§ 1. Calculating Rotational Transform Following Field Line

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Magnetic coordinates are commonly used in the study of plasma in toroidal magnetic confinement devices. To construct the magnetic coordinates, Boozer's algorithm[1] following the magnetic field line is widely used. In such calculations the rotational transform is the most important quantity of the magnetic surface. In this paper the efficient and accurate method calculating the rotational transform is described.

We consider the case in which the field line is traced in the cylindrical coordinates (r, z, ϕ). If the azimuthal component of the magnetic field does not vanish, the equations for the magnetic field line can be written in the form

$$\frac{\mathrm{d}r}{\mathrm{d}\phi}=\frac{rB_r}{B_\phi},\quad \frac{\mathrm{d}z}{\mathrm{d}\phi}=\frac{rB_z}{B_\phi}. \tag{1}$$

If we accept the definition of the rotational angle

$$\tan\vartheta=\frac{z(\phi)-z_A(\phi)}{r(\phi)-r_A(\phi)}, \tag{2}$$

where $(r_A(\phi),z_A(\phi))$ stands for a closed curve contained inside the magnetic surface, e.g., the magnetic axis, the rotational angle can be calculated by integrating the equation

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}\phi}=\frac{(r-r_A)}{(r-r_A)^2+(z-z_A)^2}\left\{\frac{\mathrm{d}z}{\mathrm{d}\phi}-\frac{\mathrm{d}z_A}{\mathrm{d}\phi}\right\}-\frac{(z-z_A)}{(r-r_A)^2+(z-z_A)^2}\left\{\frac{\mathrm{d}r}{\mathrm{d}\phi}-\frac{\mathrm{d}r_A}{\mathrm{d}\phi}\right\}. \tag{3}$$

This equation is solved simultaneously with the equations for the field line.

On the magnetic surface, the angle ϑ is expressed in terms of the magnetic coordinates θ and ϕ as

$$\vartheta=\theta+\sum_{m,n}c_{m,n}\exp(\mathrm{i}[m\theta-n\phi]). \tag{4}$$

We assume that the angle variable θ is chosen so that the magnetic field line is straight in θ - ϕ plane. Since $\theta=\iota\phi$ on a field line, the rotational angle behaves along the field line as

$$\vartheta=\iota\phi+\sum_{m,n}c_{m,n}\exp(\mathrm{i}\,\omega_{m,n}\phi), \tag{5}$$

where $\omega_{m,n}\equiv m\iota-n$.

Differentiating eq.(5) with respect to ϕ, we obtain

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}\phi}=\iota+\mathrm{i}\sum_{m,n}c_{m,n}\omega_{m,n}\exp(\mathrm{i}\,\omega_{m,n}\phi). \tag{6}$$

The rotational transform ι is the DC part of the function in the right hand side of eq.(6). Hence by posing the Gaussian window in order to eliminate the end effects, we can calculate using the relation

$$\iota\cong\frac{\alpha}{\sqrt{\pi}}\int_{-\phi_{\max}}^{\phi_{\max}}\frac{\mathrm{d}\vartheta}{\mathrm{d}\phi}\exp\left(-\frac{\alpha^2\phi^2}{\phi_{\max}{}^2}\right)\frac{\mathrm{d}\phi}{\phi_{\max}}, \tag{7}$$

with α a number chosen to achieve a desired level of accuracy. In practice, α is about 3. In the actual calculation, we consider the following differential equation

$$\frac{\mathrm{d}w}{\mathrm{d}\phi}=\frac{\mathrm{d}\vartheta}{\mathrm{d}\phi}\frac{\alpha}{\sqrt{\pi}}\exp\left(-\frac{\alpha^2\phi^2}{\phi_{\max}{}^2}\right) \tag{8}$$

with the initial condition $w(0)=0$. The equation is solved simultaneously with the other equations following the magnetic line of force. The equations for the magnetic axis are also solved simultaneously, if required. After the integration is finished, the rotational transform is obtained by the formula

$$\iota=\frac{w(\phi_{\max})-w(-\phi_{\max})}{\phi_{\max}}. \tag{9}$$

The same technique can be used to obtain the other surface quantities, such as the specific volume

$$U\cong\frac{\alpha}{\sqrt{\pi}}\int_{-\phi_{\max}}^{\phi_{\max}}\frac{r}{B_\phi}\exp\left(-\frac{\alpha^2\phi^2}{\phi_{\max}{}^2}\right)\frac{\mathrm{d}\phi}{\phi_{\max}}. \tag{10}$$

Finally, it is worth noting that the method described in this paper has no difficulty concerning the rational surface.

[1] A. Boozer, Phys. Fluids **25**, 520 (1982).