§12. Conservation of Energy and Momentum in Nonrelativistic Plasmas

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Conservation laws of energy and momentum are fundamental properties of general physical systems which have symmetries with respect to translations in time and space as shown by Noether's theorem. Recently, based on gyrokinetic formulations, energy and momentum conservation in tokamak plasmas has been actively investigated to accurately describe transport processes determining energy and flow profiles. The energy-momentum conservation laws for the Vlasov-Maxwell equations were derived by Brizard using the Noether method while the gyrokinetic model is an approximate representation of the Vlasov-Poisson-Ampère system in which electromagnetic waves propagating at the light speed c are removed. For a useful reference to the gyrokinetic conservation laws, the present work [1] derives conservation laws for the Vlasov-Poisson-Ampère system and show how they differ from those for the full Vlasov-Maxwell system.

The governing equations for the Vlasov-Poisson-Ampère system are given by

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \frac{e_a}{m_a}\left\{\mathbf{E}(\mathbf{x},t) + \frac{1}{c}\mathbf{v}\times\mathbf{B}(\mathbf{x},t)\right\}\cdot\frac{\partial}{\partial \mathbf{v}}\right] f_a(\mathbf{x},\mathbf{v},t) = 0, \tag{1}$$

$$\nabla^2\phi(\mathbf{x},t) = -4\pi\sum_a e_a\int f_a(\mathbf{x},\mathbf{v},t)d^3\mathbf{v}, \tag{2}$$

and

$$\nabla^2\mathbf{A}(\mathbf{x},t) = -\frac{4\pi}{c}\mathbf{j}_T. \tag{3}$$

Here, $f_a(\mathbf{x},\mathbf{v},t)$ denotes the particle distribution function for species a and the electromagnetic fields are written as $\mathbf{E} = -\nabla\phi - c^{-1}\partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla\times\mathbf{A}$. Here, we use the Coulomb gauge condition $\nabla\cdot\mathbf{A} = 0$. The current density defined by $\mathbf{j} \equiv \sum_a e_a\int f_a d^3\mathbf{v}$ can be written as $\mathbf{j} = \mathbf{j}_L + \mathbf{j}_T$, where $\mathbf{j}_L \equiv -(4\pi)^{-1}\nabla\int d^3\mathbf{x}'(\nabla'\cdot\mathbf{j})/|\mathbf{x}-\mathbf{x}'|$ and $\mathbf{j}_T \equiv (4\pi)^{-1}\nabla\times(\nabla\times\int d^3\mathbf{x}'\,\mathbf{j}/|\mathbf{x}-\mathbf{x}'|)$ represent the longitudinal and transverse parts, respectively. The longitudinal and transverse parts of $\mathbf{E}$ are given by $\mathbf{E}_L = -\nabla\phi$ and $\mathbf{E}_T = -c^{-1}\partial\mathbf{A}/\partial t$, respectively.

Conservation laws of energy and momentum for nonrelativistic plasmas are derived [1] from applying Noetherfs theorem to the action integral for the Vlasov-Poisson-Ampère system [2]. The symmetric pressure tensor is obtained from modifying the asymmetric canonical pressure tensor with using the rotational symmetry of the action integral [1].

Defining the particle energy density $\mathcal{E}_p$ and the particle energy flux $\mathbf{Q}_p$ by

$$\begin{aligned}\mathcal{E}_p &= \sum_a\int d^3\mathbf{v}\, f_a(\mathbf{x},\mathbf{v},t)\frac{1}{2}m_a|\mathbf{v}|^2,\\ \mathbf{Q}_p &= \sum_a\int d^3\mathbf{v}\, f_a(\mathbf{x},\mathbf{v},t)\frac{1}{2}m_a|\mathbf{v}|^2\mathbf{v},\end{aligned} \tag{4}$$

the energy conservation law is written as

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\mathcal{E}_p + \frac{|\mathbf{E}_L|^2+|\mathbf{B}|^2}{8\pi}\right)\\ &\quad+\nabla\cdot\left(\mathbf{Q}_p + \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} - \frac{1}{4\pi}\frac{\partial\phi}{\partial t}\mathbf{E}_T\right)\\ =\;&\frac{\partial}{\partial t}\left(\mathcal{E}_p + \frac{|\mathbf{E}_L|^2+2\mathbf{E}_L\cdot\mathbf{E}_T+|\mathbf{B}|^2}{8\pi}\right)\\ &\quad+\nabla\cdot\left(\mathbf{Q}_p + \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} + \frac{1}{4\pi}\phi\frac{\partial\mathbf{E}_T}{\partial t}\right)\\ =\;&0.\end{aligned} \tag{5}$$

The momentum conservation law is given by

$$\frac{\partial}{\partial t}(\mathbf{P}_p + \mathbf{P}_f) + \nabla\cdot(\mathbf{\Pi}_p + \mathbf{\Pi}_f) = 0, \tag{6}$$

where the particle and field parts of the momentum density and the pressure tensor are defined by

$$\begin{aligned}\mathbf{P}_p &\equiv \sum_a\int d^3\mathbf{v}\, f_a(\mathbf{x},\mathbf{v},t)m_a\mathbf{v},\\ \mathbf{P}_f &\equiv \frac{\mathbf{E}_L\times\mathbf{B}}{4\pi c},\\ \mathbf{\Pi}_p &\equiv \sum_a\int d^3\mathbf{v}\, f_a(\mathbf{x},\mathbf{v},t)m_a\mathbf{v}\mathbf{v},\\ \mathbf{\Pi}_f &\equiv \frac{1}{8\pi}(|\mathbf{E}_L|^2+2\mathbf{E}_L\cdot\mathbf{E}_T+|\mathbf{B}|^2)\mathbf{I}\\ &\quad-\frac{1}{4\pi}(\mathbf{E}_L\mathbf{E}_L+\mathbf{E}_L\mathbf{E}_T+\mathbf{E}_T\mathbf{E}_L+\mathbf{B}\mathbf{B}).\end{aligned} \tag{7}$$

Differences between the resultant conservation laws and those for the Vlasov-Maxwell system including the Maxwell displacement current are clarified by Eqs.(4)–(7). These results provide a useful basis for gyrokinetic conservation laws because gyrokinetic equations are derived as an approximation of the Vlasov-Poisson-Ampère system.

1) H. Sugama, T.-H. Watanabe, M. Nunami, Phys. Plasmas **20**, 024503 (2013).
2) H. Sugama, Phys. Plasmas **7**, 466 (2000).