performed for a given Q. The existence of the distribution free bounds on Q for the continuous review (r, Q) inventory system gives rise to a joint optimization procedure in this paper. A similar approach is applicable as long as a finite search space is obtained through similar distribution free bounds for other inventory systems such as (s, S). Therefore, the future research is to investigate developing a similar optimization procedure for other inventory systems.

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A APPENDIX

<u>*Proof of Expression (4) and (5)*</u>: Let Y be a random variable that has a mixture distribution having k mixing distributions with cumulative distribution function $F_Y(y) = \sum_{i=1}^{k} q_i F_{W_i}(y)$ where $0 < q_i < 1$, $\sum_{i=1}^{k} q_i = 1$ $k \geq 2$ and $F_{W_i}(y)$ is the cumulative distribution function of a random variable W_i, $i = 1, \ldots, k$. Let h be the function of a given random variable, then clearly,

$$E[h(Y)] = \sum_{i=1}^{k} q_i E_{W_i}[h(W_i)] \tag{99}$$

where E is the expectation of a given function. Then we can prove (4) in the following way.

$$G_1^1(x) = \sum_{x\in\chi} x f_1(x) - \sum_{0\leq y<x} (1 - F_1(y))$$

$$G_2^1(x) = \sum_{x\in\chi} x f_2(x) - \sum_{0\leq y<x} (1 - F_2(y))$$

$$G_{MD}^1(x) = E_f[X] - \sum_{0\leq y<x} (1 - F_Y(y))$$

and by expression (99):

$$G_{MD}^1(x) = \left[(1-q)\sum_{x\in\chi} x f_1(x) + q \sum_{x\in\chi} x f_2(x)\right] - \sum_{0\leq y<x} (1 - F_Y(y))$$

$$= (1-q)\sum_{x\in\chi} x f_1(x) + q \sum_{x\in\chi} x f_2(x) - \sum_{0\leq y<x} 1 + \sum_{0\leq y<x} F_Y(y)$$

$$= (1-q)\sum_{x\in\chi} x f_1(x) + q \sum_{x\in\chi} x f_2(x) - \sum_{0\leq y<x} ((1-q)+q) + \sum_{0\leq y<x} [(1-q)F_1(y) + qF_2(y)]$$

$$= (1-q)\sum_{x\in\chi} x f_1(x) + q \sum_{x\in\chi} x f_2(x) - \sum_{0\leq y<x} (1-q) - \sum_{0\leq y<x} q + \sum_{0\leq y<x} (1-q)F_1(y) + \sum_{0\leq y<x} qF_2(y)$$

$$= (1-q)\left[\sum_{x\in\chi} x f_1(x) + \sum_{0\leq y<x} (1-F_1(y))\right] + q\left[\sum_{x\in\chi} x f_2(x) + \sum_{0\leq y<x} (1-F_2(y))\right]$$

$$= (1-q)\, G_1^1(x) + q G_2^1(x)_\square$$

We can prove (5) in the following way.

$$G_{MD}^2(x) = 0.5\left[E_f\left[X^2\right] - E_f[X]\right] - \sum_{0<y\leq x} G_{MD}^1(y)$$

and by expression (99):

$$\begin{aligned} G_{MD}^2(x) \;&=\; 0.5\left[(1-q)\sum_{x\in\chi} x^2 f_1(x) + q\sum_{x\in\chi} x^2 f_2(x) - (1-q)\sum_{x\in\chi} x f_1(x) - q\sum_{x\in\chi} x f_2(x)\right] \\ &\quad - \sum_{0<y\leq x}\left[(1-q)\, G_1^1(x) + q G_2^1(x)\right] \end{aligned}$$

$$\begin{aligned} =\;& 0.5(1-q)\sum_{x\in\chi} x^2 f_1(x) + 0.5q\sum_{x\in\chi} x^2 f_2(x) - 0.5(1-q)\sum_{x\in\chi} x f_1(x) - 0.5q\sum_{x\in\chi} x f_2(x) \\ & -(1-q)\sum_{0<y\leq x} G_1^1(x) - q\sum_{0<y\leq x} G_1^1(x) \end{aligned}$$

$$\begin{aligned} =\;& (1-q)\left[0.5\left[\sum_{x\in\chi} x^2 f_1(x) - \sum_{x\in\chi} x f_1(x)\right] - \sum_{0<y\leq x} G_1^1(x)\right] \\ & + q\left[0.5\left[\sum_{x\in\chi} x^2 f_2(x) - \sum_{x\in\chi} x f_2(x)\right] - \sum_{0<y\leq x} G_2^1(x)\right] \end{aligned}$$

$$= (1-q)\left[0.5\left[E_{f_1}\left[X^2\right] - E_{f_1}[X]\right] - \sum_{0<y\leq x} G_1^1(y)\right] + q\left[0.5\left[E_{f_2}\left[X^2\right] - E_{f_2}[X]\right] - \sum_{0<y\leq x} G_2^1(y)\right]$$

$$= (1-q)\, G_1^2(x) + q G_2^2(x)_\square$$

For the algorithm that generates test cases in analytical and simulation evaluation, the following notation is used.

Notation:

$\mathbb{Z}^+$: positive integers.

n: number of generated demand scenarios.

$\hat{p}$: determines the percentage of the data for cases in Group 1. $\hat{p}$ is set to 0.92 since approximately 92% of all test cases fall into Group 1.

w_L: element from the set of target ready rates.

Q_{min}: minimum re-order quantity.

Q_{max}: maximum re-order quantity.

Q_R: set of integer re-order quantity values such that $Q_R = \left\{Q_j \in \mathbb{Z}^+ | Q_{\min} \leq Q_j \leq Q_{\max}\right\}$.

$|Q_R|$: number of elements in the set of Q_R.

z: step size parameter used to determine the next enumerated re-order quantity from set Q_R.

m_q: input parameter used to determine the maximum number of re-order quantity values enumerated from Q_R. If $z = \left\lfloor \frac{|Q_R|}{m_q} \right\rfloor \leq 1$, then each re-order quantity value in Q_R is enumerated. Otherwise, m_q re-order quantity values are enumerated. The enumeration is performed by skipping z successive re-order quantity values in Q_R every time. We set m_q equal to 10 for convenience.

M: a large number.

ε: maximum tolerance between the desired ready rate and actually hit by the corresponding test case.

$r_{(w_L(Q))}$: computed re-order point with respect to corresponding w_L and Q.

$W\left(\mu, \sigma, r_{(w_L(Q))}, Q\right)$: achieved RR value under the gamma distribution with respect to the specified test case.

Algorithm A.1 Test Case Generation Algorithm for Analytical Evaluation

Set $\hat{p} = 0.92$ and $m_q = 10$
For $i = 1$ to n do
  Generate $u \sim U(0,1)$
  If $u < \hat{p}$ then
    Generate $(\mu,\sigma) \sim BVML$
  Else
    Generate $(\mu,\sigma) \sim BVMH$
  End-if
  For $W_L \in W$ do
    If $\mu - 3\sigma < 0$ then
      $Q_{min} = 1$
    Else
      $Q_{min} = \left\lceil (\mu - 3\sigma)^{0.5} \right\rceil$
    End-if
    $Q_{max} = max\left(Q_{min} + 1, \left\lceil (\mu - 3\sigma)^{0.5} \right\rceil\right)$, $z = \left\lfloor \frac{|Q_R|}{m_q} \right\rfloor$ and
    $Q_R = \left\{Q_j \in \mathbb{Z}^+ | Q_{min} \le Q_j \le Q_{max}\right\}$
    If $z \le 1$ then $\tau = 1$
    else $\tau = z$
    End-if
    For $j = 1$ to $|Q_R|$ do $Q = Q_j$ and $j = j + \tau$
      $r_{(w_L(Q))} = -Q$
      while $\left(r_{(w_L(Q))} < M\right)$ loop
        If $(W\left(\mu,\sigma,r_{(w_L(Q))},Q\right) - W_L \le \varepsilon)$ then
          Accept test case
          Set test case $\left(\mu,\sigma,r_{(w_L(Q))},Q\right)$
        Else
          Reject test case
        End if
        $r_{(w_L(Q))} = r_{(w_L(Q))} + 1$
      End-loop
    End-do
   End-if
 End-do
End-do

Algorithm A.2 Demand Generator

***Initialization*:**

$t \leftarrow$current time,

$E_I \leftarrow$demand event for the end of a period in OFF state,

$E_B \leftarrow$demand event for the end of a period in ON state,

$E_Y \leftarrow$demand arrival event

$(\mu_{NZ}, \sigma_{NZ}, \mu_B, \sigma_B, \mu_I, \sigma_I) \leftarrow$demand generator parameters set

Set $S(t)$ equal to 0 or 1

If $(S(t) = 0)$ then

    Generate $X_I \sim$gamma(fitted parameters of μ_I and σ_I)

    Schedule E_I at time $t = t + X_I$

End-if

If $(S(t) = 1)$ then

    Generate $X_B \sim$gamma(fitted parameters of μ_B and σ_B)

    Schedule E_B at time $t = t + X_B$

End-if

Generate $Y_i \sim$exponential(1)

Schedule E_Y at time $t = t + Y_i$

Event E_Y**:**

If $(S(t) = 1)$ then

    Generate $D_i \sim$gamma(fitted parameters of μ_{NZ} and σ_{NZ})(rounded up to nearest integer)

End-if

Generate $Y_i \sim$exponential(1)

Schedule E_Y at time $t = t + Y_i$

Event E_B**:**

Set $(S(t) = 0)$

Generate $X_I \sim$gamma(fitted parameters of μ_I and σ_I)

Schedule E_I at time $t = t + X_I$

Event E_I**:**

Set $(S(t) = 1)$

Generate $X_B \sim$gamma(fitted parameters of μ_B and σ_B)

Schedule E_B at time $t = t + X_B$

Algorithm A.3 Test Case Generation Algorithm for Simulation Evaluation

Set $\hat{p} = 0.92$ and $m_q = 10$
For $i = 1$ to n do
    Generate $u \sim U(0,1)$
    If $u < \hat{p}$ then
        Generate $(\mu_{NZ}, \sigma_{NZ}, \mu_B, \sigma_B, \mu_I, \sigma_I) \sim BVML$
    Else
        Generate $(\mu_{NZ}, \sigma_{NZ}, \mu_B, \sigma_B, \mu_I, \sigma_I) \sim BVMH$
    End-if
    $P_b = \frac{\mu_B}{\mu_B + \mu_I}$
    Generate $L \sim gamma\,(fitted\, parameters\, of\, \mu_L\, and\, \sigma_L)$
    $\mu_A = P_b L \mu_{NZ}$
    $\sigma_A = \sqrt{\lambda P_b L \left[\sigma_{NZ}^2 + (\mu_{NZ})^2\right]}$
    Determine μ and σ from simulation (through LTD capturer)
    For $W_L \in W$ do
        If $\mu_A - 3\sigma_A < 0$ then
            $Q_{min} = 1$
        Else
            $Q_{min} = \left\lceil (\mu_A - 3\sigma_A)^{0.5} \right\rceil$
        End-if
        $Q_{max} = max\left(Q_{min} + 1, \left\lceil (\mu_A - 3\sigma_A)^{0.5} \right\rceil\right)$, $z = \left\lfloor \frac{|Q_R|}{m_q} \right\rfloor$ and
        $Q_R = \left\{Q_j \in \mathbb{Z}^+ | Q_{min} \le Q_j \le Q_{max}\right\}$
        If $z \le 1$ then $\tau = 1$
        else $\tau = z$
        End-if
        For $j = 1$ to $|Q_R|$ do $Q = Q_j$ and $j = j + \tau$
            $r_{(w_L(Q))} = -Q$
            while $\left(r_{(w_L(Q))} < M\right)$ loop
                If ($W\left(\mu, \sigma, r_{(w_L(Q))}, Q\right) - W_L \le \varepsilon$) then
                    Accept test case
                    Set test case $\left(\mu, \sigma, r_{(w_L(Q))}, Q\right)$
                Else
                    Reject test case
                End if
                $r_{(w_L(Q))} = r_{(w_L(Q))} + 1$
            End-loop
        End-do
    End-do
End-do

Exhibit A.1: Adan *et al*. Distribution Selection Rule

$$a = \frac{\sigma^2 - \mu}{\mu^2}$$
$$\text{if } a < 0 \text{ then } F = MB$$
$$\text{else if } a > 0 \text{ and } a < 1 \text{ then } F = MNB$$
$$\text{else if } a = 0 \text{ then } F = P$$
$$\text{else (i.e. } a \geq 1) \text{ then } F = MG$$

Exhibit A.2: Axsäter's Distribution Selection Rule

$$VMR = \frac{\sigma^2}{\mu}$$
$$\text{if } VMR < 0.9 \text{ and } a < 0 \text{ then } F = MB$$
$$\text{else if } VMR < 0.9 \text{ and } a \geq 0 \text{ then } F = P$$
$$\text{else if } VMR \geq 0.9 \text{ and } VMR < 1.1 \text{ then } F = P$$
$$\text{else (i.e. if } VMR \geq 1.1) \text{ then } F = NB$$

Exhibit A.3: Gamma-Adan Distribution Selection Rule

$$\text{if } r > 0 \text{ then determine } F \text{ with } ADR$$
$$\text{else } F = G$$

Exhibit A.4: *MNNB* Distribution Selection Rule

$$\text{if } \mu < \sigma^2 \text{ then}$$
$$G^1_{MNNB}(x) = qG^1_N(x) + (1-q)\,G^1_{NB}(x)$$
$$G^2_{MNNB}(x) = qG^2_N(x) + (1-q)\,G^2_{NB}(x)$$
$$\text{else}$$
$$G^1_{MNNB}(x) = qG^1_N(x) + (1-q)\,G^1_G(x)$$
$$G^2_{MNNB}(x) = qG^2_N(x) + (1-q)\,G^2_G(x)$$

Exhibit A.5: *MGNBA* Distribution Selection Rule

$$\begin{gathered} if\ \mu < \sigma^2\ then \\ G^1_{MGNBA}(x) = qG^1_G(x) + (1-q)\,G^1_{NB}(x) \\ G^2_{MGNBA}(x) = qG^2_G(x) + (1-q)\,G^2_{NB}(x) \\ else \\ G^1_{MGNBA}(x) = qG^1_G(x) + (1-q)\,G^1_{ADR}(x) \\ G^2_{MGNBA}(x) = qG^2_G(x) + (1-q)\,G^2_{ADR}(x) \end{gathered}$$

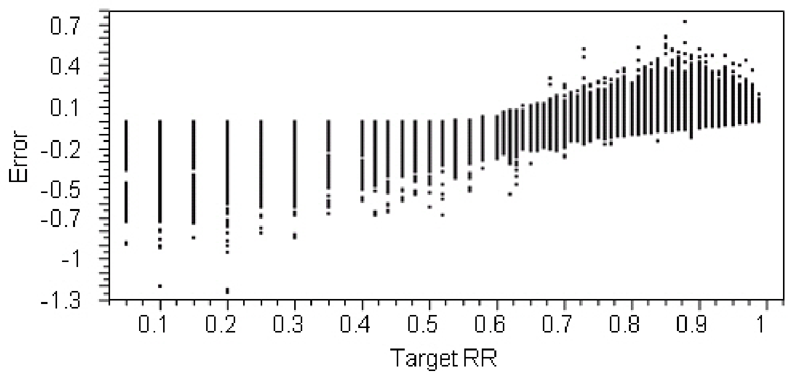

Figure A.6: Backorder Error versus Target RR When the LTD is Approximated by N

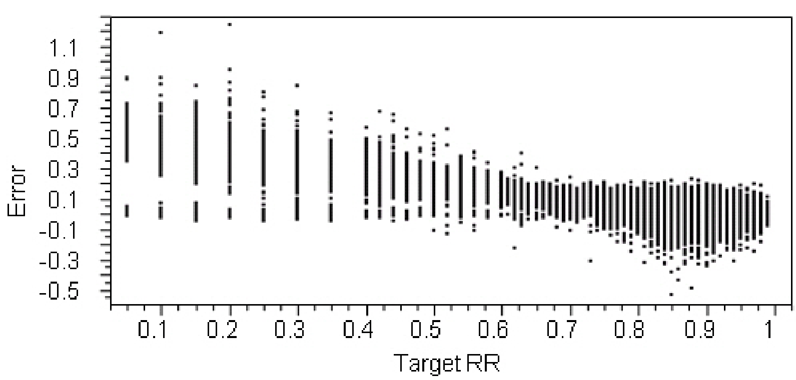

Figure A.7: Backorder Error versus Target RR when the LTD is Approximated by NB

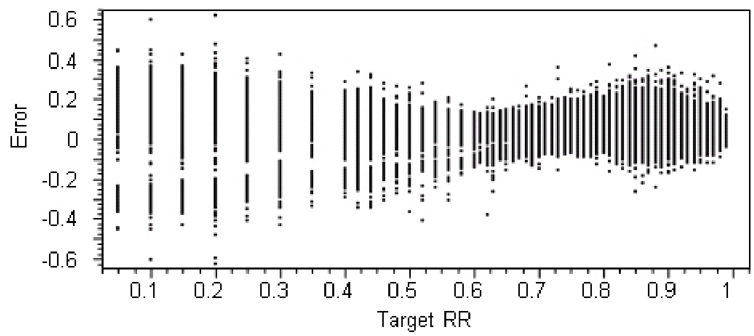

Figure A.8: Backorder Error versus Target RR when the LTD is Approximated by MNNB

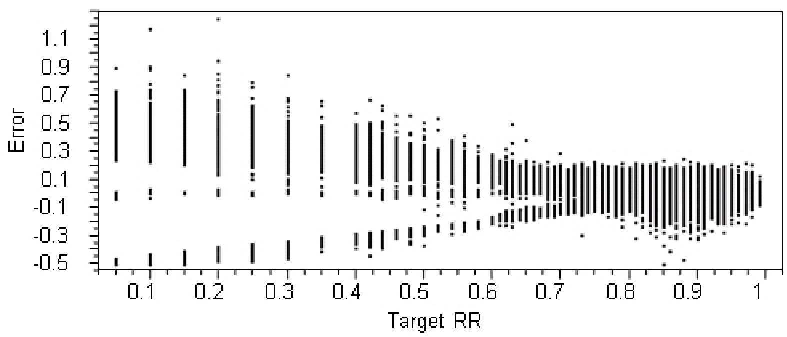

Figure A.9: Backorder Error versus Target RR when the LTD is Approximated by ADR

Table A.1: Error Types in Approximating the LTD

	Reality	
Decision	True LTD is F	True LTD is not F
F is picked	no error	type II error
F is not picked	type I error	no error

Table A.2: Statistical Summary of the Generated Data in Group 1 for Analytical Evaluation

Statistics	μ	σ	r	Q	w_L	CV
Mean	6.78	3.34	1	24	0.71	0.74
Std Dev	6.66	2.07	12	11	0.24	0.53
Min.	0.01	0.10	-37	1	0.05	0.13
25%ile	2.37	1.87	-6	15	0.58	0.44
Median	4.81	2.87	0	25	0.76	0.60
75%ile	8.96	4.29	6	34	0.91	0.85
Max.	139.67	30.43	137	40	0.99	0.85

Table A.3: Statistical Summary of the Generated Data in Group 2 for Analytical Evaluation

Statistics	μ	σ	r	Q	w_L	CV
$Mean$	3676.00	1458.70	4360	63	0.67	0.73
$Std Dev.$	5625.62	1231.47	6394	47	0.23	0.58
$Min.$	1.52	32.51	-79	1	0.05	0.06
$25\%ile$	826.41	664.19	945	27	0.54	0.39
$Median$	1940.98	1119.52	2410	54	0.71	0.58
$75\%ile$	4283.62	1847.97	5246	87	0.85	0.87
$Max.$	217780.96	19319.47	223750	547	0.99	22.40

Table A.4: For Group 1, Error Results (Type-I) in Analytical Evaluation

	G			LN			N			NB			P		
	B	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.540	0.980	0.896	0.515	0.974	0.896	0.501	0.969	0.892	0.502	0.980	0.888	0.504	0.997	0.890
PRE(.05)	0.412	0.956	0.828	0.378	0.947	0.826	0.359	0.921	0.824	0.345	0.958	0.821	0.367	0.991	0.828
PRE(.01)	0.320	0.827	0.590	0.256	0.763	0.556	0.237	0.708	0.575	0.232	0.829	0.548	0.299	0.940	0.589
Mean	0.122	-0.001	0.122	0.122	0.001	0.122	0.103	-0.005	0.103	-0.026	-0.002	-0.026	-0.100	0	-0.100
StdDev.	0.210	0.013	0.210	0.204	0.014	0.204	0.220	0.017	0.220	0.212	0.013	0.212	0.125	0.004	0.125
Min.	-0.279	-0.116	-0.279	-0.417	-0.105	-0.417	-0.475	-0.153	-0.475	-0.720	-0.117	-0.720	-0.502	-0.059	-0.502
25%ile	0	-0.001	0	0	0	0	-0.003	-0.006	-0.003	-0.119	-0.001	-0.119	-0.159	0	-0.159
Median	0.042	0	0.042	0.054	0	0.054	0.012	0	0.012	0	0	0	-0.047	0	-0.047
75%ile	0.174	0	0.174	0.172	0.003	0.172	0.165	0	0.165	0.015	0	0.015	0	0	0
Max.	3.341	0.114	3.341	3.179	0.153	3.179	3.299	0.108	3.299	3.158	0.111	3.158	0.004	0.043	0.004

Table A.5: For Group 2, Error Results (Type-I) in Analytical Evaluation

	G			LN			N			NB			P		
	B	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.630	0.703	0.712	0.479	0.644	0.589	0.335	0.459	0.616	0.630	0.703	0.710	0.462	0.891	0.724
PRE(.05)	0.557	0.631	0.660	0.316	0.546	0.499	0.202	0.314	0.548	0.557	0.630	0.658	0.448	0.884	0.719
PRE(.01)	0.437	0.465	0.503	0.083	0.130	0.281	0.035	0.085	0.124	0.439	0.465	0.497	0.405	0.872	0.708
Mean	73.301	-0.001	73.301	60.666	0.003	60.666	72.378	-0.050	72.378	71.107	-0.001	71.107	53.192	0.034	53.192
StdDev.	209.00	0.152	209.00	208.60	0.148	208.60	212.90	0.156	212.90	226.40	0.153	226.44	439.60	0.181	439.60
Min.	-541.9	-0.438	-541.9	-637.2	-0.523	-637.2	-567.6	-0.477	-567.6	-15138	-1.230	-15138	-0.508	-0.131	-0.508
25%ile	0	-0.020	0	-17.09	-0.027	-17.09	-21.36	-0.104	-21.36	-0.178	-0.020	-0.178	-0.017	0	-0.017
Median	0.417	0	0.417	2.387	0.012	2.387	12.137	-0.042	12.137	0.240	0	0.240	0	0	0
75%ile	45.274	0.002	45.274	42.660	0.032	42.660	80.332	0.005	80.332	45.052	0.002	45.052	0	0	0
Max.	15102	1.116	15102	15154	1.115	15154	14969	1.119	14969	4349.4	0.881	4349.4	18931	1.128	18931

Table A.6: For Group 1, Error Results (Type-II) in Analytical Evaluation (1)

	GADR			ADR			AXR			G		
	B	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.730	0.993	0.964	0.586	0.995	0.901	0.400	0.953	0.838	0.549	0.992	0.925
PRE(.05)	0.651	0.980	0.933	0.410	0.985	0.844	0.285	0.908	0.743	0.423	0.978	0.871
PRE(.01)	0.557	0.885	0.796	0.314	0.902	0.594	0.210	0.724	0.454	0.307	0.867	0.647
Mean	-0.036	-0.002	-0.036	0.079	0.002	0.079	0.187	-0.004	0.187	-0.072	-0.004	-0.072
StdDev	0.085	0.007	0.085	0.136	0.006	0.136	0.290	0.019	0.290	0.111	0.007	0.111
Min.	-0.470	-0.013	-0.470	-0.496	-0.090	-0.496	-0.477	-0.137	-0.477	-0.491	-0.113	-0.491
25%ile	-0.036	-0.001	-0.036	0	0	0	0.003	-0.004	0.003	-0.115	-0.008	-0.115
Median	0	0	0	0.033	0	0.033	0.101	0	0.101	-0.022	0	-0.022
75%ile	0	0.001	0	0.142	0.000	0.142	0.275	0	0.275	0	0.000	0
Max.	0.278	0.039	0.278	0.716	0.062	0.716	3.340	0.110	3.340	0.278	0.039	0.278

Table A.7: For Group 1, Error Results (Type-II) in Analytical Evaluation (2)

	LN			N			NB			P		
	B	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.545	0.982	0.924	0.549	0.984	0.917	0.465	0.991	0.877	0.273	0.941	0.801
PRE(.05)	0.420	0.962	0.871	0.411	0.953	0.865	0.223	0.975	0.802	0.130	0.882	0.680
PRE(.01)	0.296	0.809	0.628	0.279	0.771	0.639	0.096	0.850	0.469	0.035	0.643	0.331
Mean	-0.072	-0.002	-0.072	-0.054	0.003	-0.054	0.123	0.002	0.123	0.247	-0.005	0.247
StdDev	0.111	0.010	0.111	0.130	0.012	0.130	0.127	0.008	0.127	0.296	0.022	0.296
Min.	-0.489	-0.153	-0.489	-0.720	-0.052	-0.720	-0.299	-0.093	-0.299	-0.031	-0.137	-0.031
25%ile	-0.121	-0.002	-0.121	-0.101	0	-0.101	0.024	-0.005	0.024	0.059	-0.008	0.059
Median	-0.029	-0.000	-0.029	-0.008	0.002	-0.008	0.096	0	0.096	0.166	0	0.166
75%ile	0	0	0	0.006	0.003	0.006	0.181	0.001	0.181	0.334	0.001	0.334
Max.	0.416	0.052	0.416	0.475	0.153	0.475	0.720	0.059	0.720	3.340	0.110	3.340

Table A.8: For Group 2, Error Results (Type-II) in Analytical Evaluation (1)

	GADR			ADR			AXR			G		
	B	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.787	0.870	0.869	0.715	0.833	0.794	0.261	0.402	0.434	0.628	0.850	0.795
PRE(.05)	0.688	0.807	0.826	0.607	0.762	0.749	0.238	0.340	0.367	0.523	0.784	0.750
PRE(.01)	0.541	0.576	0.632	0.412	0.539	0.551	0.210	0.268	0.276	0.363	0.548	0.552
Mean	-3.446	-0.010	-3.446	53.297	0.031	53.297	189.72	-0.039	189.72	-3.485	-0.010	-3.485
StdDev	71.565	0.045	71.565	451.92	0.187	451.92	291.17	0.227	291.17	71.560	0.045	71.560
Min.	-15086	-1.451	-15086	-438.5	-0.546	-438.5	-10849	-1.654	-10849	-15086	-1.456	-15086
25%ile	-8.632	0	-8.632	-3.307	-0.001	-3.307	0	-0.200	0	-8.632	-0.001	-8.632
Median	-0.095	0	-0.095	0	0	0	87.438	-0.014	87.438	-0.162	0	-0.162
75%ile	0	0.001	0.000	1.566	0.006	1.566	267.27	0.000	267.27	0	0.001	0
Max.	541.9	0.347	541.9	18931.6	1.127	18931	6565	0.6357	6565	541.9	0.3472	541.9

Table A.9: For Group 2, Error Results (Type-II) in Analytical Evaluation (2)

	LN			N			NB			P		
	B	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.530	0.792	0.719	0.426	0.660	0.721	0.716	0.827	0.825	0.080	0.170	0.242
PRE(.05)	0.366	0.695	0.651	0.287	0.511	0.678	0.584	0.742	0.768	0.055	0.085	0.160
PRE(.01)	0.138	0.338	0.420	0.110	0.286	0.269	0.389	0.434	0.510	0.023	0.012	0.055
Mean	7.571	-0.013	7.571	-2.677	0.033	-2.677	-2.034	-0.013	-2.034	251.82	-0.025	251.82
StdDev	76.650	0.056	76.660	82.647	0.063	82.647	135.04	0.053	135.04	300.91	0.287	300.91
Min.	-15138	-1.457	-15138	-14953	-1.457	-14953	-445.1	-0.543	-445.1	-10849	-1.210	-10849
25%ile	-4.510	-0.029	-4.510	-16.010	0	-16.010	-16.45	-0.019	-16.47	64.503	-0.250	64.503
Median	0	-0.007	0	0	0.013	0	0.088	0	0.088	160.21	-0.100	160.21
75%ile	18.651	0	18.651	22.448	0.064	22.448	0.410	0.011	0.410	331.36	0.197	331.36
Max.	637.2	0.5234	637.2	567.6	0.4774	567.6	15138	1.119	15138	6565	0.8809	6565

Table A.10: Statistical Summary of Generated Cases within Group 1 for Simulation Evaluation

Statistics	μ_{NZ}	σ_{NZ}	μ_B	σ_B	μ_I	σ_I	μ	σ	r	Q	L	w_L	CV
Mean	1.83	1.26	2.33	2.15	1.78	1.43	8.42	4.43	7	22	8.33	0.82	0.75
Std Dev	0.53	0.63	0.60	0.92	0.32	0.55	7.22	2.40	12	12	6.43	0.17	0.51
Min.	1.11	0.25	1.23	0.37	1.19	0.38	0.07	0.37	-28	1	0.05	0.20	0.22
25%ile	1.49	0.82	1.90	1.51	1.55	1.03	3.44	2.79	0	12	3.63	0.74	0.47
Median	1.70	1.13	2.21	1.98	1.73	1.32	6.63	4.04	6	23	6.74	0.90	0.62
75%ile	2.02	1.53	2.63	2.57	1.95	1.73	11.24	5.57	13	32	11.32	0.95	0.85
Max.	9.13	6.88	5.53	8.31	3.66	4.66	61.60	21.28	80	40	44.85	0.99	8.73

Table A.11: Statistical Summary of Generated Cases within Group 2 for Simulation Evaluation

Statistics	μ_{NZ}	σ_{NZ}	μ_B	σ_B	μ_I	σ_I	μ	σ	r	Q	L	w_L	CV
Mean	1938.64	841.70	2.17	1.46	2.01	1.26	6548.50	3609.20	9769	77	8.10	0.77	0.91
Std Dev	4357.47	1356.76	1.67	2.09	0.89	1.06	14979.45	6914.34	20261	82	6.01	0.20	0.65
Min.	28.32	22.55	1.08	0.34	1.04	0.24	29.30	63.20	-46	1	0.05	0.20	0.28
25%ile	235.82	167.30	1.39	0.61	1.42	0.64	602.78	531.14	976	23	3.95	0.66	0.49
Median	606.78	375.27	1.79	0.93	1.72	0.93	1890.00	1357.04	2910	53	6.73	0.82	0.73
75%ile	1812.02	974.51	2.32	1.53	2.27	1.55	5076.02	3400.59	7968	106	10.65	0.94	1.11
Max.	38112.17	10243.04	20.38	22.19	6.16	7.53	136760.42	54478.69	146031	518	36.23	0.99	4.75

Table A.12: For Group 1, Error Results in Simulation Evaluation When the LTD is Approximated by Classic Distributions

	G			LN			N			NB			P		
	B	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.024	0.993	0.794	0.029	0.993	0.790	0.152	0.975	0.755	0.161	0.993	0.961	0.194	0.963	0.855
PRE(.05)	0.001	0.973	0.290	0.001	0.977	0.310	0.070	0.916	0.257	0.065	0.970	0.809	0.126	0.880	0.639
PRE(.01)	0	0.600	0	0	0.706	0	0.013	0.541	0	0.001	0.597	0.096	0.027	0.368	0.065
Mean	-0.15	0.010	0.807	-0.159	0.006	0.798	-0.097	0.011	0.860	-0.08	0.010	0.376	0.115	-0.008	0.572
StdDev	0.202	0.009	0.312	0.179	0.011	0.313	0.237	0.022	0.363	0.152	0.010	0.298	0.270	0.031	0.555
Min.	-3.628	-0.004	0.124	-3.411	-0.038	0.127	-4.230	-0.017	0.080	-3.237	0	0	-2.277	-0.166	0.005
25%ile	-0.195	0.004	0.627	-0.202	0.002	0.620	-0.140	-0.002	0.653	-0.093	0.004	0.189	0.007	-0.020	0.249
Median	-0.09	0.007	0.750	-0.112	0.005	0.735	-0.009	0.006	0.790	-0.038	0.007	0.299	0.040	-0.005	0.413
75%ile	-0.037	0.012	0.915	-0.057	0.009	0.895	0.008	0.016	0.985	-0.013	0.012	0.467	0.132	0.006	0.683
Max.	0	0.134	3.949	0	0.149	3.990	0.225	0.279	4.605	0	0.136	3.539	2.852	0.338	6.947

Table A.13: For Group 1, Error Results in Simulation Evaluation When the LTD is Approximated by Distribution Selection Rules

	ADR			GADR			AXR			MNNB			MGNBA		
	B	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.161	0.993	0.961	0.059	0.993	0.906	0.166	0.966	0.871	0.315	0.988	0.964	0.054	0.993	0.969
PRE(.05)	0.065	0.970	0.809	0.007	0.971	0.646	0.105	0.884	0.661	0.146	0.944	0.812	0.011	0.971	0.840
PRE(.01)	0.001	0.594	0.096	0	0.594	0.072	0.024	0.377	0.068	0.028	0.596	0.134	0	0.600	0.151
Mean	-0.08	0.010	0.377	-0.117	0.010	0.471	0.095	-0.006	0.552	-0.089	0.010	0.368	-0.115	0.010	0.342
StdDev	0.151	0.010	0.3	0.196	0.010	0.314	0.278	0.03	0.542	0.188	0.016	0.326	0.175	0.010	0.303
Min.	-3.268	-0.002	0	-3.628	-0.002	0.039	-2.277	-0.166	0	-3.734	-0.005	-0.188	-3.433	0	-0.188
25%ile	-0.092	0.004	0.189	-0.150	0.004	0.250	-0.002	-0.018	0.238	-0.108	0	0.172	-0.146	0.004	0.156
Median	-0.037	0.007	0.299	-0.047	0.007	0.407	0.029	-0.003	0.395	-0.026	0.006	0.294	-0.064	0.007	0.275
75%ile	-0.013	0.012	0.468	-0.015	0.012	0.607	0.123	0.008	0.657	-0.002	0.014	0.472	-0.026	0.012	0.439
Max.	0	0.139	3.559	0	0.134	3.559	2.852	0.338	6.947	0.028	0.207	3.822	0	0.135	3.494

Table A.14: For Group 2, Error Results in Simulation Evaluation When the LTD is Approximated by Classic Distributions

	G			LN			N			NB			P		
	B	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.31	0.89	0.63	0.42	0.87	0.51	0.30	0.56	0.66	0.31	0.89	0.64	0.05	0.26	0.26
PRE(.05)	0.12	0.82	0.42	0.22	0.67	0.41	0.13	0.43	0.33	0.13	0.82	0.43	0.03	0.14	0.14
PRE(.01)	0.03	0.47	0.09	0.06	0.26	0.14	0.03	0.18	0.02	0.04	0.47	0.09	0	0.03	0.01
Mean	-70.66	0.02	119.11	-51.43	0.02	138.33	-36.91	0.08	152.86	-70.53	0.02	118.74	584.08	0.01	773.34
StdDev	168.92	0.06	244.62	250.1	0.10	307.27	207.81	0.1	418.29	168.92	0.06	244.63	1808	0.32	2086
Min.	-1535	-0.03	-529.80	-2324.28	-0.07	-777.46	-1328.01	-0.02	-553.89	-1535.49	-0.03	-530.25	-301.47	-0.40	3.94
25%ile	-65.84	0	15.54	-53.95	-0.03	15.09	-44.89	0	10.75	-65.69	0	15.15	22.77	-0.19	70.02
Median	-20.73	0.01	47.49	-12.01	-0.01	47.58	0.01	0.05	47.94	-20.59	0.01	47.05	85.22	-0.08	172.71
75%ile	-8.12	0.02	114.29	0.47	0.01	129.31	18.25	0.12	151.84	-7.96	0.02	113.81	353.55	0.02	534.35
Max.	340.61	0.34	1414.01	1569.08	0.62	2421.59	1102.23	0.56	3969.12	340.80	0.34	1414.0	14257.00	0.98	15774.01

Table A.15: For Group 2, Error Results in Simulation Evaluation When the LTD is Approximated by Distribution Selection Rules

	ADR			GADR			AXR			MNNB			MGNBA		
	B	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.36	0.90	0.62	0.36	0.90	0.62	0.09	0.40	0.33	0.50	0.72	0.69	0.31	0.89	0.64
PRE(.05)	0.18	0.83	0.43	0.18	0.83	0.43	0.02	0.26	0.19	0.33	0.58	0.40	0.13	0.82	0.43
PRE(.01)	0.04	0.45	0.09	0.04	0.45	0.09	0	0.11	0.02	0.06	0.33	0.05	0.03	0.47	0.09
Mean	-66.93	0.03	122.34	-67.01	0.02	122.26	555.89	-0.02	745.16	-53.72	0.05	135.55	-70.59	0.02	118.67
StdDev	165.55	0.09	243.33	165.57	0.07	243.36	1816.03	0.23	2093.9	124.18	0.07	311.39	168.92	0.06	244.62
Min.	-1498	-0.04	-500.66	-1498	-0.04	-500.66	-330.37	-0.4	3.17	-1000.17	-0.01	-265.16	-1535.41	-0.03	-530.27
25%ile	-64	0	15.45	-64.01	0	15.42	-2.57	-0.18	47.79	-49.56	0	13.31	-65.76	0	15.09
Median	-19.86	0.01	52.36	-19.86	0.01	52.36	50.96	-0.04	118.48	-8.07	0.02	50.16	-20.66	0.01	47.02
75%ile	-4.94	0.02	116.83	-4.94	0.02	116.83	321.87	0.04	478.74	-0.55	0.07	123.29	-8.04	0.02	113.80
Max.	234.00	0.75	1403.12	234.23	0.39	1403.57	14257.72	0.63	15774.85	59.94	0.38	2649.52	340.70	0.34	1414.82

Table A.16: Comparisons for all Model Pairs using Tukey-Kramer HSD in Simulation Evaluation for Group 1

Models	Category 1	Category 2	Category 3	Category 4	Category 5	Mean
N	A					0.257
P		B				0.226
G		B				0.222
LN		B				0.215
AXR		B				0.213
GADR			C			0.121
ADR			C	D		0.102
NB			C	D		0.101
MNNB				D	E	0.096
MGNBA					E	0.078

Table A.17: Error Results (Type-II) in Analytical Evaluation for Poisson Model ($\mu \leq 1$, $\sigma^2 \leq 1$ and $0.9 \leq \sigma^2/\mu \leq 1.1$)

Statistics	*B*	*RR*	*I*
PRE(.10)	0.729	1.000	0.860
PRE(.05)	0.612	0.999	0.760
PRE(.01)	0.448	0.971	0.454
Mean	0.129	0.000	0.129
StdDev.	0.124	0.003	0.124
Min.	-0.001	-0.051	-0.001
25%ile	0.018	0.000	0.018
Median	0.091	0.000	0.091
75%ile	0.201	0.000	0.201
Max.	0.476	0.014	0.476

Table A.18: ($\mu \leq 1$, $\sigma^2 \leq 1$ and $0.9 \leq \sigma^2/\mu \leq 1.1$)

Statistics	*B*	*RR*	*I*
PRE(.10)	0.527	1.000	1.000
PRE(.05)	0.367	1.000	1.000
PRE(.01)	0.187	0.833	0.333
Mean	-0.019	0.004	0.178
StdDev.	0.041	0.003	0.088
Min.	-0.102	0.000	0.084
25%ile	-0.031	0.001	0.105
Median	-0.003	0.005	0.160
75%ile	0.001	0.006	0.255
Max.	0.002	0.006	0.321

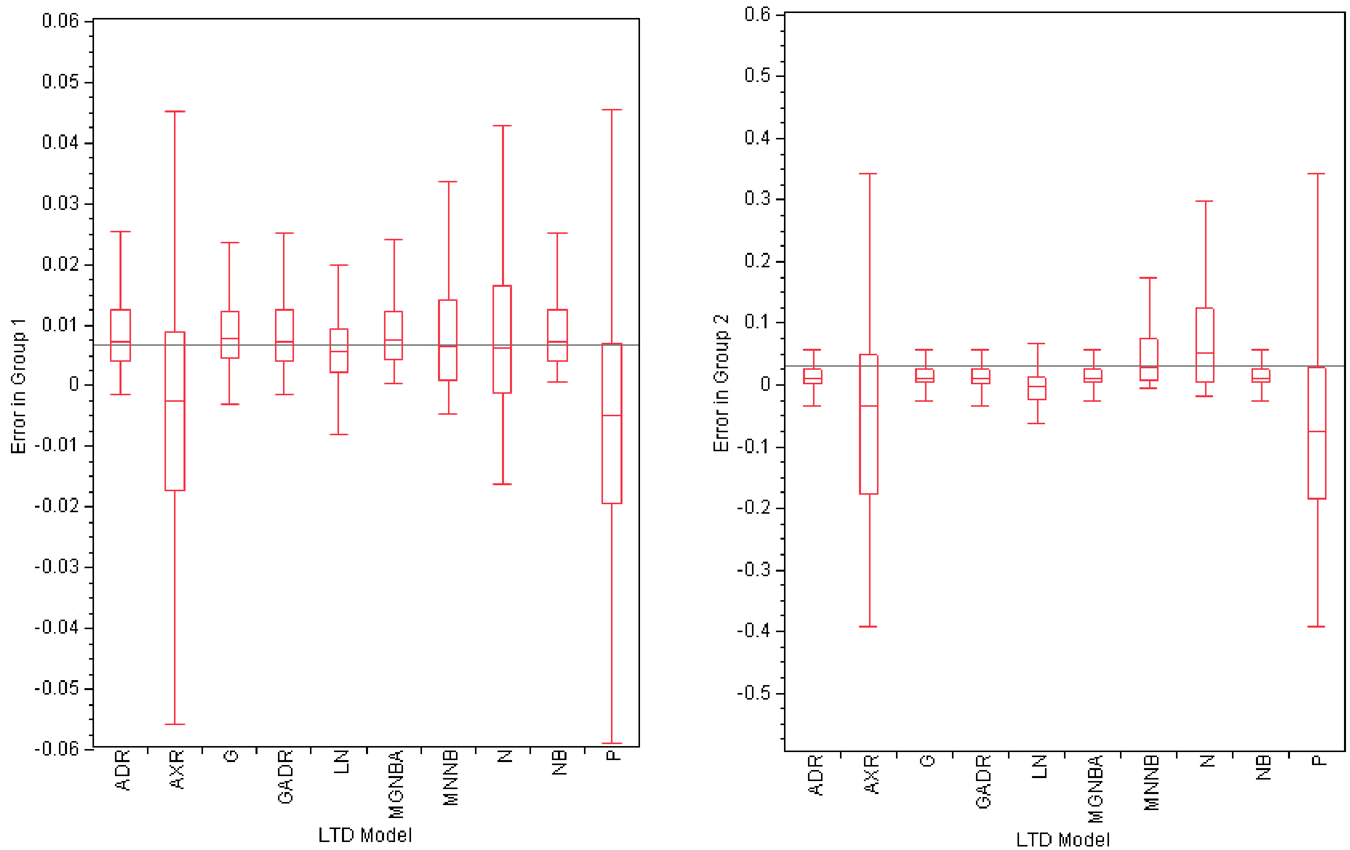

Figure A.10: Box Plots of RR Error Results in Simulation Evaluation for Group 1 and Group 2

Exhibit A.11: For Group 1, Hsu Individual 95% CIs For Mean Based on Pooled StDev

```
Level      N    Mean   StDev  ------+---------+---------+---------+---
ADR     6024  0.1024  0.2769       (*-)
AXR     6024  0.2138  0.4279                           (-*-)
G       6024  0.2220  0.4703                            (-*-)
GADR    6024  0.1212  0.3310          (-*-)
LN      6024  0.2150  0.4667                           (-*-)
MGNBA   6024  0.0786  0.2791  (-*-)
MNNB    6024  0.0965  0.2929     (-*-)
N       6024  0.2578  0.4959                                    (-*-)
NB      6024  0.1019  0.2763      (-*-)
P       6024  0.2262  0.4354                             (-*-)
                              ------+---------+---------+---------+---
                                  0.100     0.150     0.200     0.250
```

Exhibit A.12: For Group 1, Hsu Intervals for Level Mean Minus Smallest of Other Level Means

```
Level    Lower   Center   Upper  ------+---------+---------+---------+---
ADR     0.0000   0.0238  0.0408        (---*--)
AXR     0.0000   0.1352  0.1522        (----------------------*-)
G       0.0000   0.1434  0.1604        (-----------------------*--)
GADR    0.0000   0.0426  0.0596        (------*--)
LN      0.0000   0.1364  0.1534        (----------------------*--)
MGNBA  -0.0349  -0.0179  0.0000  (--*--)
MNNB    0.0000   0.0179  0.0349        (--*--)
N       0.0000   0.1792  0.1962        (-----------------------------*--)
NB      0.0000   0.0233  0.0402        (---*--)
P       0.0000   0.1476  0.1646        (------------------------*-)
                                 ------+---------+---------+---------+---
                                     0.000     0.060     0.120     0.180
```

Exhibit A.13: For Group 2, Hsu Individual 95% CIs For Mean Based on Pooled StDev

```
Level    N   Mean   StDev  ---+---------+---------+---------+------
ADR     525   18.5   186.9 (---*----)
AXR     525  433.7  1628.3                             (---*---)
G       525   16.2   188.4 (---*---)
GADR    525   18.4   186.9 (---*----)
LN      525   29.0   242.0  (---*---)
MGNBA   525   16.0   188.3 (---*---)
MNNB    525   27.3   209.0  (---*---)
N       525   38.7   281.4  (----*---)
NB      525   16.1   188.3 (---*---)
P       525  452.5  1624.7                              (---*---)
                           ---+---------+---------+---------+------
                              0       150       300       450
```

Exhibit A.14: For Group 2, Hsu Intervals for Level Mean Minus Smallest of Other Level Means

```
Level    Lower  Center  Upper  ------+---------+---------+---------+---
ADR     -109.7     2.4  114.6   (----*-----)
AXR        0.0   417.6  529.8        (--------------------*----)
G       -112.0     0.1  112.2  (-----*-----)
GADR    -109.7     2.4  114.5   (----*-----)
LN       -99.2    12.9  125.0   (-----*----)
MGNBA   -112.2    -0.0  112.1  (-----*-----)
MNNB    -100.9    11.3  123.4   (-----*----)
N        -89.5    22.6  134.7    (----*-----)
NB      -112.1     0.0  112.2  (-----*-----)
P          0.0   436.4  548.6        (---------------------*----)
                               ------+---------+---------+---------+---
                                     0       200       400       600
```

B APPENDIX

Proposition: Let G^1_{ZM} and G^2_{ZM} be the first and second order loss functions of a zero-modified distribution, respectively. In addition, let G^1_O and G^2_O be the first and second order loss functions of the unmodified probability distribution, respectively. Then, the following expressions hold for G^1_{ZM} and G^2_{ZM}:

$$G^1_{ZM}(x) = w + (1-w)\,G^1_O(x) \tag{100}$$

$$G^2_{ZM}(x) = w + (1-w)\,G^2_O(x) \tag{101}$$

where w is the parameter of mixing component and $x \in \chi$.

Proof: Let Y be a random variable that has a mixture distribution having k mixing distributions with cumulative distribution function $F(y) = \sum_{i=1}^{k} q_i F_{\omega_i}(y)$ where $0 < q_i < 1$, $\sum_{i=1}^{k} q_i = 1$ for $k \geq 2$ and $F_{\omega_i}(y)$ is the cumulative distribution function of a discrete random variable $\omega_i = 1, 2, ..., k$. Let h be the function of a given random variable, then clearly,

$$E_{MD}[h(Y)] = \sum_{i=1}^{k} q_i E_{\omega_i}[h(\omega_i)] \tag{102}$$

and thus,

$$E_{ZM}[X] = w + (1-w)\sum_{x\in\chi} x f_O(x) \tag{103}$$

where E_{MD} and E_{ZM} are the expectation of a given function under a mixture distribution and zero-modified distribution, respectively. In addition, $f_O(x)$ is the probability mass function of the original distribution. Then, we can prove (100) by the following way. Clearly, the following expressions hold due to the properties of loss functions:

$$G_O^1(x) = \sum_{x \in \chi} x f_O(x) - \sum_{0 \le y < x} (1 - F_O(y)) \tag{104}$$

$$G_{ZM}^1(x) = E_{ZM}[X] - \sum_{0 \le y < x} (1 - F_{ZM}(y)) \tag{105}$$

where F_O and F_{ZM} are the cumulative distributions of the original and zero-modified distributions, respectively. Then, by (103) and (105):

$$G_{ZM}^1(x) = \left[w + (1-w) \sum_{x \in \chi} x f_O(x) \right] - \sum_{0 \le y < x} (1 - F_{ZM}(y))$$

It follows that

$$G_{ZM}^1(x) = w + (1-w) \sum_{x \in \chi} x f_O(x) - \sum_{0 \le y < x} (1) + \sum_{0 \le y < x} (F_{ZM}(y))$$

$$G_{ZM}^1(x) = w + (1-w) \sum_{x \in \chi} x f_O(x) - \sum_{0 \le y < x} [w + (1-w)] + \sum_{0 \le y < x} (w + (1-w) F_O(y))$$

After some elaborations, we will obtain

$$G_{ZM}^1(x) = w + (1-w) \left[\sum_{x \in \chi} x f_O(x) - \sum_{0 \le y < x} (1 - F_O(y)) \right]$$

By expression (104), we show that

$$G_{ZM}^1(x) = w + (1-w) G_O^1(x)$$

We can prove (101) by the following way. Clearly, the following expressions hold due to the properties of loss functions:

$$G_{ZM}^{2}(x)=\frac{1}{2}\left(E_{ZM}\left[X^{2}\right]-E_{ZM}[X]\right)-\sum_{0<y\leq x}G_{ZM}^{1}(y) \tag{106}$$

Suppose that $f_D(x)$ is a probability mass function whose second order loss function $G_D^2(x)=1$ for $x\in\chi$. Suppose also that we construct a mixture distribution using $f_D(x)$ and f_O as follows:

$$G_{MD}^{2}(x)=wG_{D}^{2}(x)+(1-w)\,G_{O}^{2}(x) \tag{107}$$

Since $G_D(x)=1$ for $x\in\chi$, (107) will be equivalent to the zero-modified distribution. One can express the right hand side of (107) as follows:

$$=w\left[\frac{1}{2}\left(E_D\left[X^2\right]-E_D[X]\right)-\sum_{0<y\leq x}G_D^1(y)\right]+(1-w)\left[\frac{1}{2}\left(E_O\left[X^2\right]-E_O[X]\right)-\sum_{0<y\leq x}G_O^1(y)\right]$$

After some elaboration, it follows that

$$\begin{aligned}
= \; & \left[\left(\frac{1}{2}w\sum_{x\in\chi}x^2f_D(x)+\frac{1}{2}(1-w)\sum_{x\in\chi}x^2f_O(x)-\frac{1}{2}w\sum_{x\in\chi}xf_D(x)-\frac{1}{2}(1-w)\sum_{x\in\chi}xf_O(x)\right)\right] \\
& -\sum_{0<y\leq x}\left[wG_D^1(y)+(1-w)\,G_O^1(y)\right]
\end{aligned}$$

$$\begin{aligned}
= \; & \frac{1}{2}\left[\left(w\sum_{x\in\chi}x^2f_D(x)+(1-w)\sum_{x\in\chi}x^2f_O(x)\right)-\left(w\sum_{x\in\chi}xf_D(x)+(1-w)\sum_{x\in\chi}xf_O(x)\right)\right] \\
& -\sum_{0<y\leq x}\left[wG_D^1(y)+(1-w)\,G_O^1(y)\right]
\end{aligned}$$

$$\begin{aligned}
= \; & \frac{1}{2}\left[\left(wE_D\left[X^2\right]+(1-w)\,E_O\left[X^2\right]\right)-\left(wE_D[X]+(1-w)\,E[X]\right)\right] \\
& -\sum_{0<y\leq x}\left[wG_D^1(y)+(1-w)\,G_O^1(y)\right]
\end{aligned}$$

By expression (102), it follows that

$$= \frac{1}{2}\left(E_{ZM}\left[X^2\right] - E_{ZM}\left[X\right]\right) - \sum_{0<y\leq x} G^1_{ZM}(y)$$

By expression (106), the above resultant expression will be equivalent to G^2_{ZM} and this completes the proof □.

*<u>*Second Order Loss Function of the S-D Distribution*</u>:

Second order loss function (G^2_{SD}) of the S-D distribution can be derived as follows:

$$G^2_{SD}(r) = \frac{1}{2}\int_r^{\infty}(x-r)^2 f(x)\,dx$$

It follows that

$$\begin{aligned}\int_r^{\infty}(x-r)^2 f(x)\,dx &= \int_r^{\infty}\left(x^2-2xr+r^2\right)f(x)\,dx \\ &= \int_r^{\infty}x^2 f(x)\,dx - \int_r^{\infty}2xrf(x) + \int_r^{\infty}rf(x)\,dx\end{aligned}$$

Therefore, the second order loss function consists of 3 integrals such that:

$$G^2_{SD}(r) = \frac{1}{2}\left(A_{SD}-B_{SD}+C_{SD}\right)$$

<u>*First Integral*</u> (A_{SD}):

$$A_{SD} = \int_r^{\infty}x^2 f(x)\,dx$$

$$= \int_{F(r)}^{1}\left[F^{-1}(p)\right]^2 dF(x)$$

For $r \geq a$:

$$= \int_{F(r)}^{1}\left[a+b\left(F(x)-d\right)^c\right]^2 dF(x)$$

$$= \int_p^1\left[a+b\left(p-d\right)^c\right]^2 dp$$

$$= \int_p^1\left[a^2+2ab\left(p-d\right)^c+b^2\left(p-d\right)^{2c}\right]dp$$

$$= \left[a^2 p + 2ab\frac{(p-d)^{c+1}}{c+1} + b^2 \frac{(p-d)^{2c+1}}{2c+1} \right]_{F(r)}^{1}$$

Since $r \geq a$ and $F(r) \geq d$ then

$$p = F(r) = d + \left(\frac{r-a}{b} \right)^{1/c}$$

$$= a^2 \left[1 - \left[d + \frac{r-a}{b} \right]^{1/c} \right] + \frac{2ab}{c+1} \left[d + \left[\frac{r-a}{b} \right]^{1/c} - d \right]^{c+1} + \frac{b^2}{2c+1} \left[d + \left[\frac{r-a}{b} \right]^{1/c} - d \right]^{2c+1}$$

$$= a^2 \left[1 - \left[d + \frac{r-a}{b} \right]^{1/c} \right] + \frac{2ab}{c+1} \left[\frac{r-a}{b} \right]^{\frac{c+1}{c}} + \frac{b^2}{2c+1} \left[\frac{r-a}{b} \right]^{\frac{2c+1}{c}}$$

For $r \leq a$:

$$= \int_{F(r)}^{1} \left[a + b \left(d - F(x) \right)^c \right]^2 dF(x)$$

$$= \int_{p}^{1} \left[a^2 + 2ab (d-p)^c + b^2 (d-p)^{2c} \right] dp$$

$$= \left[a^2 p + 2ab\frac{(d-p)^{c+1}}{c+1} + b^2 \frac{(d-p)^{2c+1}}{2c+1} \right]_{F(r)}^{1}$$

Since $r < a$ and $F(r) < d$ then

$$p = F(r) = d - \left(\frac{a-r}{b} \right)^{1/c}$$

$$= a^2 \left[1 - \left[d - \frac{a-r}{b} \right]^{1/c} \right] + \frac{2ab}{c+1} \left[d + \left[\frac{a-r}{b} \right]^{1/c} - d \right]^{c+1} + \frac{b^2}{2c+1} \left[d + \left[\frac{a-r}{b} \right]^{1/c} - d \right]^{2c+1}$$

$$= a^2 \left[1 - \left[d - \frac{a-r}{b} \right]^{1/c} \right] + \frac{2ab}{c+1} \left[\frac{a-r}{b} \right]^{\frac{c+1}{c}} + \frac{b^2}{2c+1} \left[\frac{a-r}{b} \right]^{\frac{2c+1}{c}}$$

Thus, for $r \geq a$

$$A_{SD} = a^2 \left[1 - \left[d + \frac{r-a}{b} \right]^{1/c} \right] + \frac{2ab}{c+1} \left[\frac{r-a}{b} \right]^{\frac{c+1}{c}} + \frac{b^2}{2c+1} \left[\frac{r-a}{b} \right]^{\frac{2c+1}{c}}$$

and for $r < a$

$$A_{SD} = a^2 \left[1 - \left[d - \frac{a-r}{b} \right]^{1/c} \right] + \frac{2ab}{c+1} \left[\frac{a-r}{b} \right]^{\frac{c+1}{c}} + \frac{b^2}{2c+1} \left[\frac{a-r}{b} \right]^{\frac{2c+1}{c}}$$

<u>*Second Integral*</u> (B_{SD}):

$$B_{SD} = \int_r^\infty 2xrf(x)$$

This integral is calculated by as follows. If $r \geq a$, then equivalently $F(r) \geq d$.

Clearly, the following holds.

$$\int_r^\infty 2xrf(x) = \int_r^{B_2} 2xrf(x)$$

In this form, this integral represented by density function, it can also be represented by cdf $F(x)$ and it follows that

$$\int_r^{B_2} xf(x) = \int_{F(r)}^1 F^{-1}(F(x))\, dF(x)$$

$$= \int_{F(r)}^1 [a + b(F(x) - d)^c]\, dF(x)$$

$$= \int_{F(r)}^{1} a dF(x) + \int_{F(r)}^{1} b(F(x) - d)^c dF(x)$$

Since it is assumed that $F(r) \geq d$, then

$$F(x) = d + \left[\frac{x-a}{b}\right]^{1/c}$$

and

$$F(r) = d + \left[\frac{r-a}{b}\right]^{1/c}$$

It follows that

$$\int_{F(r)}^{1} a dF(x) + \int_{F(r)}^{1} b(F(x) - d)^c dF(x) = a[dF(x)]_{F(r)}^{1} + \left[\frac{b[F(x)-d]^{c+1}}{c+1}\right]_{F(r)}^{1}$$

$$= a\left[1 - \left(d + \left[\frac{r-a}{b}\right]^{1/c}\right)\right] + \frac{b}{c+1}\left[(1-d)^{c+1} - \left(d + \left[\frac{r-a}{b}\right]^{1/c} - d\right)\right]$$

$$= a\left[1 - \left(d + \left[\frac{r-a}{b}\right]^{1/c}\right)\right] + \frac{b}{c+1}\left[(1-d)^{c+1} - \left[\frac{r-a}{b}\right]^{\frac{c+1}{c}}\right]$$

For $r < a$, then equivalently $F(r) < d$.

Clearly, the following holds.

$$\int_{r}^{\infty} xf(x) = \int_{r}^{B_2} 2xrf(x)$$

and it follows that

$$\int_{r}^{B_2} 2xrf(x) = \int_{F(r)}^{1} F^{-1}(F(x))\, dF(x)$$

$$= \int_{F(r)}^{1} \left[a + b\left(d - F(x)\right)^{c}\right] dF(x)$$

$$= \int_{F(r)}^{1} adF(x) + \int_{F(r)}^{1} b\left(d - F(x)\right)^{c} dF(x)$$

Since it is assumed that $F(r) < d$, then

$$F(x) = d - \left[\frac{a-x}{b}\right]^{1/c}$$

and

$$F(r) = d - \left[\frac{a-r}{b}\right]^{1/c}$$

It follows that

$$\int_{F(r)}^{1} adF(x) + \int_{F(r)}^{1} b\left(F(x) - d\right)^{c} dF(x) = a\left[dF(x)\right]_{F(r)}^{1} + \left[\frac{b\left[d - F(x)\right]^{c+1}}{c+1}\right]_{F(r)}^{1}$$

$$= a\left[1 - \left(d - \left[\frac{a-r}{b}\right]^{1/c}\right)\right] + \frac{b}{c+1}\left[(1-d)^{c+1} - \left(d - \left(d - \left[\frac{a-r}{b}\right]^{1/c}\right)\right)\right]$$

$$= a\left[1 - \left(d - \left[\frac{r-a}{b}\right]^{1/c}\right)\right] + \frac{b}{c+1}\left[(1-d)^{c+1} - \left[\frac{a-r}{b}\right]^{\frac{c+1}{c}}\right]$$

Thus, for $r \geq a$:

$$B_{SD} = 2r\left[a\left[1 - \left(d + \left[\frac{r-a}{b}\right]^{1/c}\right)\right] + \frac{b}{c+1}\left[(1-d)^{c+1} - \left[\frac{r-a}{b}\right]^{\frac{c+1}{c}}\right]\right]$$

and for $r < a$:

$$B_{SD} = 2r\left[a\left[1 - \left(d - \left[\frac{r-a}{b}\right]^{1/c}\right)\right] + \frac{b}{c+1}\left[(1-d)^{c+1} - \left[\frac{a-r}{b}\right]^{\frac{c+1}{c}}\right]\right]$$

Third Integral (C_{SD}):

$$C_{SD} = \int_{r}^{\infty} r f(x)\, dx$$

Clearly, the following holds.

$$\int_{r}^{\infty} r f(x)\, dx = \int_{r}^{B_2} r f(x)\, dx$$

It follows that

$$\int_{r}^{B_2} r f(x)\, dx = [rF(x)]_{r}^{B_2} = r[F(B_2) - F(r)]$$

$$F(B_2) = 1$$

For $a \geq r$

$$F(r) = d + \left[\frac{a-r}{b}\right]^{1/c}$$

and

$$r[F(B_2) - F(r)] = r\left[1 - \left(d + \left[\frac{a-r}{b}\right]^{1/c}\right)\right]$$

For $a < r$

$$F(r) = d - \left[\frac{r-a}{b}\right]^{1/c}$$

and

$$r[F(B_2) - F(r)] = r\left[1 - \left(d - \left[\frac{r-a}{b}\right]^{1/c}\right)\right]$$

Thus, for $a \geq r$

$$C_{SD} = r\left[1 - \left(d + \left[\frac{a-r}{b}\right]^{1/c}\right)\right]$$

for $a < r$

$$C_{SD} = r\left[1 - \left(d - \left[\frac{r-a}{b}\right]^{1/c}\right)\right]$$

All in all, G^2_{SD} is expressed by using A_{SD}, B_{SD} and C_{SD} as follows:

For $r \geq a$

$$G^2_{SD}(r) = 0.5\left[\begin{array}{c} a^2\left[1-\left[d+\frac{r-a}{b}\right]^{1/c}\right]+\frac{2ab}{c+1}\left[\frac{r-a}{b}\right]^{\frac{c+1}{c}}+\frac{b^2}{2c+1}\left[\frac{r-a}{b}\right]^{\frac{2c+1}{c}} \\ -2r\left[a\left[1-\left(d+\left[\frac{r-a}{b}\right]^{1/c}\right)\right]+\frac{b}{c+1}\left[(1-d)^{c+1}-\left[\frac{r-a}{b}\right]^{\frac{c+1}{c}}\right]\right] \\ +r\left[1-\left(d+\left[\frac{a-r}{b}\right]^{1/c}\right)\right] \end{array}\right]$$

and for $r < a$

$$G^2_{SD}(r) = 0.5\left[\begin{array}{c} a^2\left[1-\left[d-\frac{a-r}{b}\right]^{1/c}\right]+\frac{2ab}{c+1}\left[\frac{a-r}{b}\right]^{\frac{c+1}{c}}+\frac{b^2}{2c+1}\left[\frac{a-r}{b}\right]^{\frac{2c+1}{c}} \\ -2r\left[a\left[1-\left(d-\left[\frac{a-r}{b}\right]^{1/c}\right)\right]+\frac{b}{c+1}\left[(1-d)^{c+1}-\left[\frac{a-r}{b}\right]^{\frac{c+1}{c}}\right]\right] \\ +r\left[1-\left(d-\left[\frac{r-a}{b}\right]^{1/c}\right)\right] \end{array}\right]$$

**Second Order Loss Function of the 4 Parameter Pearson Distribution*:

Second order loss function (G^2_{PD}) of the 4 parameter Pearson distribution can be derived as follows:

$$G^2_{PD}(R) = \frac{1}{2}\int_R^\infty (x-R)^2 f(x)\,dx$$

It follows that

$$\begin{aligned}\int_R^\infty (x-R)^2 f(x)\,dx &= \int_R^\infty \left(x^2 - 2xR + R^2\right) f(x)\,dx \\ &= \int_R^\infty x^2 f(x)\,dx - \int_R^\infty 2xRf(x) + \int_R^\infty R^2 f(x)\,dx\end{aligned}$$

Therefore, the second order loss function consists of 3 integrals such that:

$$G^2_{PD}(R) = \frac{1}{2}\left(A_{PD} - B_{PD} + C_{PD}\right)$$

First Integral (A_{PD}):

$$A_{PD} = \int_R^\infty x^2 f(x)\,dx$$

The following transformation is applied:

$$x = (b-a)t + a$$

Then,

$$x^2 = [(b-a)t + a]^2$$

It follows that

$$\int_R^\infty x^2 f(x)\,dx = \int_r^1 \left[(b-a)^2 t^2 + 2(b-a)ta + a^2\right] g_I(t)\,dt$$

$$= (b-a)^2 \int_r^1 t^2 g_I(t)\,dt + 2(b-a)a \int_r^1 t g_I(t)\,dt + a^2\left[1 - I_r(p,q)\right]$$

$$= (b-a)^2 \int_r^1 t^2 g_I(t)\,dt + 2(b-a)a\left(\frac{p}{p+q}\right) I_r(p+1,q) + a^2\left[1 - I_r(p,q)\right]$$

Notice that

$$\int_r^1 t^2 g_I(t)\,dt = \int_r^1 t^2 \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} t^{p-1}(1-t)^{q-1}\,dt$$

It follows that

$$= \frac{\Gamma(p+2)\Gamma(q)}{\Gamma(p+q+2)} \cdot \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_r^1 \frac{\Gamma(p+q+2)}{\Gamma(p+2)\Gamma(q)} t^{p+2-1}(1-t)^{q-1}\,dt$$

Thus,

$$A_{PD} = \frac{(b-a)^2 p(p+1)}{(p+q)(p+q+1)} I_r(p+2,q) + 2(b-a)a\left(\frac{p}{p+q}\right) I_r(p+1,q) + a^2\left[1 - I_r(p,q)\right]$$

<u>*Second Integral*</u> (B_{PD}):

$$B_{PD} = 2R\int_R^\infty x f(x)$$

By the same transformation

$$\int_R^\infty x f(x) = \int_r^1 \left[(b-a)t + a\right] g_I(t)\,dt$$

It follows that

$$= (b-a)\int_r^1 t g_I(t)\,dt + a\left[1 - I_r(p,q)\right]$$

Notice that

$$\int_r^1 t g_I(t)\,dt = \int_r^1 t \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} t^{p-1}(1-t)^{q-1}\,dt$$

It follows that

$$= \frac{\Gamma(p+1)\Gamma(q)}{\Gamma(p+q+1)} . \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_r^1 \frac{\Gamma(p+q+1)}{\Gamma(p+1)\Gamma(q)} t^{p}(1-t)^{q-1}\,dt$$

$$= \left(\frac{p}{p+q}\right) I_r(p+1,q)$$

Thus,

$$B_{PD} = 2R\left[(b-a)\left[\left(\frac{p}{p+q}\right) I_r(p+1,q)\right] + a\left[1 - I_r(p,q)\right]\right]$$

<u>*Third Integral*</u> (C_{PD}):

$$C_{PD} = \int_R^\infty R^2 f(x)\,dx$$

Clearly, the following holds under the same transformation:

$$\int_R^\infty f(x)\,dx = 1 - I_R(p,q)$$

Thus,

$$C_{PD} = R^2\left[1 - I_R(p,q)\right]$$

All in all, G^2_{PD} is expressed by using A_{PD}, B_{PD} and C_{PD} as follows:

$$G^2_{PD}(R) = 0.5\left[\begin{array}{c} \frac{(b-a)^2 p(p+1)}{(p+q)(p+q+1)} I_r(p+2,q) + 2(b-a)a\left(\frac{p}{p+q}\right) I_r(p+1,q) + a^2\left[1 - I_r(p,q)\right] \\ -2R\left[(b-a)\left[\left(\frac{p}{p+q}\right) I_r(p+1,q)\right] + a\left[1 - I_r(p,q)\right]\right] \\ +R^2\left[1 - I_R(p,q)\right] \end{array}\right]$$

Table B.1: For Case-I: Error Results When the LTD is Approximated by Distributions (1)

	G			GL			LN			N			NB			P		
Statistics	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.62	1.00	0.97	0.45	1.00	1.00	0.52	1.00	0.95	0.38	1.00	0.92	1.00	1.00	1.00	0.44	1.00	1.00
PRE(.05)	0.45	1.00	0.66	0.09	1.00	1.00	0.55	1.00	0.64	0.25	1.00	0.61	1.00	1.00	1.00	0.28	0.97	0.97
PRE(.01)	0.12	1.00	0.00	0.00	0.78	0.33	0.07	0.86	0.00	0.10	0.81	0.00	0.98	1.00	1.00	0.11	0.66	0.81
Mean	-0.038	0.000	0.462	0.001	-0.005	0.151	-0.039	-0.005	0.461	-0.008	0.004	0.493	0.000	0.000	0.000	0.063	-0.010	0.063
Std Dev	0.013	0.002	0.014	0.024	0.008	0.024	0.018	0.007	0.018	0.035	0.007	0.035	0.001	0.000	0.004	0.055	0.016	0.054
Min	-0.066	-0.004	0.434	-0.053	-0.034	0.097	-0.081	-0.032	0.419	-0.047	-0.021	0.444	-0.002	0.000	-0.012	0.004	-0.081	0.005
25%ile	-0.049	-0.001	0.452	-0.014	-0.008	0.136	-0.048	-0.006	0.451	-0.034	0.001	0.466	0.000	0.000	-0.001	0.018	-0.013	0.018
Median	-0.040	-0.001	0.461	-0.005	0.000	0.146	-0.040	-0.003	0.461	-0.012	0.003	0.488	0.000	0.000	0.000	0.052	-0.003	0.051
75%ile	-0.025	0.000	0.474	0.015	0.000	0.165	-0.025	0.000	0.474	0.006	0.007	0.507	0.000	0.000	0.002	0.087	0.000	0.088
Max	-0.019	0.006	0.497	0.084	0.002	0.236	0.000	0.007	0.501	0.137	0.026	0.632	0.003	0.000	0.021	0.260	0.000	0.255

Table B.2: For Case-I: Error Results When the LTD is Approximated by Distributions (2)

	PT			SD			TPM			ZIP			ZMG			ZMNB		
Statistics	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.38	1.00	1.00	0.47	1.00	0.98	0.30	1.00	1.00	0.05	1.00	1.00	0.00	1.00	1.00	0.17	1.00	1.00
PRE(.05)	0.38	0.80	0.68	0.19	1.00	0.92	0.14	1.00	1.00	0.00	0.89	0.97	0.00	0.95	0.97	0.03	0.81	1.00
PRE(.01)	0.00	0.38	0.38	0.16	0.97	0.22	0.00	0.72	0.97	0.00	0.42	0.83	0.00	0.72	0.73	0.00	0.27	0.86
Mean	-0.346	0.024	0.286	0.035	0.000	0.185	0.012	-0.008	0.012	0.042	-0.020	0.042	-0.073	0.009	-0.073	0.023	-0.024	0.023
Std Dev	0.409	0.021	0.277	0.100	0.003	0.099	0.017	0.006	0.017	0.075	0.020	0.075	0.090	0.016	0.090	0.042	0.025	0.042
Min	-1.419	0.000	-0.004	-0.062	-0.014	0.088	-0.010	-0.020	-0.012	-0.046	-0.091	-0.050	-0.352	-0.008	-0.352	-0.043	-0.087	-0.054
25%ile	-0.530	0.000	-0.003	-0.011	-0.001	0.139	-0.005	-0.013	-0.006	-0.020	-0.027	-0.023	-0.109	-0.001	-0.107	-0.014	-0.037	-0.013
Median	-0.215	0.025	0.277	0.006	0.000	0.156	0.012	-0.007	0.012	0.016	-0.013	0.017	-0.051	0.003	-0.051	0.027	-0.020	0.028
75%ile	-0.003	0.044	0.507	0.028	0.000	0.180	0.022	0.000	0.022	0.094	-0.005	0.093	-0.018	0.013	-0.018	0.049	-0.004	0.045
Max	-0.002	0.063	0.963	0.438	0.010	0.588	0.053	0.000	0.053	0.263	0.000	0.263	0.086	0.066	0.081	0.129	0.011	0.128

Table B.3: For Case-I, Error Results When the LTD is Approximated by Distribution Selection Rules (1)

	ADR			GADR			AXR			ZMADR			ZMADR2		
Statistics	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	1.00	1.00	1.00	0.56	1.00	1.00	0.98	1.00	1.00	0.14	1.00	0.72	0.83	1.00	1.00
PRE(.05)	1.00	1.00	1.00	0.55	1.00	1.00	0.70	1.00	1.00	0.02	0.95	0.50	0.75	1.00	1.00
PRE(.01)	0.92	1.00	1.00	0.48	0.88	0.95	0.67	1.00	0.89	0.00	0.39	0.44	0.52	0.97	1.00
Mean	0.000	0.000	0.000	0.020	-0.004	0.020	-0.004	0.000	0.041	0.024	-0.017	-0.651	0.005	-0.001	0.005
Std Dev	0.001	0.001	0.002	0.032	0.010	0.033	0.006	0.001	0.063	0.042	0.016	0.651	0.009	0.003	0.008
Min	-0.003	-0.006	-0.005	-0.002	-0.046	-0.012	-0.015	-0.004	-0.003	-0.140	-0.051	-1.549	0.000	-0.013	-0.001
25%ile	0.000	0.000	0.000	0.000	-0.003	-0.001	-0.007	0.000	0.000	0.014	-0.024	-1.330	0.000	-0.002	0.000
Median	0.000	0.000	0.000	0.004	0.000	0.006	0.000	0.000	0.000	0.022	-0.015	-0.640	0.001	0.000	0.001
75%ile	0.000	0.000	0.001	0.041	0.000	0.040	0.000	0.000	0.135	0.044	-0.003	0.012	0.009	0.000	0.008
Max	0.001	0.000	0.008	0.139	0.000	0.139	0.001	0.000	0.149	0.107	0.000	0.040	0.032	0.001	0.032

Table B.4: For Case-I, Error Results When the LTD is Approximated by Distribution Selection Rules (2)

	ZMADR2ADR			ZMADR2PT			MGNBA			MNNB		
Statistics	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	1.00	1.00	1.00	0.42	1.00	1.00	0.16	1.00	1.00	0.52	1.00	1.00
PRE(.05)	1.00	1.00	1.00	0.39	0.82	0.72	0.00	1.00	1.00	0.17	1.00	1.00
PRE(.01)	0.95	1.00	1.00	0.21	0.42	0.39	0.00	1.00	1.00	0.02	0.95	0.97
Mean	0.000	0.000	0.000	-0.002	0.000	-0.002	-0.019	0.000	-0.019	-0.004	0.002	-0.004
Std Dev	0.000	0.000	0.000	0.001	0.001	0.001	0.007	0.001	0.008	0.017	0.004	0.018
Min	-0.001	-0.001	-0.001	-0.004	-0.003	-0.004	-0.033	-0.002	-0.036	-0.024	-0.011	-0.034
25%ile	0.000	0.000	0.000	-0.003	-0.001	-0.003	-0.024	-0.001	-0.024	-0.017	0.001	-0.017
Median	0.000	0.000	0.000	-0.002	0.000	-0.002	-0.020	0.000	-0.019	-0.006	0.002	-0.006
75%ile	0.000	0.000	0.000	-0.001	0.000	-0.001	-0.013	0.000	-0.013	0.003	0.003	0.003
Max	0.000	0.000	0.002	0.000	0.000	0.000	-0.009	0.003	0.009	0.069	0.013	0.064

Table B.5: Statistical Summary of Generated Cases within Group 1

Statistics	μ_{NZ}	σ_{NZ}	CV_{NZ}^2	μ_B	σ_B	μ_I	σ_I	μ	σ	CV	r	Q	L	w_L
Mean	3.64	2.49	0.66	2.31	2.12	1.79	1.43	8.49	6.02	1.16	7.00	32.00	4.27	0.85
Std Dev	0.99	1.20	0.17	0.60	0.88	0.33	0.54	9.71	3.65	0.75	17.00	18.00	4.67	0.17
Min.	2.17	0.42	0.19	1.24	0.40	1.32	0.30	0.07	0.48	0.21	-45.00	1.00	0.05	0.20
25%ile	2.97	1.64	0.54	1.89	1.50	1.56	1.05	2.72	3.55	0.71	-2.00	19.00	1.44	0.76
Median	3.41	2.25	0.65	2.20	1.96	1.73	1.34	5.64	5.21	0.96	5.00	34.00	2.96	0.92
75%ile	4.04	3.05	0.76	2.59	2.55	1.96	1.72	9.70	7.50	1.36	13.00	48.00	4.73	0.97
Max.	18.41	14.26	1.52	7.66	10.67	4.55	5.43	110.93	37.52	7.43	135.00	60.00	48.48	1.00

Table B.6: Statistical Summary of Generated Cases within Group 2

Statistics	μ_{NZ}	σ_{NZ}	CV_{NZ}^2	μ_B	σ_B	μ_I	σ_I	μ	σ	CV	r	Q	L	w_L
Mean	2791.32	1420.31	0.71	2.24	1.54	2.00	1.32	4097.63	3474.02	1.45	7606	654	3.72	0.82
Std Dev	6407.94	3212.69	0.63	1.42	1.57	1.06	1.31	7511.23	5785.22	1.13	13601	646	4.12	0.18
Min.	39.53	37.03	0.03	1.04	0.32	1.32	0.23	10	33.91	0.25	-415	1	0.1	0.2
25%ile	434.28	233.25	0.31	1.43	0.65	1.38	0.60	388.6	577.64	0.77	572	234	1.23	0.72
Median	1135.34	573.45	0.54	1.81	1.02	1.69	0.91	1368.73	1598.06	1.13	2410	462	2.56	0.88
75%ile	2670.92	1420.18	0.86	2.58	1.73	2.22	1.52	4067.51	3575.96	1.69	8275	864	4.42	0.96
Max.	143810.23	47182.73	5.28	15.06	19.10	10.23	14.88	60557.3	79653.78	9.3	97408	7250	28.23	0.99

Table B.7: Statistical Summary of Generated Cases within Erratic

Statistics	μ_{NZ}	σ_{NZ}	CV_{NZ}^2	μ_B	σ_B	μ_I	σ_I	μ	σ	CV	r	Q	L	w_L
Mean	5.21	3.88	0.77	5.19	5.72	1.25	0.70	20.64	8.53	0.59	22.00	21.00	8.15	0.82
Std Dev	1.37	1.16	0.09	1.06	1.29	0.05	0.13	16.88	4.11	0.47	22.00	12.00	6.28	0.16
Min.	3.31	2.32	0.70	1.11	3.30	1.08	0.29	0.01	0.18	0.20	-29.00	1.00	0.01	0.20
25%ile	4.24	3.07	0.72	2.43	4.82	1.22	0.62	8.52	5.62	0.37	7.00	11.00	0.99	0.74
Median	4.86	3.60	0.75	4.86	5.47	1.26	0.70	16.09	7.87	0.49	17.00	22.00	6.53	0.87
75%ile	5.76	4.36	0.82	5.60	6.36	1.29	0.78	27.86	10.73	0.68	33.00	32.00	11.28	0.94
Max.	16.25	12.48	1.19	13.67	14.98	1.32	1.17	159.52	43.94	16.85	194.00	40.00	40.93	1.00

Table B.8: Statistical Summary of Generated Cases within Smooth

Statistics	μ_{NZ}	σ_{NZ}	CV^2_{NZ}	μ_B	σ_B	μ_I	σ_I	μ	σ	CV	r	Q	L	w_L
Mean	3.19	2.10	0.53	5.99	6.73	1.22	0.66	14.71	5.38	0.53	14.00	22.00	8.38	0.83
Std Dev	0.73	0.56	0.09	1.88	2.00	0.06	0.16	11.91	2.47	0.34	16.00	12.00	6.49	0.17
Min.	1.87	1.01	0.20	1.01	3.39	1.05	0.25	0.08	0.42	0.15	-30.00	1.00	0.05	0.20
25%ile	2.70	1.71	0.48	2.66	5.31	1.18	0.55	6.13	3.63	0.33	4.00	12.00	0.99	0.75
Median	3.11	2.07	0.55	5.43	6.18	1.23	0.66	11.65	5.03	0.44	11.00	23.00	6.73	0.90
75%ile	3.55	2.42	0.61	6.67	7.65	1.27	0.78	19.94	6.80	0.60	22.00	32.00	11.56	0.95
Max.	12.86	8.93	0.70	19.20	18.47	1.32	1.14	103.33	20.83	5.34	105.00	40.00	54.13	1.00

Table B.9: For 100 test cases and Fixed Lead-Time~Gamma(1.5,5.01), the Zero-Modified Distribution Selection Rules Results

		ADR			ZMP			ZMADR			ZMG			ZMNB			ZMADR2		
Statistics	$\hat{f}_0$	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	-	0.71	1.00	1.00	0.12	0.94	0.95	0.32	1.00	0.70	0.15	0.97	0.85	0.19	0.97	0.97	0.52	0.98	0.98
PRE(.05)	-	0.53	1.00	1.00	0.03	0.81	0.87	0.23	0.96	0.46	0.03	0.85	0.76	0.11	0.85	0.94	0.26	0.95	0.97
PRE(.01)	-	0.16	0.87	0.94	0.01	0.29	0.59	0.02	0.64	0.24	0.00	0.50	0.49	0.01	0.57	0.73	0.04	0.67	0.83
Mean	0.214	0.025	-0.004	0.025	0.144	-0.028	0.144	0.061	-0.010	-1.050	-0.529	0.016	-0.529	-0.001	-0.017	-0.002	0.055	-0.010	0.055
Std Dev	0.224	0.061	0.006	0.061	0.367	0.028	0.364	0.108	0.012	1.377	1.094	0.029	1.097	0.139	0.025	0.139	0.095	0.012	0.094
Min	0.000	-0.021	-0.027	-0.067	-0.380	-0.112	-0.361	-0.050	-0.058	-7.448	-7.073	-0.040	-7.086	-0.370	-0.095	-0.359	-0.015	-0.056	-0.023
25%ile	0.011	-0.001	-0.005	-0.003	0.000	-0.042	-0.006	0.002	-0.014	-1.442	-0.565	-0.001	-0.582	-0.070	-0.026	-0.077	0.003	-0.011	0.006
Median	0.154	0.004	-0.002	0.008	0.055	-0.018	0.057	0.017	-0.005	-0.569	-0.113	0.004	-0.116	0.000	-0.007	-0.003	0.024	-0.006	0.025
75%ile	0.352	0.025	0.000	0.031	0.178	-0.007	0.181	0.073	-0.001	-0.053	-0.036	0.028	-0.036	0.034	0.000	0.038	0.064	-0.002	0.062
Max	0.897	0.401	0.002	0.387	2.244	0.000	2.230	0.549	0.000	0.117	0.059	0.104	0.069	0.546	0.026	0.583	0.629	0.001	0.590

Table B.10: For 100 test cases and Fixed Lead-Time~Uniform(2, 5), the Zero-Modified Distribution Selection Rules Results

		ADR			ZMP			ZMADR			ZMG			ZMNB			ZMADR2		
Statistics	$\hat{f}_0$	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	-	0.74	1.00	1.00	0.13	0.94	0.95	0.45	1.00	0.96	0.23	1.00	0.98	0.29	0.99	0.98	0.56	1.00	0.97
PRE(.05)	-	0.52	1.00	1.00	0.03	0.85	0.93	0.34	0.99	0.84	0.08	1.00	0.97	0.10	0.94	0.95	0.44	0.99	0.89
PRE(.01)	-	0.18	0.88	0.93	0.01	0.37	0.69	0.07	0.87	0.34	0.02	0.86	0.78	0.01	0.55	0.79	0.17	0.89	0.30
Mean	0.315	0.013	-0.003	0.013	0.045	-0.026	0.046	0.018	-0.005	-0.288	-0.049	0.000	-0.049	-0.005	-0.010	-0.004	0.039	-0.008	0.037
Std Dev	0.113	0.033	0.004	0.037	0.180	0.030	0.181	0.040	0.007	0.262	0.068	0.006	0.071	0.089	0.019	0.091	0.061	0.008	0.064
Min	0.123	-0.010	-0.020	-0.067	-0.393	-0.157	-0.383	-0.034	-0.043	-0.952	-0.363	-0.021	-0.371	-0.352	-0.119	-0.360	-0.007	-0.036	-0.036
25%ile	0.238	0.000	-0.004	-0.005	-0.033	-0.036	-0.029	0.000	-0.007	-0.479	-0.072	-0.003	-0.074	-0.016	-0.017	-0.019	0.004	-0.012	0.001
Median	0.305	0.004	-0.001	0.007	0.015	-0.014	0.018	0.005	-0.002	-0.262	-0.030	0.001	-0.031	0.001	-0.004	0.007	0.014	-0.007	0.018
75%ile	0.392	0.013	0.000	0.025	0.092	-0.006	0.097	0.025	-0.001	-0.010	-0.010	0.003	-0.010	0.027	0.000	0.038	0.044	-0.002	0.043
Max	0.634	0.277	0.004	0.266	1.157	0.009	1.167	0.268	0.001	0.026	0.142	0.016	0.131	0.362	0.016	0.371	0.389	0.000	0.391

Table B.11: For 100 test cases and Fixed Lead-Time~Uniform(0.1,2), the Zero-Modified Distribution Selection Rules Results

		ADR			ZMP			ZMADR			ZMG			ZMNB			ZMADR2		
Statistics	$\hat{f}_0$	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	-	0.73	1.00	1.00	0.16	0.95	0.92	0.84	1.00	0.97	0.20	0.99	0.93	0.23	0.99	0.93	0.93	1.00	0.99
PRE(.05)	-	0.64	0.98	0.99	0.07	0.89	0.89	0.57	0.99	0.94	0.08	0.97	0.92	0.11	0.97	0.91	0.61	0.99	0.98
PRE(.01)	-	0.38	0.92	0.96	0.01	0.18	0.73	0.09	0.97	0.91	0.01	0.52	0.75	0.01	0.72	0.73	0.19	0.86	0.95
Mean	0.694	0.002	-0.002	0.000	-0.032	-0.021	-0.034	-0.011	-0.002	-0.014	-0.055	-0.009	-0.057	-0.055	-0.002	-0.057	0.012	-0.003	0.012
Std Dev	0.133	0.012	0.007	0.022	0.137	0.025	0.139	0.021	0.002	0.026	0.103	0.011	0.105	0.103	0.013	0.106	0.025	0.004	0.025
Min	0.437	-0.026	-0.044	-0.089	-0.434	-0.182	-0.435	-0.076	-0.020	-0.103	-0.381	-0.099	-0.388	-0.387	-0.075	-0.388	-0.010	-0.017	-0.008
25%ile	0.583	-0.002	-0.002	-0.014	-0.068	-0.022	-0.077	-0.016	-0.003	-0.019	-0.073	-0.013	-0.082	-0.083	-0.004	-0.091	0.001	-0.005	0.000
Median	0.684	0.000	0.000	-0.001	-0.010	-0.014	-0.017	-0.003	-0.001	-0.005	-0.016	-0.007	-0.021	-0.018	-0.001	-0.025	0.004	-0.002	0.004
75%ile	0.801	0.003	0.000	0.014	0.012	-0.009	0.012	0.001	-0.001	0.002	0.004	-0.003	0.008	0.001	0.003	0.003	0.013	0.000	0.013
Max	0.940	0.080	0.021	0.080	0.490	-0.002	0.490	0.014	0.001	0.019	0.072	0.010	0.072	0.174	0.043	0.174	0.181	0.004	0.185

Table B.12: For Case-II and Group 1: Error Results When the LTD is Approximated by Distributions (1)

	G			LN			N			NB			P			PT		
Statistics	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.12	0.87	0.48	0.11	0.88	0.48	0.11	0.85	0.47	0.09	0.87	0.54	0.08	0.89	0.53	0.35	0.97	0.98
PRE(.05)	0.08	0.72	0.32	0.07	0.74	0.32	0.08	0.72	0.32	0.04	0.72	0.37	0.04	0.78	0.38	0.29	0.96	0.97
PRE(.01)	0.02	0.36	0.1	0.02	0.35	0.1	0.02	0.44	0.1	0.01	0.36	0.08	0.01	0.51	0.09	0.15	0.87	0.62
Mean	-0.390	0.030	2.940	-0.100	0.030	2.940	-0.350	0.040	2.980	0.190	0.030	2.500	-0.160	0.020	2.800	0.240	0.000	0.450
Std Dev	0.940	0.060	3.980	50.130	0.060	3.970	0.960	0.070	4.020	62.190	0.060	3.980	0.750	0.070	21.670	317	0.030	317
Min	-25.400	-0.140	-1.630	-25.290	-0.200	-1.590	-25.650	-0.080	-1.150	-24.990	-0.130	-2.090	-24.000	-0.230	-1.530	-9142	0.000	-9142
25%ile	-0.390	0.000	0.450	-0.400	0.000	0.450	-0.300	0.000	0.450	-0.320	0.000	-0.010	-0.060	0.000	0.010	-0.030	0.000	0.000
Median	-0.070	0.010	2.030	-0.100	0.010	2.040	-0.020	0.000	2.050	-0.040	0.010	1.600	0.000	0.000	1.700	-0.010	0.000	0.130
75%ile	0.000	0.050	3.890	-0.010	0.040	3.890	0.000	0.050	3.950	0.000	0.050	3.450	0.020	0.030	3.670	0.000	0.000	0.250
Max	1.140	0.970	69.710	8543	0.980	69.670	1.100	0.950	70.410	7728	0.970	69.340	2.180	1.000	3628	9508	0.970	9508

Table B.13: For Case-II and Group 1: Error Results When the LTD is Approximated by Distributions (2)

	TPM			ZIP			ZMG			ZMNB			SD			GL		
Statistics	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.21	1	0.97	0.11	0.89	0.53	0.1	0.84	0.54	0.09	0.88	0.53	0.19	0.95	1	0.21	0.98	0.91
PRE(.05)	0.17	0.98	0.85	0.08	0.78	0.36	0.07	0.67	0.36	0.06	0.73	0.36	0.1	0.99	0.87	0.15	0.91	0.83
PRE(.01)	0.07	0.83	0.36	0.03	0.49	0.08	0.02	0.27	0.07	0.02	0.33	0.07	0.05	0.71	0.28	0.04	0.74	0.18
Mean	-0.040	0.000	0.520	-0.250	0.010	2.590	-0.500	0.040	2.610	-0.350	0.030	2.480	-0.040	0.010	0.500	-0.070	0.010	0.460
Std Dev	0.160	0.010	0.860	0.890	0.070	4.170	1.200	0.060	39.430	0.910	0.160	4.020	0.220	0.010	0.440	0.170	0.010	0.400
Min	-5.310	-0.060	-1.240	-26.480	-0.340	-5.940	-32.300	-0.410	-2.110	-26.550	-0.260	-17.150	-3.110	-0.010	-2.020	-3.450	-0.120	-2.930
25%ile	-0.010	0.000	-0.010	-0.090	-0.010	-0.040	-0.540	0.000	-0.050	-0.330	0.000	-0.090	-0.060	0.000	0.150	-0.070	0.000	0.140
Median	0.000	0.000	0.320	0.000	0.000	1.580	-0.100	0.010	1.540	-0.060	0.010	1.550	-0.010	0.000	0.450	-0.010	0.000	0.440
75%ile	0.000	0.000	0.720	0.010	0.010	3.590	0.000	0.060	3.330	0.000	0.040	3.460	0.010	0.010	0.760	0.000	0.010	0.720
Max	0.500	0.240	15.240	2.270	1.000	75.790	1.050	0.650	6458.000	1.540	0.320	69.100	1.510	0.060	5.050	0.980	0.080	3.540

Table B.14: For Case-II and Group 1, Error Results When the LTD is Approximated by Distribution Selection Rules (1)

	ADR			AXR			GADR			MGNBA			MNNB		
Statistics	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.18	0.97	0.86	0.09	0.88	0.53	0.24	0.99	0.92	0.11	0.87	0.53	0.1	0.86	0.53
PRE(.05)	0.11	0.9	0.61	0.05	0.77	0.37	0.16	0.94	0.7	0.07	0.72	0.36	0.06	0.72	0.36
PRE(.01)	0.02	0.57	0.22	0.01	0.47	0.08	0.06	0.64	0.28	0.02	0.36	0.08	0.01	0.4	0.08
Mean	-0.130	0.010	1.000	-0.210	0.030	2.620	-0.110	0.010	0.790	-0.130	0.030	2.480	-0.340	0.030	2.490
Std Dev	0.370	0.030	1.600	0.790	0.070	4.160	0.290	0.020	1.180	37.910	0.060	3.980	0.930	0.070	4.000
Min	-10.000	-0.060	-0.870	-24.000	-0.130	-2.090	-7.620	-0.050	-0.650	-25.200	-0.130	-2.110	-25.320	-0.090	-1.870
25%ile	-0.130	0.000	0.000	-0.160	0.000	-0.010	-0.110	0.000	0.090	-0.350	0.000	-0.030	-0.300	0.000	-0.030
Median	-0.020	0.000	0.640	0.000	0.000	1.640	-0.020	0.000	0.520	-0.060	0.010	1.570	-0.030	0.010	1.580
75%ile	0.000	0.020	1.380	0.000	0.040	3.620	0.000	0.010	1.060	0.000	0.050	3.430	0.000	0.050	3.450
Max	4.180	0.410	27.740	10.440	1.000	75.780	3.130	0.310	20.810	6458.000	0.970	69.270	1.170	0.960	69.630

Table B.15: For Case-II and Group 1, Error Results When the LTD is Approximated by Distribution Selection Rules (2)

	ZMADR			ZMADR2			ZMADR2ADR			ZMADR2PT		
Statistics	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.14	0.97	0.79	0.27	0.99	0.98	0.39	1	0.99	0.4	1	0.99
PRE(.05)	0.1	0.89	0.53	0.12	0.98	0.9	0.21	0.99	0.97	0.27	0.99	0.99
PRE(.01)	0.04	0.49	0.16	0.02	0.74	0.4	0.03	0.82	0.55	0.04	0.88	0.66
Mean	-0.170	0.010	1.650	-0.030	0.000	0.400	-0.020	0.000	0.230	0.000	0.000	0.200
Std Dev	0.460	0.030	48.830	0.150	0.010	0.590	0.090	0.010	0.360	2.740	0.010	2.760
Min	-13.270	-0.200	-1.050	-3.750	-0.070	-1.650	-2.250	-0.040	-0.990	-203.760	0.000	-203.720
25%ile	-0.160	0.000	-0.020	-0.040	0.000	0.040	-0.030	0.000	0.020	-0.020	0.000	0.010
Median	-0.030	0.000	0.780	0.000	0.000	0.270	0.000	0.000	0.150	0.000	0.000	0.110
75%ile	0.000	0.020	1.730	0.000	0.010	0.530	0.000	0.000	0.310	0.000	0.000	0.240
Max	5.440	0.820	6209.000	1.570	0.150	10.400	0.940	0.090	6.240	276.400	0.550	276.930

Table B.16: For Case-II and Group 2: Error Results When the LTD is Approximated by Distributions (1)

	G			LN			N			NB			P			PT		
Statistics	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.02	0.71	0.46	0.01	0.71	0.44	0.1	0.68	0.43	0.02	0.71	0.46	0.01	0.75	0.41	0.19	0.92	0.89
PRE(.05)	0.01	0.57	0.29	0	0.58	0.29	0.09	0.58	0.3	0.01	0.57	0.3	0.01	0.68	0.29	0.17	0.87	0.86
PRE(.01)	0	0.36	0.09	0	0.36	0.09	0.07	0.44	0.09	0	0.36	0.09	0.01	0.52	0.09	0.01	0.64	0.47
Mean	-70.110	0.050	356	-60.260	0.050	366	-70.460	0.090	356	-70.050	0.050	356	9.420	0.070	435	-113	0.040	-83.850
Std Dev	214.990	0.130	649	202	0.140	647	249	0.160	676	214	0.130	649	172	0.270	715	835	0.210	837
Min	-2400	-0.490	-487	-2382	-0.500	-503	-2584	-0.470	-1159	-2399	-0.490	-488	-2380	-0.680	-459	-2071	-0.030	-20387
25%ile	-62.470	0.000	-0.220	-66.870	0.000	6.480	-20.090	0.000	-8.550	-62.350	0.000	-0.720	0.000	-0.020	21.100	-14.040	0.000	-1.870
Median	-2.030	0.000	128	-5.730	0.010	132	0.000	0.000	120	-2.020	0.000	128	1.490	0.000	168.690	-0.800	0.000	4.890
75%ile	0.000	0.080	434	0.000	0.070	451	1.920	0.150	444	0.000	0.080	433	20.370	0.000	540.060	0.000	0.010	27.690
Max	499	0.780	8205	599	0.810	8174	536	0.760	8259	499	0.780	8205	759	1.000	8293	215	1.000	574

Table B.17: For Case-II and Group 2: Error Results When the LTD is Approximated by Distributions (2)

	TPM			ZIP			ZMG			ZMNB			SD			GL		
Statistics	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.02	0.93	0.74	0.01	0.76	0.41	0.06	0.65	0.4	0.02	0.68	0.43	0.005	0.899	0.152	0.047	0.848	0.145
PRE(.05)	0.02	0.88	0.59	0.01	0.68	0.29	0.06	0.52	0.26	0.01	0.56	0.29	0.001	0.268	0.093	0.043	0.842	0.077
PRE(.01)	0.02	0.69	0.29	0.01	0.52	0.09	0.05	0.34	0.08	0.01	0.36	0.09	0.001	0.102	0.002	0	0.149	0.023
Mean	-70.280	0.030	15.000	-353	0.110	72.800	-333	0.050	93.020	8.976	0.038	435	-476	0.110	-1718	447	0.150	-793
Std Dev	1009	0.450	1020	4923	1.220	4976	3291	0.120	3329	1933	0.707	2039	1839	0.120	2490	494	0.130	1018
Min	-23684	-0.140	-2608	-1101	-0.680	-1124	-7598	-0.400	-7221	-8217	-0.912	-7915	-6690	-0.240	-6960	-2629	-0.590	-4152
25%ile	0.000	-0.010	4.220	0.000	-0.040	21.110	-109	0.000	-6.740	-52.225	0.000	11.415	-837	0.000	-3364	0.320	0.000	-1646
Median	0.510	0.000	37.460	1.560	0.000	183	-6.690	0.000	124	-2.023	0.003	150	-837	0.100	-3364	881	0.000	-1646
75%ile	6.030	0.000	119	21.310	0.000	572	0.000	0.110	403	0.000	0.064	476	11.670	0.150	86.670	881	0.080	56.430
Max	265	1.000	1658	1327	1.000	8293	1034	0.490	6646	57044	0.895	57421	4874	0.690	10298	2553	0.890	6370

Table B.18: For Case-II and Group 2, Error Results When the LTD is Approximated by Distribution Selection Rules (1)

	ADR			AXR			GADR			MGNBA			MNNB		
Statistics	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.04	0.86	0.66	0.02	0.8	0.46	0.08	0.92	0.76	0.02	0.71	0.46	0.02	0.72	0.44
PRE(.05)	0.01	0.72	0.48	0.01	0.69	0.3	0.02	0.79	0.56	0.01	0.57	0.3	0.01	0.6	0.28
PRE(.01)	0	0.46	0.18	0	0.48	0.09	0.01	0.51	0.22	0	0.36	0.09	0	0.41	0.09
Mean	-26.500	0.020	144	-31.800	0.050	394	-19.920	0.010	107	-70.080	0.050	356	-70.250	0.070	356
Std Dev	85.370	0.060	259	181	0.200	706	64.040	0.040	194	214	0.130	649	228	0.140	661
Min	-962	-0.190	-199	-2380	-0.680	-488	-721	-0.150	-149	-2399	-0.490	-488	-2491	-0.480	-580
25%ile	-25.430	0.000	1.840	-6.770	-0.010	-0.720	-19.070	0.000	0.850	-62.410	0.000	-0.720	-39.770	0.000	-7.540
Median	-1.650	0.000	51.720	0.000	0.000	129	-1.240	0.000	38.790	-2.030	0.000	128	-0.740	0.010	124
75%ile	0.000	0.030	174	4.360	0.010	478	0.000	0.020	130	0.000	0.080	433	0.000	0.090	441
Max	236	0.310	3282	499	1.000	8293	177	0.230	2461	499	0.780	8205	430	0.770	8232

Table B.19: For Case-II and Group 2, Error Results When the LTD is Approximated by Distribution Selection Rules (2)

	ZMADR			ZMADR2			ZMADR2ADR			ZMADR2PT		
Statistics	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.08	0.84	0.61	0.1	0.98	0.87	0.29	0.99	0.96	0.29	0.95	0.9
PRE(.05)	0.06	0.69	0.4	0.04	0.89	0.69	0.05	0.96	0.85	0.08	0.91	0.85
PRE(.01)	0.04	0.45	0.14	0.03	0.62	0.35	0.01	0.66	0.44	0.01	0.65	0.47
Mean	-148	0.020	65.180	-4.670	0.000	59.280	-4.500	0.000	33.870	-39.880	-0.070	-10.690
Std Dev	1644	0.060	1672.480	29.460	0.020	98.460	21.000	0.010	57.880	346.660	1.330	345
Min	-3299	-0.260	-3110	-299	-0.100	-71.250	-216	-0.060	-42.750	-9762	-0.101	-9679
25%ile	-36.980	0.000	-3.370	-6.460	0.000	3.860	-5.260	0.000	2.100	-6.110	0.000	-0.190
Median	-2.020	0.000	65.510	-0.130	0.000	24.660	-0.090	0.000	13.680	-0.060	0.000	8.190
75%ile	0.000	0.040	222.200	0.100	0.010	72.040	0.050	0.010	40.500	0.030	0.010	29.390
Max	517	0.360	4102	163.030	0.110	1230	97.820	0.070	738	215	0.660	574

Table B.20: For Case-II and Erratic: Error Results When the LTD is Approximated by Distributions (1)

	G			GL			LN			N			NB			P		
Statistics	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*BO*	*RR*	*I*
PRE(.10)	0.74	0.97	0.79	0.13	0.68	0.37	0.74	0.93	0.68	0.85	0.95	0.88	0.96	0.98	0.96	0.17	0.56	0.35
PRE(.05)	0.46	0.95	0.5	0.11	0.56	0.3	0.55	0.81	0.43	0.49	0.8	0.59	0.9	0.96	0.89	0.09	0.39	0.21
PRE(.01)	0.03	0.52	0.01	0.01	0.3	0.07	0.16	0.28	0.01	0.04	0.32	0.04	0.55	0.66	0.63	0.03	0.15	0.04
Mean	-0.198	-0.004	0.301	-4.825	-0.022	-4.675	-0.142	-0.009	0.357	-0.238	0.009	0.261	0.015	-0.002	0.014	1.114	0.013	1.114
Std Dev	0.116	0.006	0.126	10.51	0.042	10.51	0.115	0.014	0.126	0.261	0.013	0.265	0.079	0.004	0.095	0.884	0.077	0.886
Min	-0.621	-0.052	-0.306	-37.53	-0.753	-37.39	-0.561	-0.075	-0.297	-1.237	-0.03	-0.770	-0.276	-0.049	-0.707	-0.061	-0.202	-0.504
25%ile	-0.262	-0.007	0.232	-10.97	-0.045	-10.81	-0.196	-0.017	0.293	-0.448	0.001	0.049	-0.022	-0.005	-0.029	0.465	-0.035	0.461
Median	-0.185	-0.003	0.308	-1.345	-0.015	-1.208	-0.140	-0.007	0.354	-0.191	0.007	0.297	0.002	-0.002	0.010	0.911	0.000	0.911
75%ile	-0.121	-0.000	0.376	-0.002	-0.000	0.1492	-0.084	-0.001	0.418	-0.023	0.016	0.476	0.042	0.000	0.050	1.544	0.064	1.535
Max	1.035	0.023	2.532	155.61	0.174	155.785	1.224	0.057	2.639	0.720	0.121	2.489	1.356	0.015	2.244	7.344	0.280	7.255

Table B.21: For Case-II and Erratic: Error Results Evaluation When the LTD is Approximated by Distributions (2)

	PT			SD			TPM			ZIP			ZMG			ZMNB		
Statistics	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.11	1.00	0.98	0.23	0.80	0.33	0.10	0.45	0.51	0.19	0.55	0.36	0.07	0.48	0.19	0.20	0.31	0.52
PRE(.05)	0.11	0.11	0.20	0.18	0.73	0.24	0.10	0.30	0.40	0.08	0.37	0.22	0.03	0.32	0.14	0.08	0.20	0.39
PRE(.01)	0.09	0.11	0.09	0.09	0.51	0.03	0.01	0.10	0.17	0.02	0.12	0.04	0.01	0.11	0.04	0.01	0.07	0.13
Mean	-137	0.034	0.415	-1.922	0.014	-1.772	1.211	-0.078	1.211	1.048	0.009	1.048	-6.223	-0.092	-6.223	1.313	-0.168	1.313
Std Dev	146	0.021	0.376	9.230	0.12	9.23	1.762	0.058	1.762	0.991	0.079	0.992	6.859	0.1404	6.859	2.683	0.138	2.684
Min	-558	-0.002	-0.056	-163	-2.469	-163	-1.01	-0.199	-0.998	-3.182	-0.208	-3.249	-55.105	-0.476	-55.07	-5.416	-0.602	-5.411
25%ile	-219	0.017	0.103	-2.054	-0.003	-1.901	0.155	-0.124	0.154	0.422	-0.041	0.426	-9.481	-0.188	-9.497	-0.074	-0.263	-0.087
Median	-81.774	0.037	0.331	0.019	0.000	0.172	0.601	-0.073	0.600	0.922	-0.002	0.923	-4.288	-0.045	-4.281	0.364	-0.142	0.352
75%ile	-16.562	0.053	0.659	1.241	0.006	1.395	1.564	-0.031	1.568	1.552	0.062	1.548	-1.285	0.009	-1.294	1.836	-0.053	1.864
Max	0.0062	0.068	1.872	25.256	0.640	25.414	18.767	0.129	18.774	7.344	0.261	7.255	22.960	0.146	22.853	21.216	0.232	21.280

Table B.22: For Case-II and Erratic, Error Results When the LTD is Approximated by Distribution Selection Rules (1)

	ADR			AXR			GADR			MGNBA			MNNB		
Statistics	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.99	1.00	0.99	0.19	0.56	0.35	1.00	1.00	0.99	0.88	0.98	0.89	0.96	0.98	0.80
PRE(.05)	0.98	0.99	0.97	0.11	0.40	0.22	0.98	0.99	0.97	0.71	0.95	0.81	0.85	0.95	0.70
PRE(.01)	0.81	0.89	0.79	0.04	0.16	0.06	0.77	0.93	0.76	0.22	0.58	0.36	0.14	0.57	0.42
Mean	0.0059	-0.001	0.0058	1.1086	0.0138	1.1084	-0.0029	-0.0007	0.0127	-0.0918	-0.0033	-0.092	-0.1116	0.0037	-0.1118
Std Dev	0.0314	0.0019	0.0382	0.8902	0.0767	0.892	0.0343	0.0015	0.0383	0.0814	0.0056	0.0968	0.1318	0.0061	0.1411
Min	-0.1105	-0.0199	-0.2832	-0.0611	-0.2023	-0.5046	-0.1757	-0.0149	-0.1778	-0.4256	-0.0513	-0.757	-0.7013	-0.0392	-0.7551
25%ile	-0.0086	-0.002	-0.0117	0.458	-0.0335	0.4554	-0.011	-0.0015	-0.0075	-0.1262	-0.0063	-0.1365	-0.2013	-0.0002	-0.2072
Median	0.0008	-0.0008	0.004	0.9112	0.0007	0.911	-0.001	-0.0006	0.0057	-0.0835	-0.0028	-0.0919	-0.0897	0.003	-0.1022
75%ile	0.0167	0.0002	0.0199	1.5443	0.0641	1.535	0.012	0.0001	0.0224	-0.0529	0.0002	-0.0518	-0.0112	0.0069	-0.0125
Max	0.5421	0.0057	0.897	7.3442	0.2802	7.2551	0.4066	0.0045	0.6727	1.1961	0.0189	2.1382	1.0385	0.0511	2.1166

Table B.23: For Case-II and Erratic, Error Results When the LTD is Approximated by Distribution Selection Rules (2)

	ZMADR			ZMADR2			ZMADR2ADR			ZMADR2PT		
Statistics	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.03	0.22	0.05	1.00	1.00	1.00	1.00	1.00	1.00	0.12	1.00	0.98
PRE(.05)	0.02	0.11	0.04	1.00	1.00	0.99	1.00	1.00	1.00	0.12	0.13	0.21
PRE(.01)	0.00	0.03	0.03	0.91	0.96	0.91	0.96	0.98	0.97	0.10	0.12	0.10
Mean	3.2026	-0.2031	-12.6968	0.0045	-0.0007	0.0044	0.0015	-0.0002	0.0015	-133.0086	0.0341	0.4112
Std Dev	4.3402	0.1345	10.8791	0.0152	0.0014	0.0172	0.0074	0.0006	0.0089	144.4777	0.0214	0.3789
Min	-0.4506	-0.4997	-101.1715	-0.0414	-0.0175	-0.0901	-0.0249	-0.0076	-0.0637	-558.5728	-0.0059	-0.0564
25%ile	0.5866	-0.3103	-17.9361	-0.0025	-0.001	-0.0034	-0.0019	-0.0005	-0.0026	-210.9927	0.0163	0.0945
Median	1.6816	-0.1874	-9.8868	0.0021	-0.0005	0.0028	0.0002	-0.0002	0.001	-79.3061	0.0367	0.3269
75%ile	4.1271	-0.089	-4.8111	0.0088	0	0.0099	0.0039	0	0.0046	-14.2931	0.0533	0.6593
Max	46.9174	0.0546	7.8687	0.2584	0.0021	0.3368	0.122	0.0013	0.2018	0.042	0.0685	1.8382

Table B.24: For Case-II and Smooth: Error Results When the LTD is Approximated by Distributions (1)

	G			GL			LN			N			NB			P		
Statistics	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.46	0.99	0.86	0.20	0.85	0.61	0.47	0.97	0.82	0.68	0.98	0.92	0.95	0.99	0.98	0.28	0.86	0.68
PRE(.05)	0.17	0.97	0.49	0.16	0.77	0.52	0.22	0.92	0.49	0.25	0.94	0.51	0.87	0.98	0.96	0.15	0.74	0.55
PRE(.01)	0.00	0.74	0.00	0.01	0.57	0.12	0.01	0.57	0.00	0.02	0.64	0.01	0.48	0.85	0.78	0.04	0.37	0.22
Mean	-0.1703	-0.0027	0.3291	-2.0547	-0.0086	-1.9048	-0.1628	-0.0065	0.3367	-0.1616	0.0043	0.3378	-0.0002	-0.0011	-0.0008	0.374	-0.0037	0.3734
Std Dev	0.11	0.0055	0.1186	6.8367	0.041	6.8367	0.0939	0.0105	0.1043	0.163	0.0085	0.1686	0.0418	0.0036	0.0663	0.3908	0.0353	0.3951
Min	-0.6119	-0.0593	-1.471	-46.6187	-0.7751	-46.4497	-0.5776	-0.0789	-1.5014	-0.7124	-0.0196	-1.4279	-0.1797	-0.0539	-1.9574	-0.0764	-0.1193	-1.9205
25%ile	-0.2331	-0.005	0.2619	-0.496	-0.014	-0.3516	-0.2096	-0.0104	0.2833	-0.2501	-0.0005	0.2445	-0.016	-0.0027	-0.0289	0.0947	-0.0191	0.0998
Median	-0.1513	-0.0017	0.3429	-0.0375	-0.0004	0.1144	-0.1497	-0.0042	0.3449	-0.1215	0.0028	0.3735	-0.0029	-0.0007	-0.0015	0.2386	-0.0032	0.2418
75%ile	-0.085	0.0005	0.4113	0.1709	0.0005	0.3207	-0.0971	-0.0004	0.401	-0.0299	0.0074	0.467	0.0116	0.0009	0.0255	0.5451	0.0031	0.5407
Max	0.4591	0.0267	0.9044	109	0.3039	109	0.5355	0.0543	0.9808	0.4561	0.0699	0.9014	0.6701	0.0272	0.6154	3.794	0.1875	3.7796

Table B.25: For Case-II and Smooth: Error Results When the LTD is Approximated by Distributions (2)

	PT			SD			TPM			ZIP			ZMG			ZMNB		
Statistics	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.22	1.00	0.98	0.33	0.93	0.51	0.16	0.64	0.70	0.24	0.85	0.67	0.08	0.61	0.29	0.15	0.41	0.64
PRE(.05)	0.22	0.20	0.24	0.29	0.90	0.43	0.15	0.49	0.61	0.08	0.73	0.54	0.04	0.40	0.22	0.06	0.29	0.53
PRE(.01)	0.17	0.17	0.15	0.14	0.73	0.05	0.02	0.22	0.36	0.01	0.32	0.19	0.00	0.16	0.07	0.01	0.14	0.23
Mean	-25.737	0.0379	0.4007	0.2254	0.0061	0.3752	0.4794	-0.0532	0.4793	0.3183	-0.0057	0.3178	-4.0351	-0.035	-4.0356	0.7798	-0.1695	0.7792
Std Dev	30.097	0.023	0.3513	5.8635	0.06	5.8634	0.8874	0.0512	0.8876	0.4602	0.0364	0.4628	4.9263	0.1224	4.926	2.1361	0.1706	2.1372
Min	-112	-0.0007	-0.0343	-83.986	-0.718	-83.831	-0.9704	-0.1998	-0.9684	-2.1492	-0.1734	-2.1389	-33.502	-0.4961	-33.4836	-4.7417	-0.7605	-4.7196
25%ile	-42.305	0.02	0.0925	-0.0375	-0.0016	0.114	0.0232	-0.0871	0.0233	0.0545	-0.0215	0.0457	-6.1207	-0.086	-6.1338	-0.1164	-0.276	-0.1227
Median	-13.031	0.0438	0.3465	0.0898	0	0.2383	0.1601	-0.0397	0.1587	0.2291	-0.0058	0.2288	-2.5172	0.0008	-2.5311	0.0969	-0.1139	0.0938
75%ile	-0.2842	0.058	0.6277	1.3725	0.0023	1.5211	0.5898	-0.0109	0.5909	0.5489	0.0025	0.5434	-0.4262	0.0402	-0.4129	0.9015	-0.025	0.8962
Max	-0.0001	0.0692	1.568	48.907	0.6519	49.058	13.7779	0.2008	13.795	3.797	0.1874	3.7827	16.717	0.1953	16.739	27.994	0.5899	28.080

Table B.26: For Case-II and Smooth, Error Results When the LTD is Approximated by Distribution Selection Rules (1)

	ADR			AXR			GADR			MGNBA			MNNB		
Statistics	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.99	1.00	1.00	0.29	0.86	0.69	0.99	1.00	0.99	0.72	0.99	0.91	0.91	0.99	0.89
PRE(.05)	0.96	1.00	0.99	0.16	0.74	0.55	0.95	1.00	0.98	0.47	0.98	0.85	0.64	0.98	0.82
PRE(.01)	0.74	0.96	0.92	0.05	0.37	0.22	0.64	0.97	0.80	0.05	0.79	0.45	0.06	0.80	0.53
Mean	-0.0001	-0.0004	-0.0003	0.3731	-0.0036	0.3726	-0.0104	-0.0003	0.0145	-0.0853	-0.0019	-0.0858	-0.0809	0.0016	-0.0815
Std Dev	0.0167	0.0014	0.0265	0.3915	0.0352	0.3957	0.0299	0.0011	0.0413	0.0626	0.0044	0.079	0.0841	0.0044	0.0966
Min	-0.0718	-0.0216	-0.782	-0.0764	-0.1193	-1.9205	-0.1621	-0.0162	-0.5865	-0.3765	-0.0566	-1.9642	-0.4246	-0.0359	-1.9426
25%ile	-0.0063	-0.0011	-0.0115	0.0929	-0.0189	0.0986	-0.011	-0.0008	-0.0069	-0.1178	-0.0039	-0.1289	-0.1256	-0.001	-0.1357
Median	-0.0011	-0.0003	-0.0006	0.2385	-0.003	0.2415	-0.0022	-0.0002	0.0019	-0.0744	-0.0012	-0.081	-0.0615	0.0011	-0.0686
75%ile	0.0046	0.0003	0.0102	0.5451	0.0031	0.5407	0.0023	0.0003	0.0167	-0.0405	0.0007	-0.0417	-0.0156	0.0036	-0.0142
Max	0.2680	0.0109	0.2461	3.7940	0.1875	3.7796	0.2010	0.0080	0.1846	0.5646	0.0269	0.5099	0.5631	0.0275	0.5084

Table B.27: For Case-II and Smooth, Error Results When the LTD is Approximated by Distribution Selection Rules (2)

	ZMADR			ZMADR2			ZADR2ADR			ZMADR2PT		
Statistics	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*	*B*	*RR*	*I*
PRE(.10)	0.03	0.37	0.06	1.00	1.00	1.00	1.00	1.00	1.00	0.25	1.00	0.98
PRE(.05)	0.01	0.21	0.03	0.99	1.00	0.99	1.00	1.00	1.00	0.25	0.22	0.26
PRE(.01)	0.00	0.05	0.02	0.91	0.98	0.96	0.95	0.99	0.99	0.20	0.19	0.17
Mean	1.4093	-0.1477	-8.5444	0.0007	-0.0003	0.0006	0	-0.0001	0	-24.6903	0.037	0.3918
Std Dev	2.2566	0.1192	7.4477	0.008	0.0009	0.0112	0.0044	0.0005	0.0064	29.8573	0.0235	0.3537
Min	-1.7232	-0.4995	-48.7399	-0.0317	-0.0255	-0.2924	-0.019	-0.0153	-0.1759	-112	-0.0119	-0.0343
25%ile	0.1841	-0.2239	-11.9959	-0.0022	-0.0005	-0.004	-0.0014	-0.0002	-0.0026	-40.494	0.0172	0.0634
Median	0.5912	-0.1155	-6.6301	-0.0001	-0.0002	0.0001	-0.0002	-0.0001	-0.0001	-11.5887	0.0428	0.3375
75%ile	1.6528	-0.0515	-3.0877	0.0025	0.0001	0.0045	0.0011	0.0001	0.0023	-0.0325	0.0579	0.6243
Max	34.4448	0.0176	4.4557	0.179	0.0079	0.1679	0.1074	0.0047	0.1007	0.0835	0.0692	1.568

Table B.28: For Standardized Error Measure Comparisons for all Model Pairs using Tukey-Kramer HSD across All Demand Classes

Models	Mean	Categ. 1	Categ. 2	Categ. 3	Categ. 4	Categ. 5	Categ. 6	Categ. 7	Categ. 8	Categ. 9	Categ. 10	Categ. 11	Categ. 12	Categ. 13
ZMADR	1.4639	A												
ZMG	1.3232		B											
ZMNB	1.2125			C										
LN	0.9169				D									
ZIP	0.8488					E								
P	0.7848						F							
AXR	0.7719						F	G						
G	0.7664						F	G						
N	0.7332							G						
MGNBA	0.6893								H					
PT	0.6851								H					
ZMADR2PT	0.6707								H					
NB	0.6552								H					
MNNB	0.6511								H					
TPM	0.3352									I				
ADR	0.2756										J			
GADR	0.2225											K		
ZMADR2	0.1436												L	
ZMADR2ADR	0.0774													M

Exhibit B.1: For Standardized Error Measure, Hsu Individual 95% CIs For Mean Based on Pooled StDev

```
Level        Mean    StDev   --------+---------+---------+---------+-
ADR          0.2756  0.3704             *
AXR          0.7719  0.7200                         *)
G            0.7664  0.9162                           *)
GADR         0.2225  0.2837          (*
LN           0.9169  1.0485                              *
MGNBA        0.6893  0.8872                     *)
MNNB         0.6511  0.8406                     *)
N            0.7332  0.8545                          *)
NB           0.6552  0.8876                       *)
P            0.7848  0.7327                           (*
PT           0.6851  0.8077                              *
TPM          0.3352  0.1632             *)
ZIP          0.8488  0.7750                          *)
ZMADR        1.4639  0.6898                                       (*
ZMADR2       0.1436  0.2664      (*
ZMADR2ADR    0.0774  0.1462  *
ZMADR2PT     0.6707  0.7410                    *)
ZMG          1.3232  0.8016                                       *
ZMNB         1.2125  0.6179                                  *)
                             --------+---------+---------+---------+-
```

Exhibit B.2: For Standardized Error Measure, Hsu Intervals for Level Mean Minus Smallest of Other Level Means

```
Level        Lower    Center   UpperLevel   --+---------+---------+---------+-------
ADR          0.0000   0.1982   0.2272         (----*)
AXR          0.0000   0.6946   0.7235         (----------------*)
G            0.0000   0.6890   0.7180         (----------------*)
GADR         0.0000   0.1451   0.1740         (---*
LN           0.0000   0.8395   0.8684         (--------------------*)
MGNBA        0.0000   0.6119   0.6409         (--------------*)
MNNB         0.0000   0.5738   0.6027         (-------------*)
N            0.0000   0.6558   0.6848         (---------------*)
NB           0.0000   0.5778   0.6068         (-------------*)
P            0.0000   0.7074   0.7364         (-----------------*
PT           0.0000   0.8427   0.8716         (--------------------*)
TPM          0.0000   0.2578   0.2867         (-----*)
ZIP          0.0000   0.7714   0.8004         (------------------*)
ZMADR        0.0000   1.3865   1.4154         (-----------------------------------*
ZMADR2       0.0000   0.0662   0.0952         (-*
ZMADR2ADR   -0.0952  -0.0662   0.0000       *-)
ZMADR2PT     0.0000   0.6978   0.7267         (----------------*)
ZMG          0.0000   1.2459   1.2748         (------------------------------*)
ZMNB         0.0000   1.1351   1.1640         (---------------------------*)
                                            --+---------+---------+---------+-------
                                            0.00      0.40      0.80      1.20
```

Table B.29: For Absolute Error Results of BO, Comparisons for all Model Pairs using Tukey-Kramer HSD across All Demand Classes

Models	Mean	Category 1	Category 2	Category 3	Category 4	Category 5
ZIP	164	A				
ZMG	135	A	B			
ZMNB	98	A	B			
ZMADR	93	A	B			
N	59		B			
G	56		B			
MGNBA	36			C		
NB	36			C		
LN	36			C		
MNNB	36			C		
PT	34			C		
ZMADR2PT	34			C		
TPM	33			C		
AXR	21			C		
P	20			C		
ADR	13				D	
GADR	11				D	
ZMADR2	5					E
ZMADR2ADR	4					E

Exhibit B.3: For Absolute Error Results of B, Hsu Individual 95% CIs For Mean Based on Pooled StDev

```
Level          Mean  StDev  -----+---------+---------+---------+----
ADR              13     51  (-----*-----)
AXR              21     98   (-----*------)
G                36    128      (-----*-----)
GADR             11     38  (-----*-----)
LN               36    120      (-----*-----)
MGNBA            36    128      (-----*-----)
MNNB             34    133      (-----*-----)
NB               36    128      (-----*-----)
ZMADR2PT         34    140      (-----*-----)
PT               34     141     (-----*-----)
P                20     96   (-----*-----)
TPM              33    649      (-----*-----)
ZIP             164   3194                          (-----*-----)
N                59    140          (-----*-----)
ZMADR            59   1032          (-----*-----)
ZMADR2            5     17 (-----*-----)
ZMADR2ADR         4     12 (-----*-----)
ZMG             135   2068                     (-----*-----)
ZMNB             93   1487              (-----*-----)
                               -----+---------+---------+---------+----
                                    0        60       120       180
```

Exhibit B.4: For Absolute Error Results of B, Hsu Intervals for Level Mean Minus Smallest of Other Level Means

```
Level        Lower  Center  Upper  ---------+---------+---------+---------+
ADR            -57      11     79    (-------*--------)
AXR            -51      17     86     (-------*--------)
G              -36      33    101       (-------*--------)
GADR           -61       7     76   (--------*-------)
LN             -36      32    101       (-------*--------)
MGNBA          -36      33    101       (-------*--------)
MNNB           -37      31     99      (--------*-------)
N              -38      30     98      (--------*-------)
NB             -36      33    101       (-------*--------)
P              -52      17     85     (-------*--------)
PT             -16      52    120         (-------*--------)
TPM            -39      30     98      (--------*-------)
ZIP              0     160    228            (-------------------*--------)
ZMADR          -13      55    123         (--------*-------)
ZMADR2         -67       1     70   (-------*--------)
ZMADR2ADR      -70      -1     67  (--------*-------)
ZMADR2PT         0      94    162            (-----------*-------)
ZMG              0     131    199            (---------------*--------)
ZMNB             0      89    157            (----------*--------)
                                   ---------+---------+---------+---------+
                                            0        80       160       240
```

Table B.30: For Absolute Error Results of RR, Comparisons for all Model Pairs using Tukey-Kramer HSD across All Demand Classes

	Mean	Category 1	Category 2	Category 3	Category 4	Category 5	Category 6	Category 7	Category 8	Category 9	Category 10	Category 11
ZMNB	0.2204	A										
ZMADR	0.1746		B									
ZMG	0.1417			C								
ZIP	0.1245			C								
P	0.1014				D							
AXR	0.0821					E						
TPM	0.0739					E	F					
N	0.0666					E	F	G				
LN	0.058						F	G	H			
MNNB	0.0543							G	H			
G	0.0521							G	H			
MGNBA	0.0513							G	H			
NB	0.0511								H			
PT	0.0441									I		
ZMADR2PT	0.0379									I		
ADR	0.0219										J	
GADR	0.0157											K
ZMADR2	0.0097											K
ZMADR2ADR	0.0058											K

Exhibit B.5: For Absolute Error Results of RR, Hsu Individual 95% CIs For Mean Based on Pooled StDev

```
Level         Mean    StDevLevel    +---------+---------+---------+---------
ADR           0.0219  0.0384           (-*)
AXR           0.0821  0.1432                      (*)
G             0.0521  0.0887                 (*)
GADR          0.0157  0.0276          (-*)
LN            0.0580  0.0941                 (-*)
MGNBA         0.0513  0.0888                (-*)
MNNB          0.0543  0.0999                 (*)
N             0.0666  0.1159                   (*)
NB            0.0507  0.0890                (*-)
P             0.1014  0.1838                         (*)
PT            0.0441  0.0670               (*-)
TPM           0.0739  0.1836                    (*)
ZIP           0.1245  0.7169                             (*)
ZMADR         0.1746  0.2082                                     (*)
ZMADR2        0.0097  0.0143         (-*)
ZMADR2ADR     0.0058  0.0093         (*)
ZMADR2PT      0.0379  0.0520              (*)
ZMG           0.1417  0.1586                               (-*)
ZMNB          0.2204  0.4672                                             (*)
                                     +---------+---------+---------+---------
                                     0.000     0.060     0.120     0.180
```

Exhibit B.6: For Absolute Error Results of RR, Hsu Intervals for Level Mean Minus Smallest of Other Level Means

```
Level           Lower   Center    Upper   --+---------+---------+---------+-------
ADR            0.0000   0.0161   0.0293     (-*-)
AXR            0.0000   0.0763   0.0895     (----------*-)
G              0.0000   0.0463   0.0595     (------*)
GADR          -0.0033   0.0099   0.0231     (*-)
LN             0.0000   0.0522   0.0654     (------*-)
MGNBA          0.0000   0.0456   0.0588     (------*)
MNNB           0.0000   0.0485   0.0617     (------*-)
N              0.0000   0.0608   0.0740     (--------*-)
NB             0.0000   0.0449   0.0581     (-----*-)
P              0.0000   0.0956   0.1088     (-------------*-)
PT             0.0000   0.0383   0.0515     (----*-)
TPM            0.0000   0.0681   0.0813     (---------*-)
ZIP            0.0000   0.1187   0.1319     (----------------*-)
ZMADR          0.0000   0.1688   0.1820     (-----------------------*-)
ZMADR2        -0.0093   0.0039   0.0171    (-*)
ZMADR2ADR     -0.0171  -0.0039   0.0093   (*-)
ZMADR2PT       0.0000   0.0321   0.0453     (----*)
ZMG            0.0000   0.1359   0.1491     (------------------*-)
ZMNB           0.0000   0.2146   0.2278     (------------------------------*-)
                                          --+---------+---------+---------+-------
                                          0.000     0.070     0.140     0.210
```

Table B.31: For Absolute Error Results of I, Comparisons for all Model Pairs using Tukey-Kramer HSD across All Demand Classes

	Mean	Category 1	Category 2	Category 3	Category 4	Category 5
ZIP	335	A				
ZMG	228		B			
ZMNB	228		B			
P	191		B	C		
AXR	182		B	C		
N	171		B	C		
MNNB	167		B	C		
LN	167		B	C		
G	165		B	C		
MGNBA	164		B	C		
NB	164		B	C		
ZMADR	129			C		
TPM	70				D	
ADR	66				D	
PT	63				D	
GADR	50				D	
ZMADR2	26					E
ZMADR2PT	19					E
ZMADR2ADR	15					E

Exhibit B.7: For Absolute Error Results of I, Hsu Individual 95% CIs For Mean Based on Pooled StDev

```
Level        Mean  StDev  --+---------+---------+---------+-------
ADR            66    184        (---*--)
AXR           182    503                    (--*--)
G             165    462                  (--*---)
GADR           50    138       (--*--)
LN            167    462                  (---*--)
MGNBA         164    461                  (--*---)
MNNB          167    468                  (---*--)
N             171    475                   (--*--)
NB            164    461                  (--*---)
P             191    513                     (--*--)
ZMADR2PT       19     52    (--*--)
TPM            70    612         (--*--)
ZIP           335   2996                                (---*--)
ZMADR         129    968               (--*--)
ZMADR2         26     71    (---*--)
ZMADR2ADR      15     42   (---*--)
PT             63   1512        (--*---)
ZMG           228   1924                          (--*--)
ZMNB          228   1436                          (--*--)
                          --+---------+---------+---------+-------
                            0        100       200       300
```

Exhibit B.8: For Absolute Error Results of I, Hsu Intervals for Level Mean Minus Smallest of Other Level Means

```
Level         Lower  Center   Upper  -----+---------+---------+---------+----
ADR             -10      51     112      (----*----)
AXR               0     167     228       (-------------*----)
G                 0     150     211       (-----------*-----)
GADR            -26      34      95     (----*----)
LN                0     152     212       (------------*----)
MGNBA             0     149     210       (-----------*-----)
MNNB              0     152     213       (------------*----)
N                 0     155     216       (------------*----)
NB                0     149     210       (-----------*-----)
P                 0     176     237       (--------------*----)
PT              -58       3      64  (----*----)
TPM              -6      55     116      (-----*----)
ZIP               0     320     381       (--------------------------*----)
ZMADR             0     114     175       (--------*-----)
ZMADR2          -50      11      72   (----*----)
ZMADR2ADR       -64      -3      58  (----*----)
ZMADR2PT        -13      48     109      (----*----)
ZMG               0     213     274       (-----------------*----)
ZMNB              0     212     273       (-----------------*----)
                                     -----+---------+---------+---------+----
                                          0       120       240       360
```

C APPENDIX

Proof of Theorem in Section 4.4.5. Let t_i be the time of the ith demand event. Let X_i be the amount of the ith demand. Let $D(t)$ be the cumulative demand through time t. Let $N(t)$ be the number of events through time t. Then

$$D(t) = \sum_{i=1}^{N(t)} X_i \qquad t \geq 0$$

We denote the realized values of random variables with lower case letters. For example, x_i represents ith realized demand. Suppose that we are given n realized demand event times such that $t_0 < t_1 < t_2 < ... < t_n$. Observe that

$$d(t_i) - d(t_{i-1}) \;=\; \sum_{i=1}^{N(t_i)} x_i - \sum_{i=1}^{N(t_{i-1})} x_{i-1} = x_{t_i} \; for\, i = 1, 2, 3, ..., n$$

Notice that

$$in(t_0) = IN(t = 0) = v$$

and

$$in'(t_0) = IN'(t = 0) = v'$$

and

$$\triangle = v' - v$$

Then

$$in'(t_0) - in(t_0) = \triangle \tag{108}$$

We consider two cases. In the first case, we assume that no outstanding orders exist between two consequetive demand events. Then, clearly,

$$\begin{aligned} in(t_1) &= in(t_0) - (d(t_1) - d(t_0)) \\ in(t_1) &= in(t_0) - x_{t_1} \\ in(t_1) &= v - x_{t_1} \end{aligned}$$

In addition,

$$\begin{aligned} in'(t_1) &= in'(t_0) - (d(t_1) - d(t_0)) \\ in'(t_1) &= in'(t_0) - x_{t_1} \\ in'(t_1) &= v' - x_{t_1} \end{aligned}$$

Thus, observe that

$$in'(t_1) = in(t_0) - x_{t_1} + \triangle$$

$$in'(t_1) = in(t_1) + \triangle \tag{109}$$

By (108) and (109), for n demand events

$$\begin{aligned} in'(t_2) &= in(t_2) + \triangle \\ in'(t_3) &= in(t_3) + \triangle \\ &\ldots \\ in'(t_n) &= in(t_n) + \triangle \end{aligned}$$

Therefore,

$$in'(t_i) = in(t_i) + \triangle \quad for\, i = 0, 1, 2, ..., n \tag{110}$$

Note also that $IN(t)$ only changes value at demand arrivals and replenishment times. Thus,

$$IN'(t) = IN(t) + \triangle \tag{111}$$

for all t that are not replenishment event times.

In the second case, we assume that an outstanding order (Q) exists between an infinitesimal time interval. Let A_i be the time of the ith replenishment event where $i = 1,2,3,...,n$. Denote A_i^- as an infinitesimal time prior to A_i and let A_i^+ be an infinitesimal time after A_i. By the development of net inventory process, clearly,

$$in\left(A_1^+\right) - in\left(A_1^-\right) = Q$$

and

$$in'\left(A_1^+\right) - in'\left(A_1^-\right) = Q$$

Then, since $IN'(t) = IN(t) + \triangle$ from (111) above, we have:

$$in'\left(A_1^+\right) = in\left(A_1^-\right) + \triangle + Q$$

For other replenisment events

$$\begin{aligned} in'\left(A_2^+\right) &= in\left(A_2^-\right) + \triangle + Q \\ in'\left(A_3^+\right) &= in\left(A_3^-\right) + \triangle + Q \\ &\cdots \\ in'\left(A_n^+\right) &= in\left(A_n^-\right) + \triangle + Q \end{aligned}$$

Therefore,

$$in'\left(A_i^-\right) = in\left(A_i^-\right) + \triangle \quad for\, i = 1,2,...,n$$

and

$$in'\left(A_i^+\right) = in\left(A_i^+\right) + \triangle \;\; for\, i = 1,2,...,n$$

which implies that

$$in'\left(A_i\right) = in\left(A_i\right) + \triangle \;\; for\, i = 1,2,...,n \tag{112}$$

Based on (111) and (112),

$$IN_t\left(r', v'\right) = IN_t\left(r + \triangle, v + \triangle\right) = IN_t\left(r, v\right) + \triangle \quad \forall \triangle \in \mathbb{Z}\,\square$$