§4. Averaged Resistive MHD Equations

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Averaged resistive MHD equations are derived using the Lagrangian formalism. Since no ordering assumptions are not made, the obtained equations are consistent to the 3D equilibrium.

The curvilinear coordinate system (X,Y,ζ) are assumed, in which the magnetic inductions $H^\alpha \equiv \sqrt{g}B^\alpha$ are independent to the toroidal angle ζ. It is convenient to introduce the notations

$$\bar{\boldsymbol{r}}(X,Y,\zeta) \equiv X\boldsymbol{e}_R(\zeta) + Y\boldsymbol{e}_Z,$$

$$\underline{\nabla} = \underline{\nabla}x^\alpha \frac{\partial}{\partial x^\alpha} \equiv \boldsymbol{e}_r \frac{\partial}{\partial X} + \boldsymbol{e}_z \frac{\partial}{\partial Y} + \frac{1}{X}\boldsymbol{e}_\phi \frac{\partial}{\partial \zeta},$$

$$\overline{\boldsymbol{B}} = \frac{1}{\sqrt{g_0}} H^\alpha \bar{\boldsymbol{r}}_\alpha, \quad \underline{\boldsymbol{B}} = \langle B_\alpha \rangle_\zeta \underline{\nabla}x^\alpha,$$

$$\bar{\boldsymbol{J}} = \frac{1}{\sqrt{g_0}} \langle \sqrt{g} J^\alpha \rangle_\zeta \bar{\boldsymbol{r}}_\alpha,$$

and $\mathcal{J}_* \equiv \langle \sqrt{g} \rangle_\zeta / \sqrt{g_0}$, $\sqrt{g_0} \equiv X$ being the Jacobian in the cylindrical coordinates. Also we introduce the tensor

$$\underline{\underline{\mathbf{G}}} \equiv \sqrt{g_0} \left\langle \frac{g_{\alpha\beta}}{\sqrt{g}} \right\rangle_\zeta \underline{\nabla}x^\alpha \underline{\nabla}x^\beta,$$

so that $\underline{\boldsymbol{B}} = \underline{\underline{\mathbf{G}}} \cdot \overline{\boldsymbol{B}}$. Then the MHD equilibrium equations averaged over the toroidal angle can be written in the form

$$\bar{\boldsymbol{J}} \times \overline{\boldsymbol{B}} = \mathcal{J}_* \underline{\nabla}P, \quad \bar{\boldsymbol{J}} = \underline{\nabla} \times \underline{\boldsymbol{B}}, \quad \underline{\nabla} \cdot \overline{\boldsymbol{B}} = 0.$$

The perturbation with the time dependence $\exp(qt)$ is assumed. Then the linearized resistive MHD equations are derived as

$$\frac{\partial \mathcal{L}_0[\xi,\boldsymbol{a}]}{\partial \xi} - \frac{\partial \mathcal{M}[\xi,\boldsymbol{a}]}{\partial \xi} = 0,$$

$$\frac{\partial \mathcal{L}_0[\xi,\boldsymbol{a}]}{\partial \boldsymbol{a}} + \frac{\partial \mathcal{M}[\xi,\boldsymbol{a}]}{\partial \boldsymbol{a}} = 0.$$

with the Lagrangian dendities [1)]

$$\mathcal{L}_0[\xi,\boldsymbol{a}] = q^2\rho\xi^2 + q\eta\boldsymbol{a}^2 + [\boldsymbol{Q} - \nabla\times(\eta\boldsymbol{a})]\cdot[\boldsymbol{Q} + \boldsymbol{J}\times\xi - \nabla\times(\eta\boldsymbol{a})] + (\xi\cdot\nabla P)\nabla\cdot\xi + \gamma_s P(\nabla\cdot\xi)^2,$$

$$\mathcal{M}[\xi,\boldsymbol{a}] = \boldsymbol{J}\times\xi\cdot\nabla\times(\eta\boldsymbol{a}),$$

where ξ _ stands for the plasma displacement, $\boldsymbol{a}$ is the electric displacement which is related to the perturbed magnetic field $\boldsymbol{b}$ by the relation $\boldsymbol{a} = \nabla\times\boldsymbol{b}/q$, and $\boldsymbol{Q} \equiv \nabla\times(\xi\times\boldsymbol{B})$.

We assume that the coordinate are chosen so that the Jacobian $\sqrt{g}$ is independent to ζ, and the perturbation is of single toroidal harmonics on this magnetic coordinates. Then we can obtain the equations

$$q^2\rho\mathcal{J}_*^2\underline{\underline{\mathbf{G}}}\cdot\bar{\xi} = -\mathcal{J}_*\underline{\nabla}p_1 + \bar{\boldsymbol{J}}\times\bar{\boldsymbol{Q}} + \underline{\nabla}\times(\underline{\underline{\mathbf{G}}}\cdot\bar{\boldsymbol{Q}})\times\overline{\boldsymbol{B}} + \underline{\nabla}\times(\eta\underline{\boldsymbol{a}})\times\bar{\boldsymbol{J}} + \overline{\boldsymbol{B}}\times\underline{\nabla}\times[\underline{\underline{\mathbf{G}}}\cdot\underline{\nabla}\times(\eta\underline{\boldsymbol{a}})],$$

$$\mathcal{J}_*p_1 = \mathcal{J}_*(\bar{\xi}\cdot\underline{\nabla}P) + \gamma_s P\underline{\nabla}\cdot(\mathcal{J}_*\bar{\xi}),$$

$$q\underline{\underline{\mathbf{G}}}^{-1}\cdot\underline{\boldsymbol{a}} = \underline{\nabla}\times(\underline{\underline{\mathbf{G}}}\cdot\bar{\boldsymbol{Q}}) - \underline{\nabla}\times[\underline{\underline{\mathbf{G}}}\cdot\underline{\nabla}\times(\eta\underline{\boldsymbol{a}})].$$

If we use the relation $q\underline{\boldsymbol{a}} \equiv \underline{\underline{\mathbf{G}}}\cdot\underline{\nabla}\times(\underline{\underline{\mathbf{G}}}\cdot\bar{\boldsymbol{b}})$, we can write the equation in tems of the perturbed magnetic field $\boldsymbol{b}$

$$q^2\rho\mathcal{J}_*^2\underline{\underline{\mathbf{G}}}\cdot\bar{\xi} = -\mathcal{J}_*\underline{\nabla}p_1 + \bar{\boldsymbol{J}}\times\bar{\boldsymbol{b}} + \underline{\nabla}\times(\underline{\underline{\mathbf{G}}}\cdot\bar{\boldsymbol{b}})\times\overline{\boldsymbol{B}},$$

$$\bar{\boldsymbol{b}} - \underline{\nabla}\times(\bar{\xi}\times\overline{\boldsymbol{B}}) + \frac{1}{q}\underline{\nabla}\times[\eta\underline{\underline{\mathbf{G}}}\cdot\underline{\nabla}\times(\underline{\underline{\mathbf{G}}}\cdot\bar{\boldsymbol{b}})] = 0.$$

1) J.Todoroki: J.Phys. Soc. Jpn. **61** (1992) 2615